Dynamics on invariant graphs in planar continuous piecewise linear maps

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I will present results of the recent articles:

Cima, Gasull, Mañosas, VM, Invariant graphs and dynamics of a family of continuous piecewise linear planar maps. *Qualitative Theory of Dynamical Systems* **24**:70 (2025), 103 pp.

Cima, Gasull, Mañosas, VM, Further results for a family of continuous piecewise linear planar maps, arXiv:2503.02411 [math.DS]

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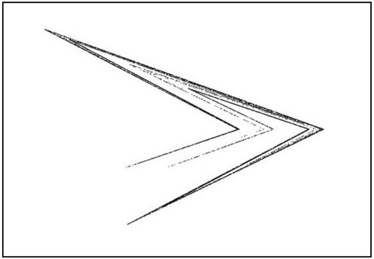
We all know that discrete **piecewise linear** dynamical systems can be very complex, even for one-dimensional maps. In the plane it is also common knowledge:



A strange attractor for the Lozi map (1978): $F(x, y) = (1 - \alpha|x| + y, \beta y)$

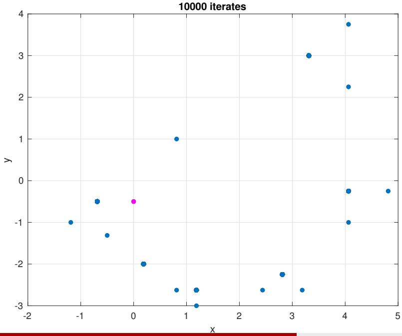
Just to highlight some recent work in our community, we cite Gardini, Sushko & Matsuyama and Gardini & Tikjha.

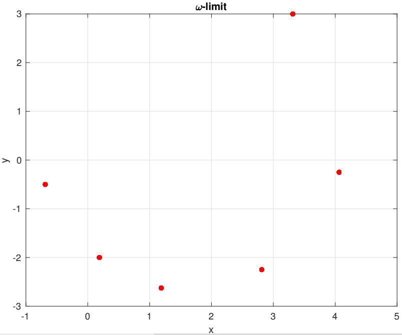
In the last years we have investigated the following family of DDS

$$F(x, y) = (|x| - y + a, x - |y| + b)\,.$$

There are some works of Tikjha, Lapierre & Lenbury..., Bula & Sīle and Aiewcharoen, Boonklurb & Konglawan focusing on periodic and pre-periodic behaviors for some values of the parameters.

Numerically, only periodic and pre-periodic behavior are founds. For instance, for $a = -1, b = -13/16$ we obtain:

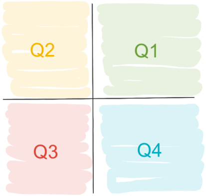


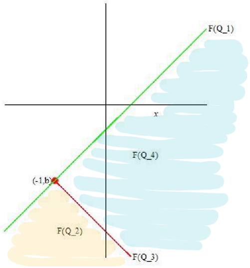


$$F(x,y) = (|x| - y + a, x - |y| + b)\,,$$

We observe that on each quadrant Q_i the map F is an affine one:

$$\begin{aligned}
Q_1 &= \{(x,y) : x \geq 0, y \geq 0\}, \quad & F_1(x,y) &= (x - y + a, x - y + b),\\
Q_2 &= \{(x,y) : x \leq 0, y \geq 0\}, \quad & F_2(x,y) &= (-x - y + a, x - y + b),\\
Q_3 &= \{(x,y) : x \leq 0, y \leq 0\}, \quad & F_3(x,y) &= (-x - y + a, x + y + b),\\
Q_4 &= \{(x,y) : x \geq 0, y \leq 0\}, \quad & F_4(x,y) &= (x - y + a, x + y + b).
\end{aligned}$$





Main results in a few words

For the map

$$F(x,y) = (|x| - y + a, x - |y| + b),$$

we prove:

Periodic and pre-periodic behavior:

- For $a \geq 0$ all the orbits are pre-periodic and moreover the set of periodic orbits has finite cardinality.

A novel behavior:

- For $a < 0$ we prove that there exists a compact graph Γ, which is invariant under the map F, such that for all $(x,y) \in \mathbb{R}^2$ there exists $n = n(x,y) \in \mathbb{N}$ such that $F^n(x,y) \in \Gamma$.
- We give explicitly all these invariant graphs Γ.
- We characterize the dynamics of the map restricted to each graph. In particular for each (a,b) **we characterize when $F|_\Gamma$ has zero or positive entropy**.

$\Rightarrow$ **Intermediate dynamical behavior between regular regime and full-plane chaos.**

Given

$$F_{a,b}(x,y) = (|x| - y + a, x - |y| + b)\,,$$

Note that for any $\lambda > 0$ we get

$$\lambda F_{a,b}(x/\lambda, y/\lambda) = F_{\lambda a,\lambda b}(x,y),$$

which says that for any $a, b \in \mathbb{R}^2$ and for any $\lambda > 0$:

The maps $F_{\lambda a,\lambda b}$ and $F_{a,b}$ are conjugated.

The above observation allows to reduce the full study when $(a,b) \in \mathbb{R}^2$ to the following five cases:

- $a = 0$ which reduces to three cases: $b = 0$; $b = -1$; $b = 1$.
- $a = 1$ and $b \in \mathbb{R}$.
- $a = -1$ and $b \in \mathbb{R}$.

Denote by $Per(F)$ the set of periodic points of F.

We will say that the dynamics of F is **pre-periodic** (in finite time) if for all $(x, y) \in \mathbb{R}^2$ there exists $n = n(x, y) \geq 0$ such that $F^n(x, y) \in Per(F)$.

Theorem A

If $a \geq 0$ then F is pre-periodic. Moreover, the set $Per(F)$ has finite cardinality.

This result extends previous ones obtained by the authors cited before.

Proposition (case $a = 0$)

Assume that $a = 0$:

(a) If $b = 0$, then $(0, 0)$ is the fixed point of F and $F^5(\mathbb{R}^2) = (0, 0)$.

(b) If $b = -1$, then $(1, 0)$ is the fixed point of F and $F^6(\mathbb{R}^2) = (1, 0)$.

(c) If $b = 1$, then F has the fixed point $p = (-1/5, 2/5)$, and the two 3-periodic orbits $\mathcal{Q}$ and $\mathcal{P}$ (explicit).

The orbit of any other point reaches $\mathcal{P} \cup \mathcal{Q}$ in finite time *(not uniform)*.

Proposition (case $a = 1$)

When $a = 1$, the following statements hold.

(a) For $b \leq 2$, F has the fixed point $p = (2 - b, 1) \in Q_1$.

For $b \in [-1/2, 2]$ and for all $(x, y) \in \mathbb{R}^2$, $F^5(x, y) = p$, while for $b < -1/2$, $F^6(x, y) = p$.

(b) For $b > 2$, F has the fixed point $q = \left(\frac{2-b}{5}, \frac{1+2b}{5}\right) \in Q_2$.

Also it has two 3-periodic orbits, $\mathcal{P}$ and $\mathcal{Q}$ (explicit).

The orbit of any other point reaches $\mathcal{P} \cup \mathcal{Q}$ in finite time *(not uniform)*.

Let us study

$$F(x, y) = (|x| - y + a, x - |y| + b)\,,$$

with $a < 0$.

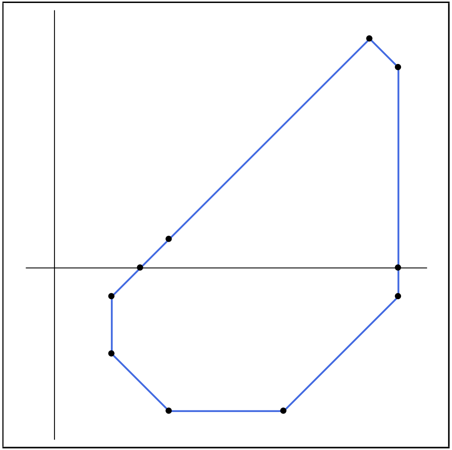
Theorem B

Set $a < 0$. For all $b \in \mathbb{R}$ there exists a compact graph Γ which is invariant under the map F such that for all $\mathbf{x} = (x, y) \in \mathbb{R}^2$ there exists $n_{\mathbf{x}} \in \mathbb{N}$ (that depends on $\mathbf{x}$) such that $F^n_{\mathbf{x}}(x, y) \in \Gamma$.

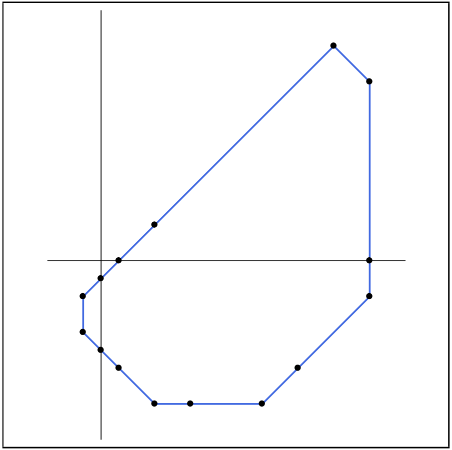
So final dynamics takes place in certain compact graphs, that we will see right now, **however**:

There are many distinct dynamics on each graph.

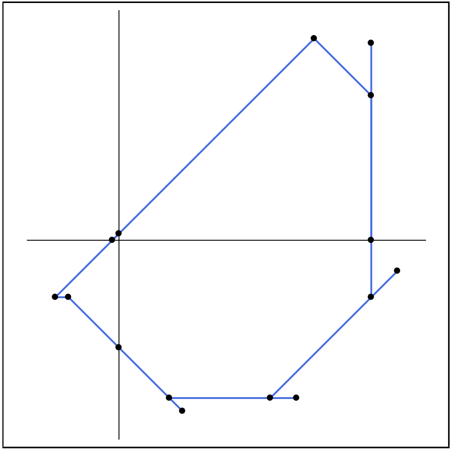
Complete catalog of graphs for the normalized case $a = -1$: $\boxed{a = -1,\, b \leq -2}$



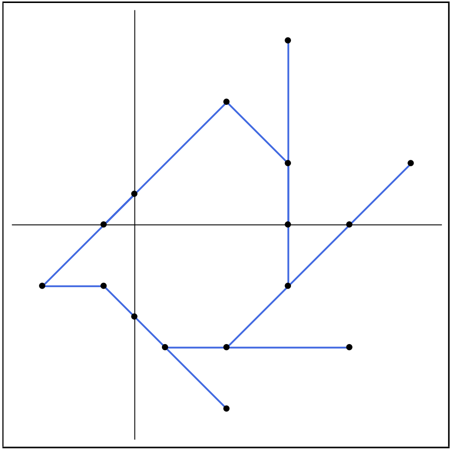
$\boxed{a = -1,\ -2 < b \leq -1}$



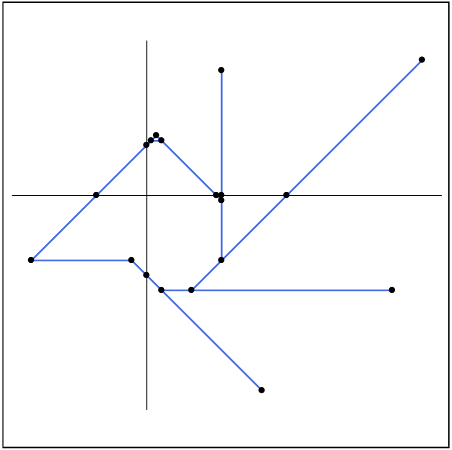
$a = -1,\ -1 < b \leq -3/4$



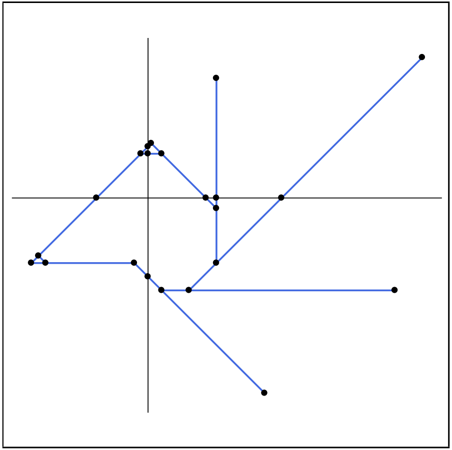
$a = -1, \ -3/4 < b \leq -1/4$



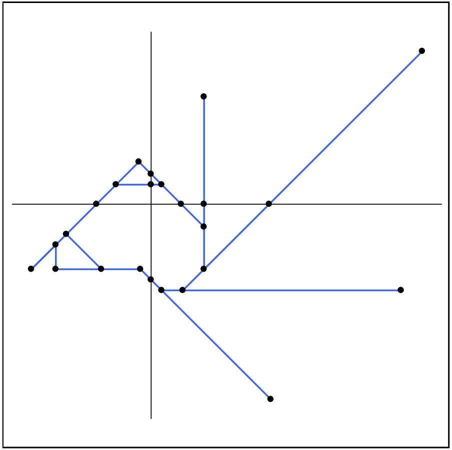
$$\boxed{a = -1,\ -1/4 < b \leq -2/9}$$



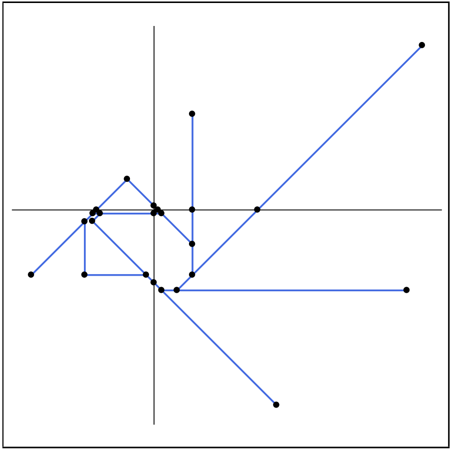
$\boxed{a = -1,\ -2/9 < b \leq -1/5}$



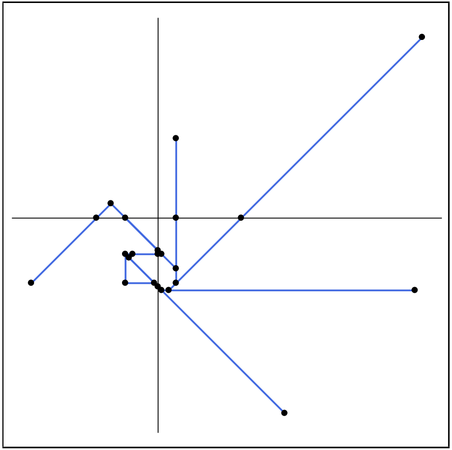
$a = -1,\ -1/5 < b \leq -1/8$



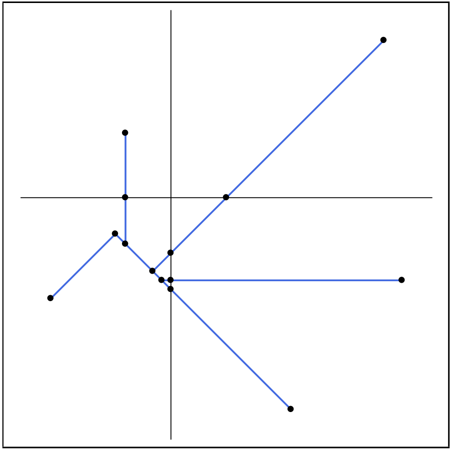
$\boxed{a = -1,\ -1/8 < b \leq -1/9}$



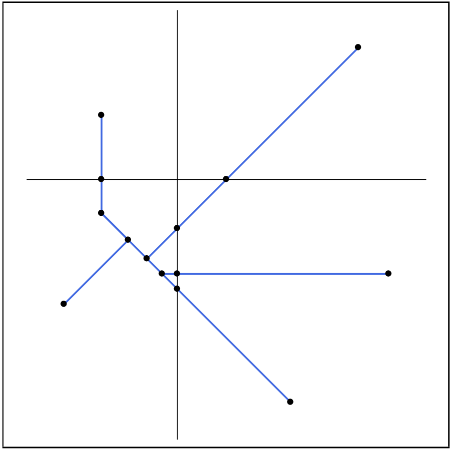
$\boxed{a=-1,\ -1/9<b<0}$



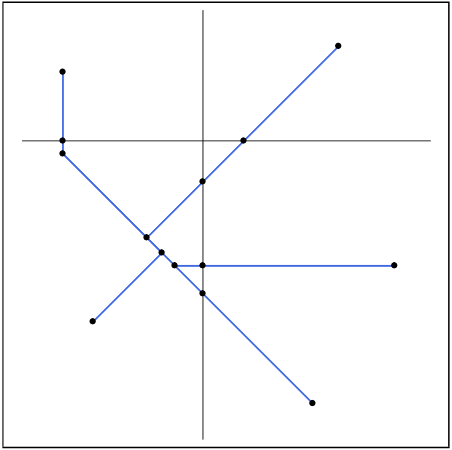
$\boxed{a = -1,\ 0 \leq b \leq 1/10}$



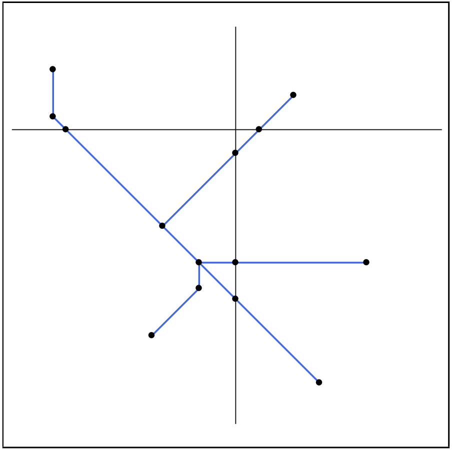
$\boxed{a = -1,\ 1/10 < b \leq 1/7}$



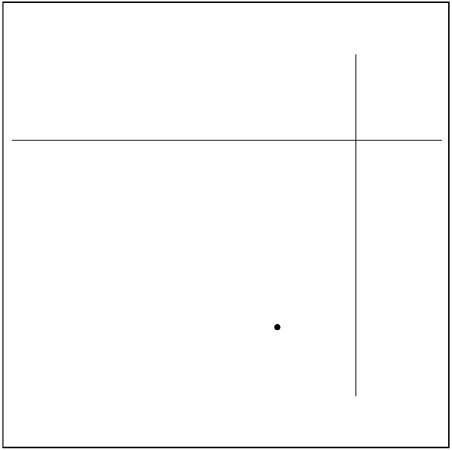
$a = -1,\ 1/7 < b \leq 1/6$



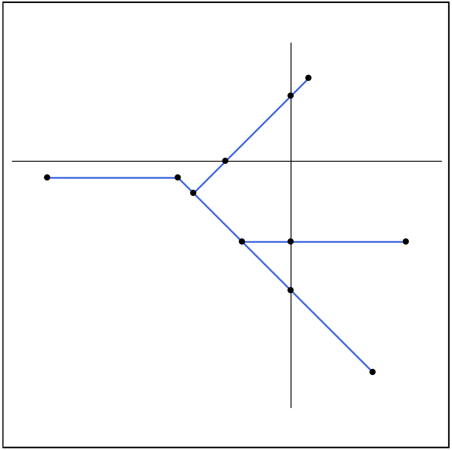
$\boxed{a = -1,\ 1/6 < b \leq 3/16}$



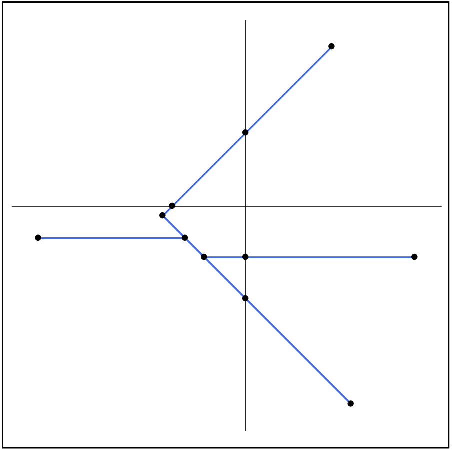
$\boxed{a=-1,\ 3/16<b<4/15}$



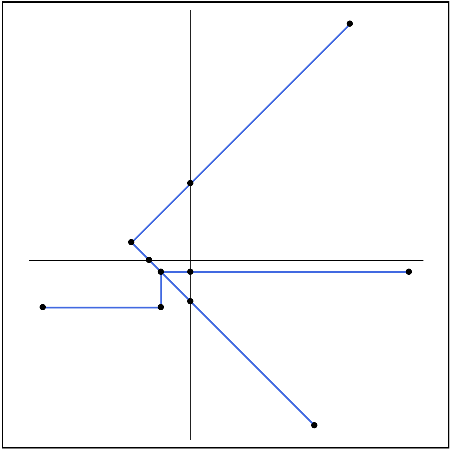
$$\boxed{a = -1,\ 4/15 < b \leq 2/7}$$



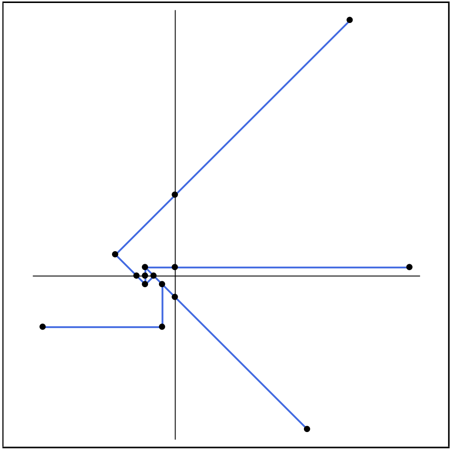
$\boxed{a = -1,\ 2/7 < b \leq 1/3}$



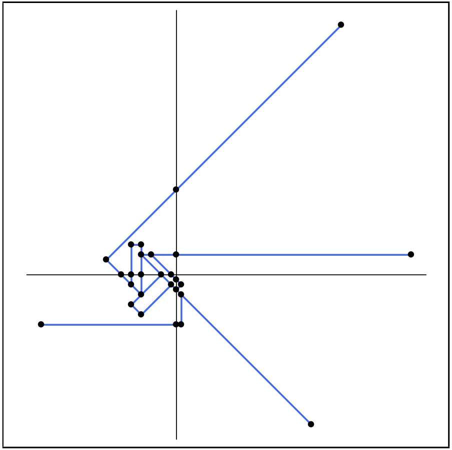
$a = -1,\ 1/3 < b \leq 1/2$



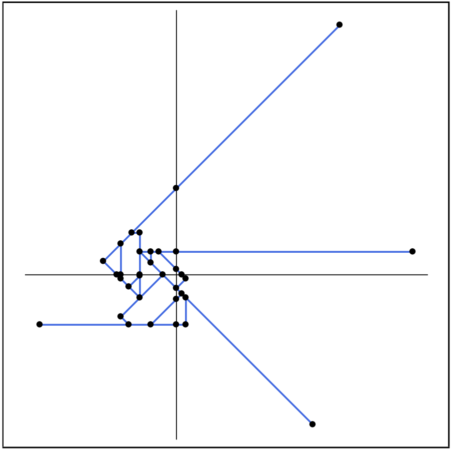
$a = -1,\ 1/2 < b \leq 2/3$



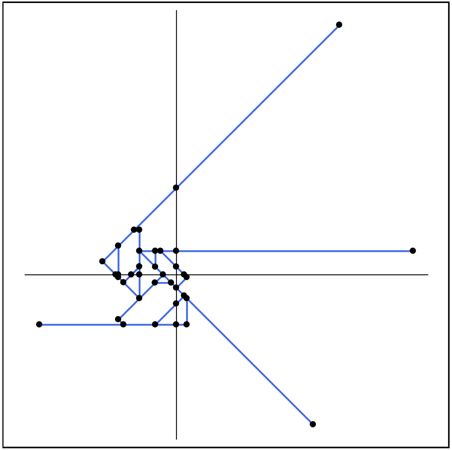
$a = -1,\ 2/3 < b \leq 5/7$



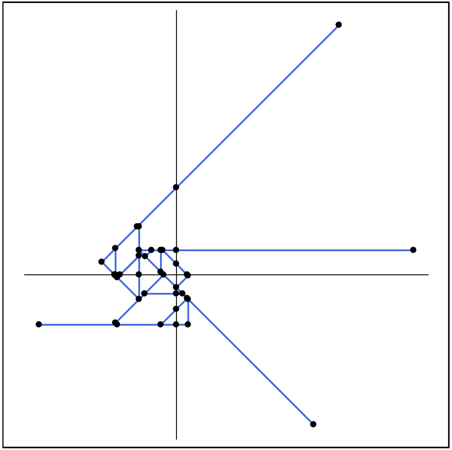
$a = -1,\ 5/7 < b \leq 19/26$



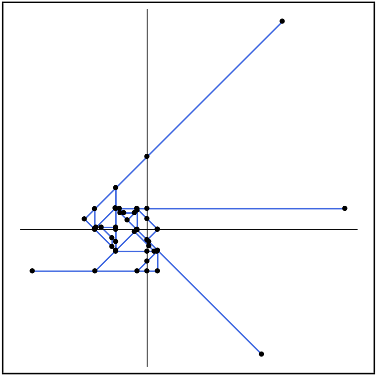
$$\boxed{a = -1,\ 19/26 < b \leq 20/27}$$



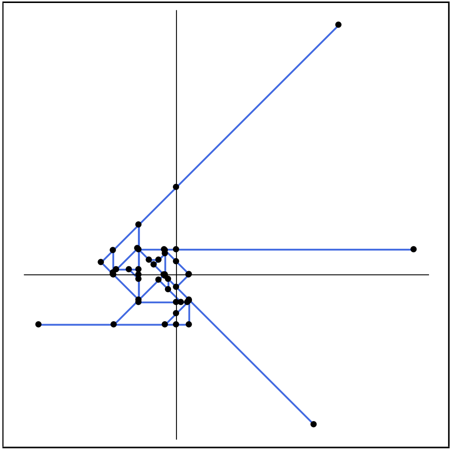
$\boxed{a = -1,\ 20/27 < b \leq 3/4}$



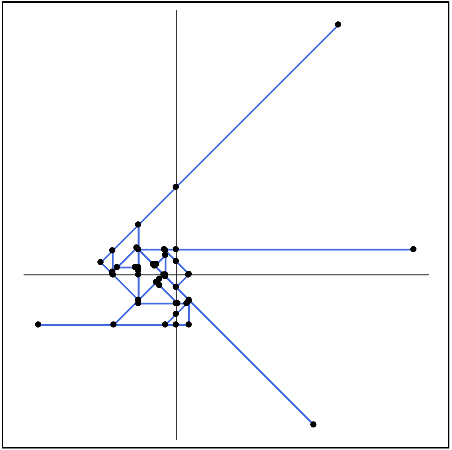
$a = -1,\ 3/4 < b \leq 154/205$



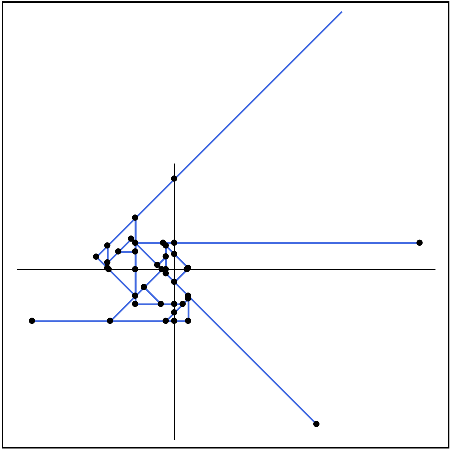
$\boxed{a = -1,\ 154/205 < b \leq 155/206}$



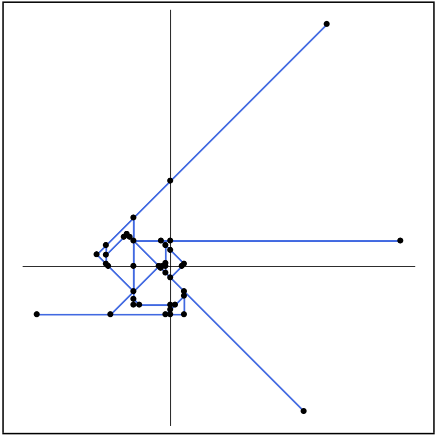
$a = -1,\ 155/206 < b \leq 58/77$



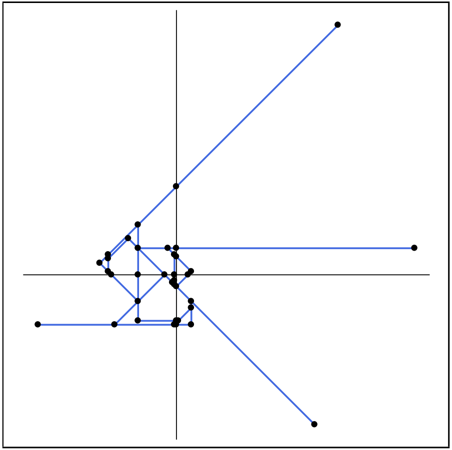
$a = -1,\ 58/77 < b \leq 19/25$



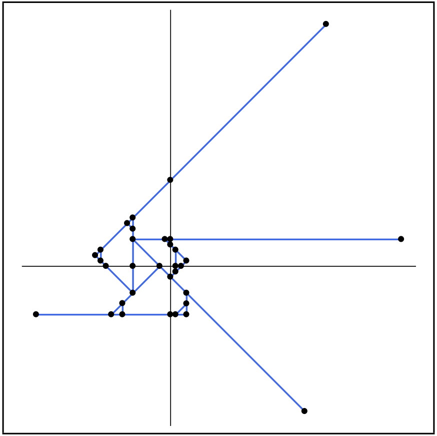
$a = -1,\ 19/25 < b \leq 29/38$



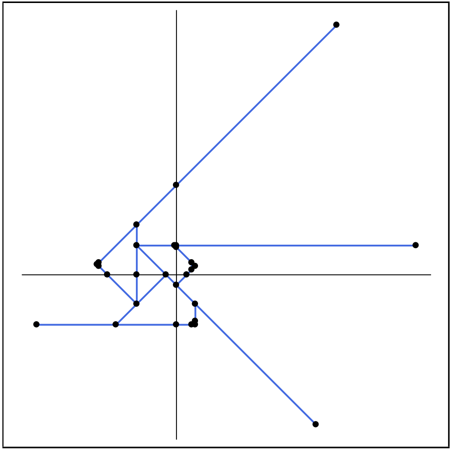
$a = -1,\ 29/38 < b \leq 10/13$



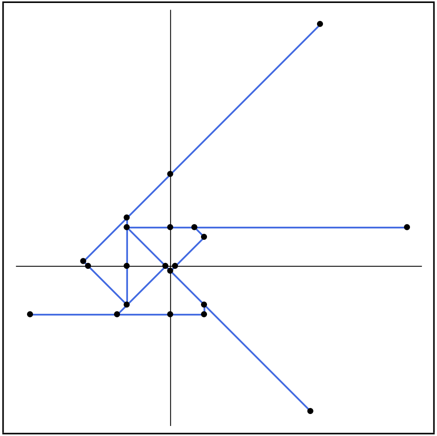
$a = -1,\ 10/13 < b \leq 11/14$



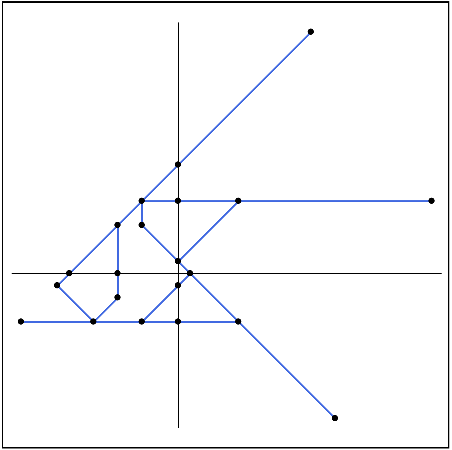
$a = -1,\ 11/14 < b \leq 4/5$



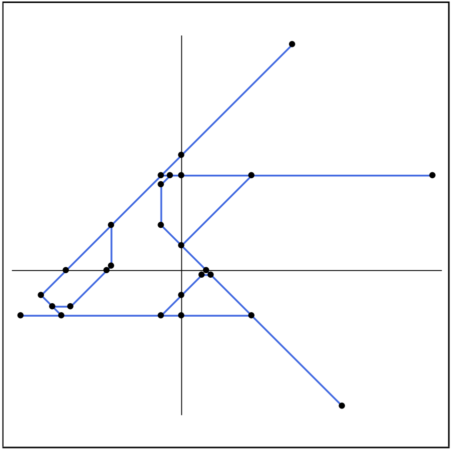
$\boxed{a=-1,\ 4/5<b\leq 1}$



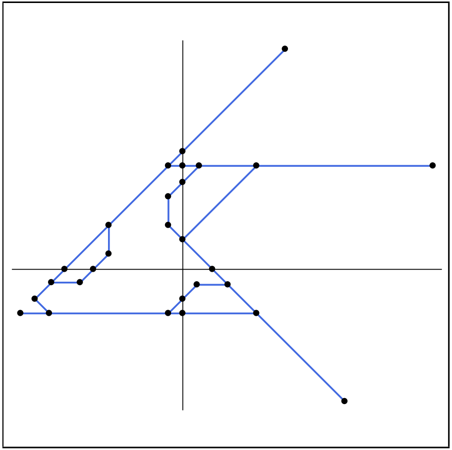
$\boxed{a = -1,\ 1 < b \leq 3/2}$



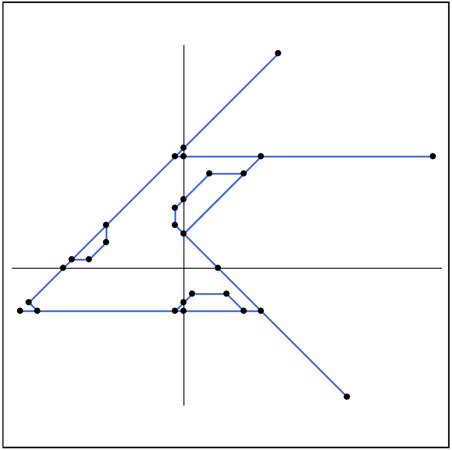
$\boxed{a = -1,\ 3/2 < b \leq 8/5}$



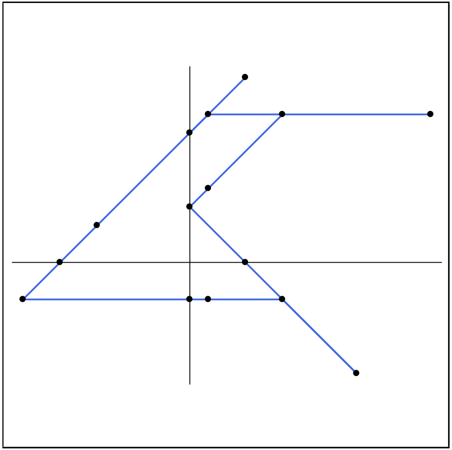
$a = -1,\ 8/5 < b \leq 7/4$



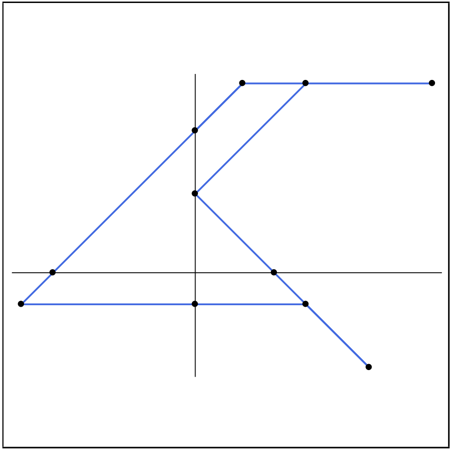
$a = -1,\ 7/4 < b < 2$



$\boxed{a = -1,\ 2 \le b < 3}$



$\boxed{a=-1,\ b\geq 3}$



To prove Theorem B we must:

1. **Step 1:** Show that for every b, the orbits of all points in $\mathbb{R}^2$ reach the set $Q_1 \cup Q_3$ at some time.
2. **Step 2:** For each $b \in \mathbb{R}$:
   2.1 We need to characterize the graph Γ (analytically).
   2.2 We need to prove its invariance.
3. **Step 3:** Show that for every b, every orbit with initial conditions in $Q_1 \cup Q_3$ reaches the invariant graph Γ (in finite time).

Proposition 1 (Step 1)

For $a = -1$ and $b \in \mathbb{R}$, **the orbit of every point in $Q_2 \cup Q_4$ reaches $Q_1 \cup Q_3$**, except the fixed point in Q_2 ($b > 1/2$); the fixed point in Q_4 ($b < 0$); and a 3-periodic orbit in $Q_2 \cup Q_4$ ($3/4 < b < 2$).

Proposition 2 (Steps 2 and 3)

Let $a = -1$, then:

(a) For every $b \in \mathbb{R}$, there exists a compact graph Γ invariant under F.

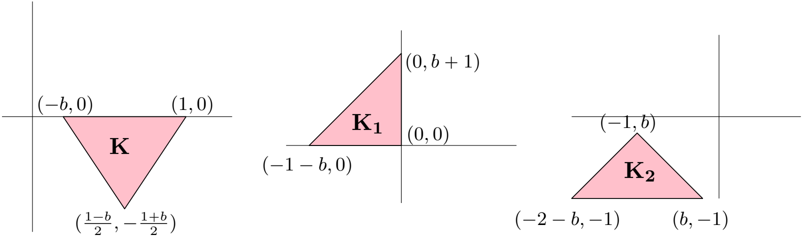
(b) **For every $b \in \mathbb{R}$ and for $(x, y) \in Q_1 \cup Q_3$, then $F^{11}(x, y) \in \Gamma$.**

Idea of the proof of Step 1

1. The fixed points of $F_2 = F_{|Q_2}$ and $F_4 = F_{|Q_4}$ (real or virtual) are unstable focus. **Any other point must leave Q_2 or Q_4, respectively**.
2. It is enough to prove the result for the points in Q_4 such that its image by F_4 is in Q_2, i.e. the iterates, $K_i := F^i(K)$, of the set

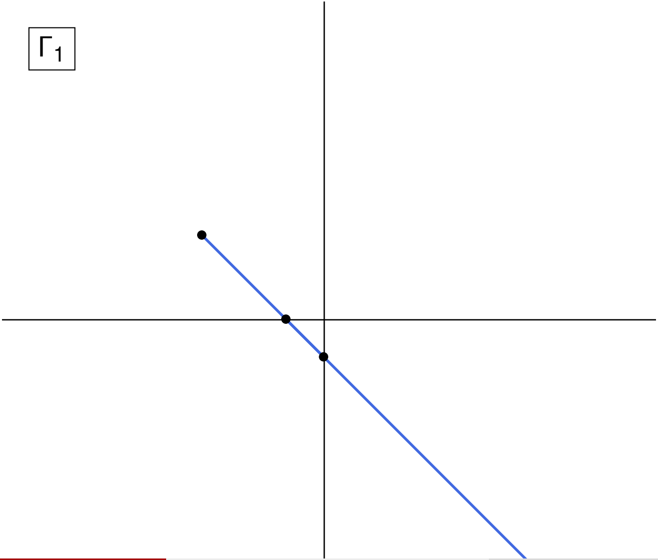
$$K := \{(x, y) \in Q_4 \text{ such that } F(x, y) \in Q_2\}.$$

A lot of cases, but the easiest one is $-1 < b \leq 0$:

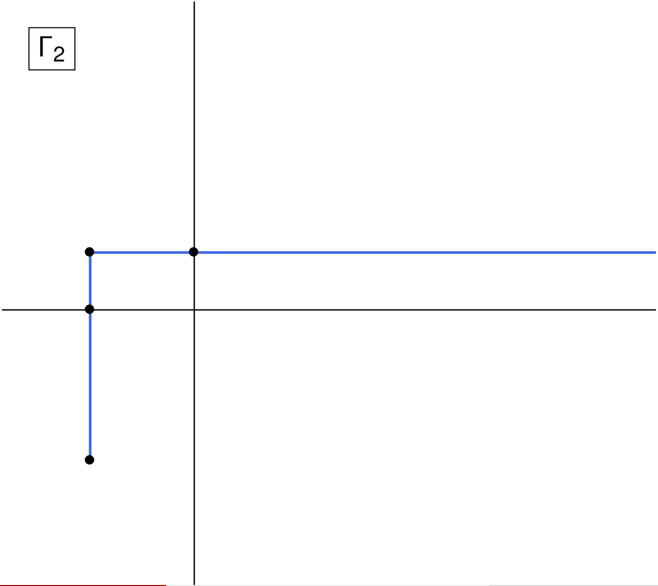


(a) The set K. (b) The set $K_1 = F(K)$. (c) The set $K_2 = F(K_1)$.

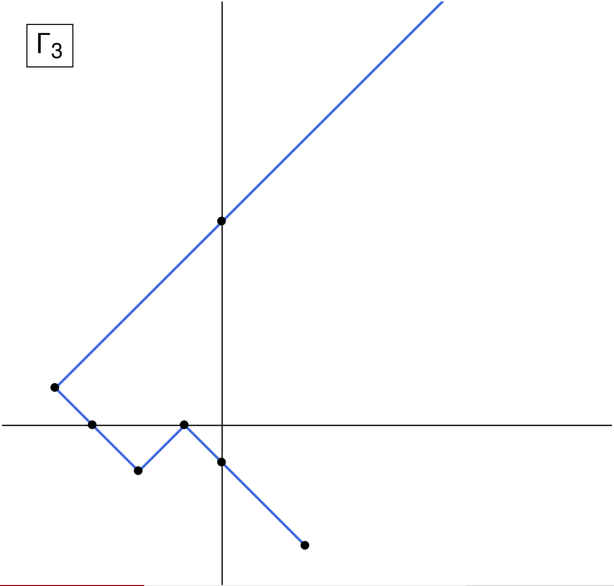
We will follow the iterates of Q_3 for $a = -1$ and $2/3 < b \leq 5/7$.



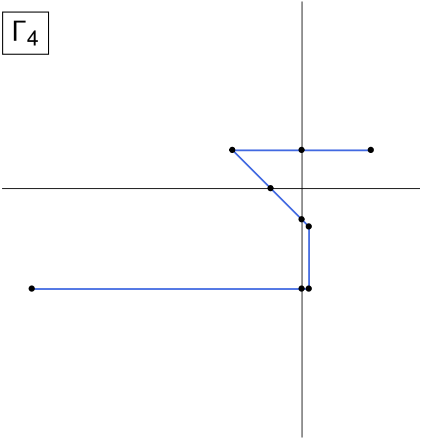
Second iterate of Q_3



Third iterate of Q_3



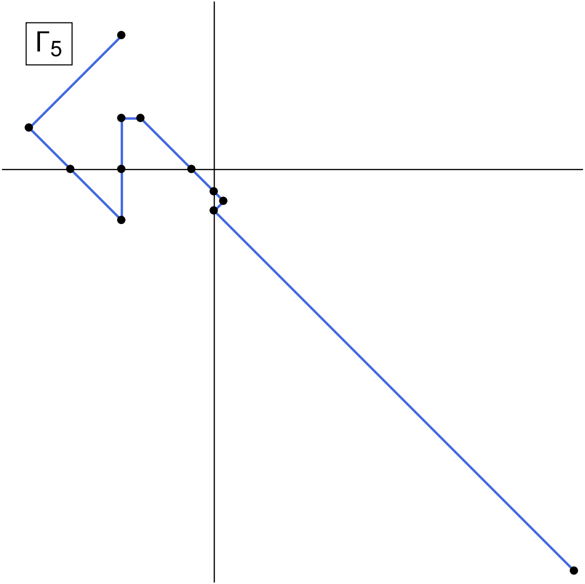
Fourth iterate of Q_3

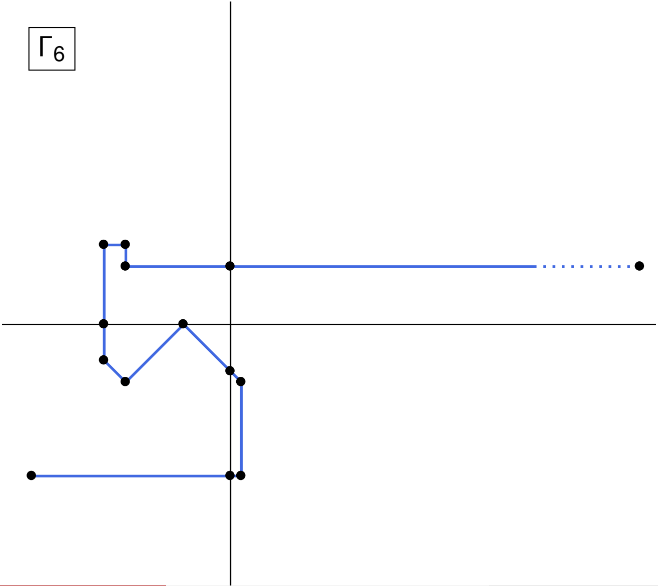


For the first time the image is compact.

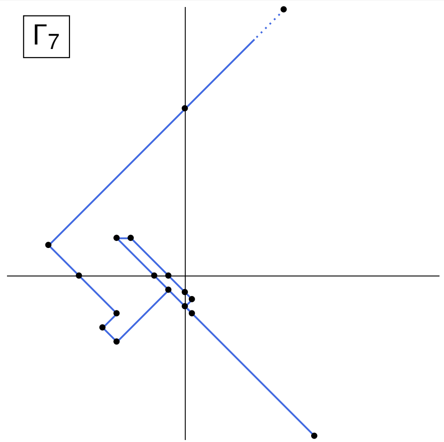
This is because the collapse of the unbounded edge.

Fifth iterate of Q_3




Γ_6

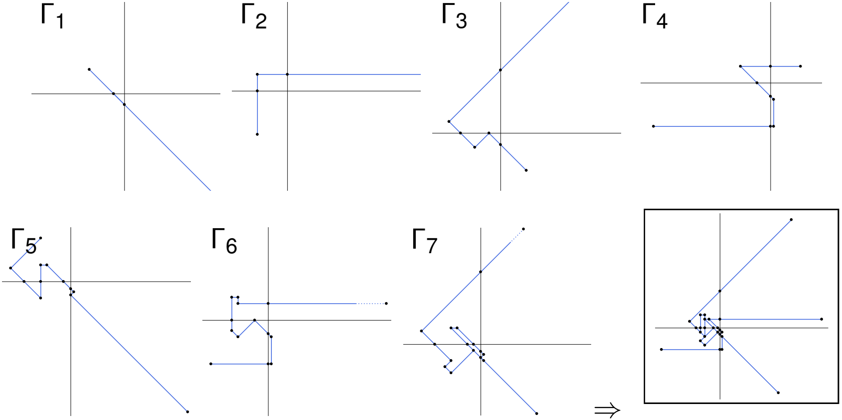
Seventh iterate of Q_3

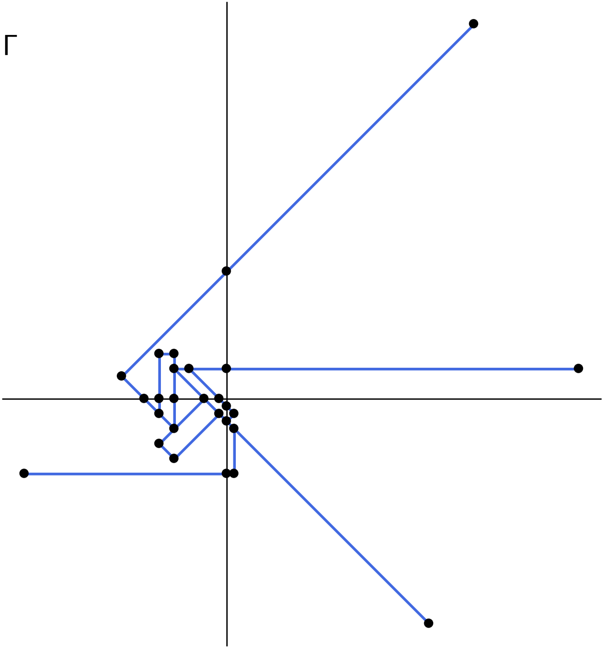


At this point we realize that the following iterates remain in

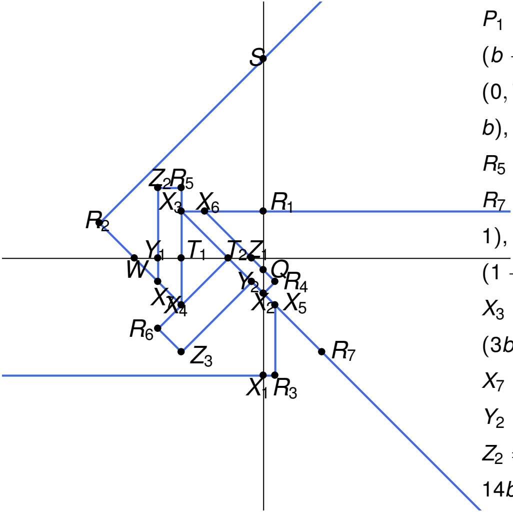
$$\Gamma = \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \Gamma_7$$

Summary of the process:




Γ

We prove the invariance by following the image of all edges



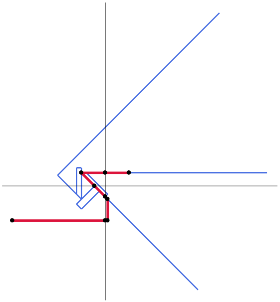
$P_1 = (-b-2, -1)$, $P_2 = (b+2, -3)$, $P_3 = (b+4, 2b-1)$, $P_4 = (-b+4, 5)$, $Q = (0, 7b-5)$, $R_1 = (0, 2b-1)$, $R_2 = (-2b, 1-b)$, $R_3 = (3b-2, -1)$, $R_4 = (3b-2, 4b-3)$, $R_5 = (-b, 8b-5)$, $R_6 = (-7b+4, -8b+5)$, $R_7 = (15b-10, -14b+9)$, $S = (0, b+1)$, $T_1 = (-b, 0)$, $T_2 = (b-1, 0)$, $W = (1-3b, 0)$, $X_1 = (0, -1)$, $X_2 = (0, b-1)$, $X_3 = (-b, 2b-1)$, $X_4 = (-b, 1-2b)$, $X_5 = (3b-2, 1-2b)$, $X_6 = (5b-4, 2b-1)$, $X_7 = (-7b+4, 4b-3)$, $Y_1 = (-7b+4, 0)$, $Y_2 = (7b-5, -6b+4)$, $Z_1 = (7b-5, 0)$, $Z_2 = (-7b+4, 8b-5)$ and $Z_3 = (-b, 9-14b)$.

	$b \leq -2$	$-2 < b \leq -1/4$	$-1/4 < b < 0$	$0 \leq b \leq 3/16$
N_1	8	6	5	6
N_3	5	5	4	4
	$3/16 < b < 4/15$	$4/15 \leq b \leq 2/3$	$2/3 < b \leq 7/4$	$b > 7/4$
N_1	**11**	6	5	5
N_3	9	4	4	5

Table: Arrival times N_1 and N_3 of points in $Q_1 \cup Q_3$ to Γ.

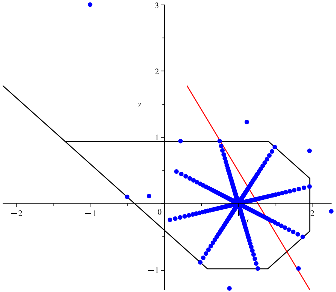
In this case $F^4(Q_3) = \Gamma_4 \subset \Gamma$.

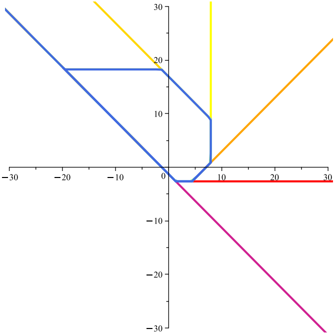
Analogously $F^5(Q_1) \subset \Gamma$



A remark

This phenomenon is not unique to the family of maps studied in this work. The following figures are drawn from ongoing research demonstrating that the phenomenon also arises in broader families of maps, in this case featuring two regions.





Dynamics of a particular map (left); Process of formation of the graph in another case (right).

Theorem D (as stated in the first work)

Set $a < 0$. For each $b \in \mathbb{R}$ consider the map F restricted to the corresponding invariant graph Γ. Set $\mathbf{c} = -\mathbf{b}/\mathbf{a}$, then

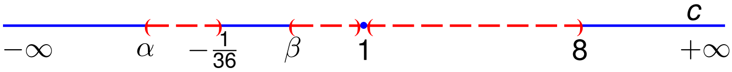
There exists α and β such that:

1. $F|_\Gamma$ has positive entropy if and only if $c \in (\alpha, -1/36) \cup (\beta, 1) \cup (1, 8)$.
2. $F|_\Gamma$ has zero entropy if and only if $c \in (-\infty, \alpha] \cup [-1/36, \beta] \cup \{1\} \cup [8, \infty)$.

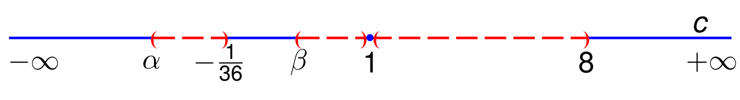
Furthermore:

- $\alpha \in (-112/137, -13/16) \approx (-0.8175, -0.8125)$,
  $\beta \in (603/874, 563/816) \approx (0.6899, 0.6900)$.

  Other rational bounds can be found with arbitrarily high precision.
- In these two intervals the entropy of $F|_\Gamma$ is non-decreasing in c.
- The entropy is discontinuous at $c = -1/36$, but continuous at $c \in \{\alpha, \beta, 1, 8\}$.



Theorem D



Proposition

$$\alpha = -0.8170016601273940755793791069223688332 40\ldots,$$
$$\beta = 0.6899324282045742867004889129507817387052 6\ldots,$$

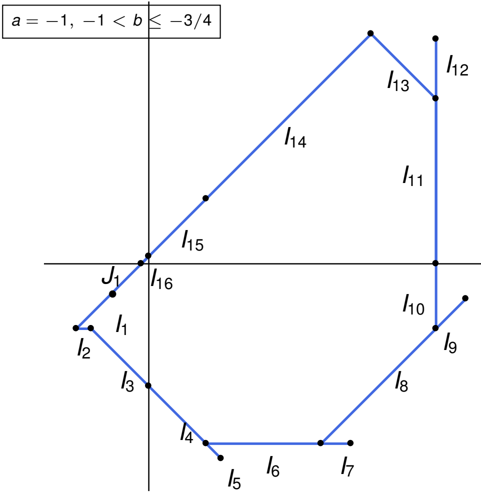
where all shown digits are correct.

Topological entropy was introduced by Adler, Konheim & McAndrew (1965) and also Bowen (1971), Misiurewicz & Ziemian (1992)

Take a partition $\mathcal{P} = I_1, \ldots, I_n$ s.t. $F(I_i)$ is an interval and $F|_{I_i}$ is continuous and *monotonic* for $i \in 1, \ldots, n$.

The ***itinerary of*** *x* ***with length*** *m is a sequence of m symbols* that explains which elements of the partition visit the first $m-1$ iterates of x.

Let $N(F, \mathcal{P}, m)$ be the number of different itineraries of length m.



Lemma (not the definition of *Topological Entropy*, but we can use it as such)

Let $F : \Gamma \longrightarrow \Gamma$ be a piecewise monotone map on a compact graph Γ. Let $\mathcal{P}$ be a monopartition. Then the topological entropy is

$$h(f) := \ln\left(\lim_{m} \sqrt[m]{N(F, \mathcal{P}, m)}\right).$$

The limit exists, and it is independent of the election of the mono-partition.

Markov partitions

$\mathcal{P}$ is a *Markov partition* if for all $I \in \mathcal{P}$, $F(I)$ is the union of some elements of $\mathcal{P}$. In this case

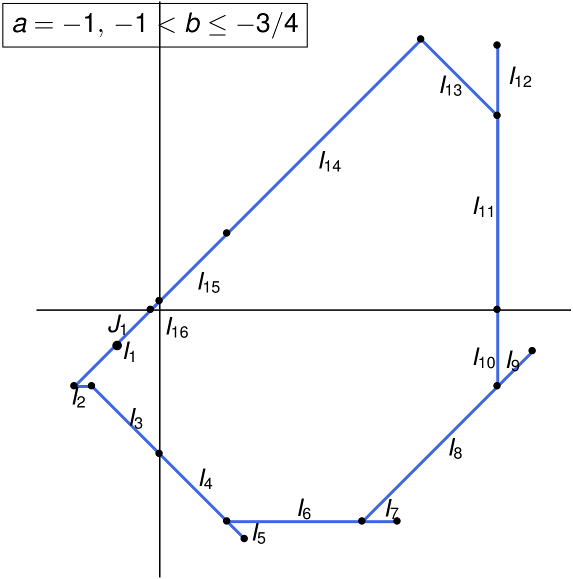
$$h(F) = \ln\left(r(\mathcal{P})\right),$$

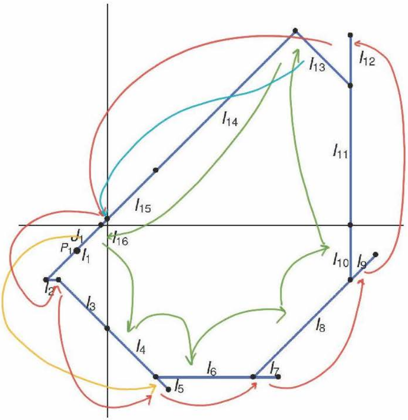
where $r(\mathcal{P})$ is the spectral radius of a Markov matrix $M(F, \mathcal{P})$ given by

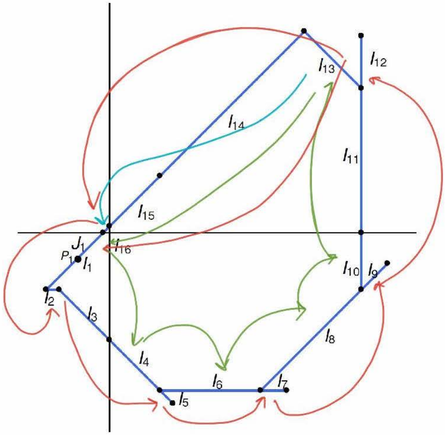
$$m_{i,j} = \begin{cases} 1, & \text{if } I_i \subset f(I_j); \\ 0, & \text{otherwise} \end{cases}$$

By the *Perron-Frobenius Theorem* is given by a positive real eigenvalue.

There are distinct dynamics on each graph! Example: $a = -1,\ -1 < b \leq -3/4$







$-1 < b \leq \mathbf{-7/8}$ $\quad \ldots \quad$ $-7/9 \leq b \leq -3/4$

For $-1 < b \leq -7/8$

$$
\begin{tikzcd}
I_1 \arrow[r] \arrow[rrrrrrrr, bend left] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left]
\end{tikzcd}
$$

For $-7/9 \leq b \leq -3/4$ *at least* we have:

$$
\begin{tikzcd}
J_1 \arrow[r] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left] \arrow[lllllllllll, bend left]
\end{tikzcd}
$$

For $-1 < b \leq -7/8$ we have “less dynamics” than for $\mathbf{b = -7/8}$. In this case:

$$\begin{tikzcd}
I_1 \arrow[r] \arrow[rrrrrrrr, bend left] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left]
\end{tikzcd}$$

$$M(F,\mathcal{P}) = \begin{array}{cc} & \begin{array}{cccccccccccc} I_1 & I_2 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{12} & I_{13} & I_{16} \end{array} \\ \begin{array}{c} I_1 \\ I_2 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \\ I_{10} \\ I_{12} \\ I_{13} \\ I_{16} \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

The characteristic polynomial is $(\lambda^6 - 1)^2$, hence, the logarithm of its spectral radius, the entropy, is zero.

For $-\mathbf{7}/\mathbf{9} < b \leq -3/4$ at least we have:

$$
\begin{tikzcd}
J_1 \arrow[r] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left] \arrow[lllllllllll, bend left]
\end{tikzcd}
$$

$$
M(F,\mathcal{P}) = \begin{array}{cc}
 & \begin{array}{cccccccccccc} J_1 & I_2 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{12} & I_{13} & I_{16} \end{array} \\
\begin{array}{c} J_1 \\ I_2 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \\ I_{10} \\ I_{12} \\ I_{13} \\ I_{16} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcolor{red}{\mathbf{1}} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\end{array}.
$$

The characteristic polynomial is $\lambda^6(\lambda^6 - 2)$. The spectral radius is $r = \sqrt[6]{2} \approx 1.122462048$, hence

$$h_b(F) \geq \ln(r) = \ln(\sqrt[6]{2}) \approx 0.1155245298.$$

A quick way to compute the entropy.

The factor giving rise the spectral radius of the Markov Matrix can be computed using only the **Romes**, taking into account the length of its cycles and the itineraries connecting them.

$$\begin{tikzcd} J_1 \arrow[r] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left] \arrow[lllllllllll, bend left] \end{tikzcd}$$

The intervals $\{J_1, I_{16}\}$ form **a Rome**.

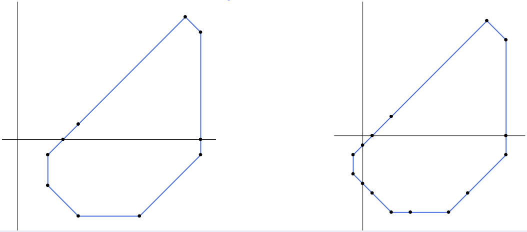
$$\begin{array}{cc} & \begin{array}{cc} J_1 & I_{16} \\ \downarrow & \downarrow \end{array} \\ \begin{array}{c} J_1 \to \\ I_{16} \to \end{array} & \left| \begin{array}{cc} \lambda^{-6} - 1 & \lambda^{-6} \\ \lambda^{-6} & \lambda^{-6} - 1 \end{array} \right| \end{array} = 1 - 2\lambda^{-6}$$

Hence the entropy is $h_F(b) \geq \ln(\sqrt[6]{2}) \approx 0.115524.$

In blue: length of the first cycle. **In green**: length of the second cycle. **In maroon**: length of the path connecting them.

What happens in the case of **zero entropy**?

1. In the majority of cases there is a finite number of **Periodic orbits (repelling or attracting), the dynamics is pre-periodic in finite time**.
2. For $b \leq -1$, $F|\Gamma$ is a degree one map of the circle (it turns around once).



Given a degree 1 map g of $\mathbb{S}^1$ it is characterized by the **rotation number** (which is the average angle modulus 2π):

(a) g has periodic orbits if and only if $\rho(g) \in \mathbb{Q}$.

(b) If $\rho(g) = p/q$ with $(p, q) = 1$ all the periodic orbits of g have period q.

(c) If $\rho(g) \in \mathbb{R} \setminus \mathbb{Q}$ then the ω-limit is the same for every point an it is either the whole $\mathbb{S}^1$ or a Cantor set.

Proposition (summary)

(a) If $b \leq -15/8$, then $\rho = 1/7$. There exists a fixed point and two 7-Periodic orbits (explicit). Each orbit reaches one of these orbits in a finite number of iterates.

(b) If $-15/8 \leq b \leq -7/4$. then ρ varies continuously from $1/7$ to $1/6$.

• For the irrational values we encounter, there is a fixed point an the ω-limit of the other points is a **Cantor set** of Γ.

• When $\rho = p/q$, $\exists$ two Periodic orbits q-periodic, that can collide for some values of b. And the set of periods is are **all the natural numbers** *except:* $2-5$, $8-12$, $14-17$, 18, $21-24$, 26, $28-30$, 35, 36, $38-40$, 42, 50, 52, 54, 57, 60, $64-66$, 78, 96, 100, 102, 138 and 220.

(c) If $-7/4 \leq b \leq -1$, then $\rho = 1/6$. $\exists$ a fixed point and two 6-Periodic orbits (explicit). Every orbit reaches one of these orbits in a finite number of iterates.

When the topological entropy is positive, $F_{|\Gamma}$ has periodic orbits with infinitely different periods and the orbits have different combinatorial behaviors.

Since, for all b, Γ is compact, when $F|_\Gamma$ has positive entropy, then it is chaotic in the sense of Li and Yorke.

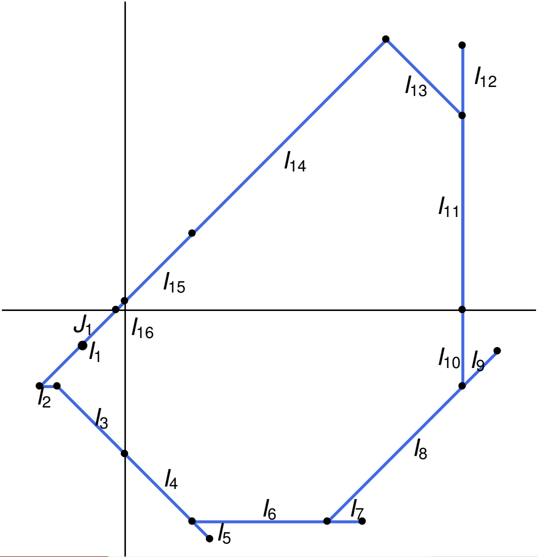
Chaos in the sense of Li and Yorke

A map f is said to be chaotic in the sense of Li and Yorke if:

1. It has periodic points with arbitrarily large periods and,
2. There exists a uncountable set $\mathcal{S}$ (*scrambled set*), so that for any $p, q \in \mathcal{S}$ and each periodic point r of f we have
   (a) $\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0$,
   (b) $\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0$,
   (c) $\limsup_{n\to\infty} |f^n(p) - f^n(r)| > 0$.

An example of more specific results

Lets go back to our base example $a = -1,\ b \in (-1, -3/4]$



An example of more specific results: $a = -1$ and $b \in (-1, -3/4]$.

Proposition 24. Zero entropy cases.

Assume $a = -1$ and $b \in (-1, -3/4]$.

(a) If $b \in (-1, -8/9]$, there exist 2 different periodic orbits , with period 6:
- $\mathcal{O}_1$ associated with a point in I_{16} (attractive).
- $\mathcal{O}_2$ associated with a point in I_1 (repellor).

(b) If $b \in (-8/9, -112/137]$,
- $\mathcal{O}_1$ with with period 12 associated with a point in I_{16} (attractive).
- $\mathcal{O}_2$ with with period 6 associated with a point in I_1 (repellor).
- $\mathcal{O}_2$ with with period 6 associated with a point inI_{16} (repellor).

All the orbits are explicit.

$$\begin{tikzcd}
I_1 \arrow[r] \arrow[rrrrrrrr, bend left] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left]
\end{tikzcd}$$

Proposition 24. Positive entropy cases.

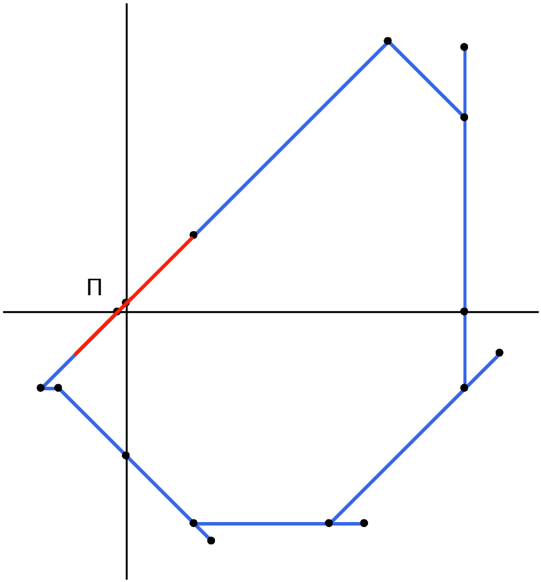
(d) If $b \in [-13/16, -3/4]$, P_2 which can follow very different itineraries. Li-Yorke Chaos

$$
\begin{tikzcd}
J_1 \arrow[r] & I_4 \arrow[r] & I_6 \arrow[r] & I_8 \arrow[r] & I_{10} \arrow[r] & I_{13} \arrow[r] \arrow[lllll, bend left] & I_{16} \arrow[r] & I_2 \arrow[r] & I_5 \arrow[r] & I_7 \arrow[r] & I_9 \arrow[r] & I_{12} \arrow[lllll, bend left] \arrow[lllllllllll, bend left]
\end{tikzcd}
$$

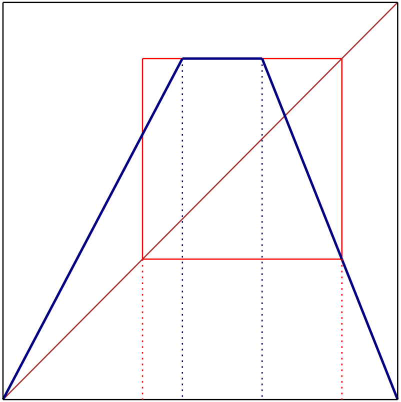
Proposition 24 (transition).

(c) When $b \in [-112/137, -13/16]$ there exists a subinterval $\Pi \subset \Gamma$ which is invariant by F^6 and such that it is visited for all elements of Γ except for the points of the repulsive orbit $\mathcal{O}_2$ that still is 6−periodic; Moreover

- $F^6|_\Pi$ is semiconjugated to trapezoidal map.
- There exists α such the entropy is zero for for $b \in [-112/137, \alpha]$ and positive and non-decreasing $b \in (\alpha, -13/16]$.
- **The orbit of certain point in Π under F^6 runs through all the dynamic situations offered by the maximum of a unimodal map.**

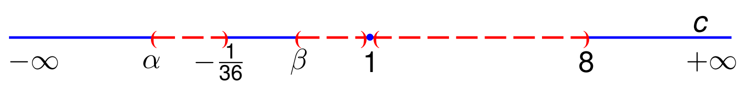

Π

After some linear changes and a "surgery" to remove a constant interval the map $F^6_{|\Pi}$ is *semiconjugate* with the map in the red box (this surgery does not affect to entropy)



This map can be extended to one of the **full families** of trapezoidal maps studied by Brucks, Misiurewicz and Tresser (1991). **They display all situations offered by the maximum of a unimodal map (logistic, tent...).**

This trapezoidal map allow us to compute rational bounds for the transition parameter between zero and positive entropy:



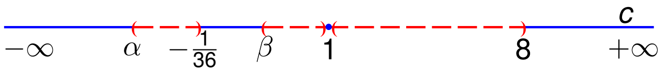
Remember

$$\alpha = -0.817001660127394075579379106922368833240\ldots$$

$$\in \left[-\frac{14085047614008594570281674616 2288}{17239925328685782866066913256 9609}, -\frac{1049417824596806956103568}{1284474531463219438945271}\right].$$

lower bounds: find an orbit with period $p = 2^N$. The Markov partition induced by this periodic orbit gives zero entropy.

Upper bounds: find an orbit with period $p = m\,2^N$ with m odd: By the *Bowen-Franks' Theorem*: the entropy satisfies: $h_\varphi > \ln(2)/p > 0$.

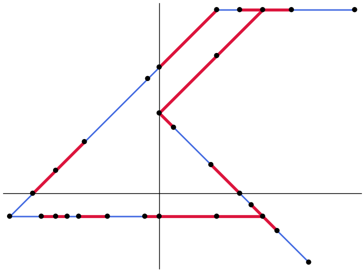


The entropy $h_F(c)$ is discontinuous at $c = -1/36$.

Theorem

The entropy function $h_F(c)$ is continuous at $c = 8$.

$\boxed{4 < c < 8}$

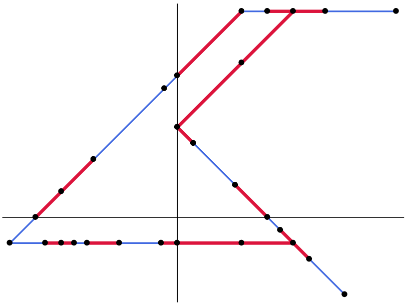


Idea:

The set $\Gamma \cap \{x = -1\}$ is invariant by F^3:

$$F^3(x, -1) = (4x - 2 + b, -1).$$

The changes in the entropy is reflected on the *bifurcations* on the directed graphs depending on the position of the iterates of $x_0 = b - 10$ by F^3.



We consider:

$$c \in (4, 8) = \bigcup_{n=0}^{\infty} (S_n \cup T_n \cup U_n \cup V_n),$$

with

$$S_n = (p_{n-1}, s_n),\ T_n = [s_n, r_n],\ U_n = (r_n, q_n),\ V_n = [q_n, p_n],$$

where

$$p_n = \frac{4(4 \cdot 4^{n+1} - 1)}{2 \cdot 4^{n+1} + 1}, \quad q_n = \frac{8 \cdot 4^{n+1} + 1}{4^{n+1} + 2}, \quad r_n = \frac{16 \cdot 4^{n+1} - 1}{2 \cdot 4^{n+1} + 4}, \quad s_n = \frac{2(4 \cdot 4^{n+2} - 1)}{4^{n+2} + 11}.$$

Proposition

The entropy $h_F(b)$ is always positive and satisfies:

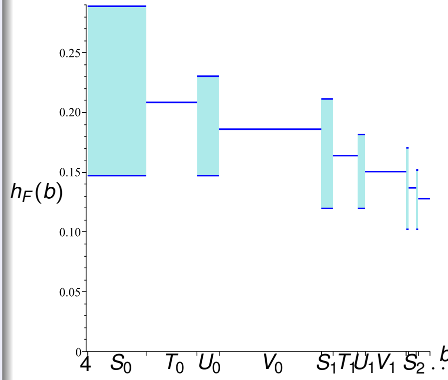
- $b \in S_n$: $\ln(\alpha_n) \le h_F(b) \le \ln(\beta_n)$
- $b \in T_n$: $h_F(b) = \ln(\delta_n)$
- $b \in U_n$: $\ln(\alpha_n) \le h_F(b) \le \ln(\gamma_n)$
- $b \in V_n$: $h_F(b) = \ln(\varphi_n)$

Each $\alpha_n, \beta_n, \gamma_n, \delta_n, \varphi_n$ is the positive root of a polynomial (see paper). Moreover:

$$1 < \alpha_n < \varphi_n < \delta_n < \gamma_n < \beta_n,$$

and all five sequences decrease to 1 as $n \to \infty$.

Hence $\lim_{b \to 8^-} h_F(b) = 0$



Theorem C

For all $a < 0$ and $b \in \mathbb{R}$, **for an open and dense set of initial conditions in Γ.** there are at most three possible ω-limit sets
Moreover if $b/a \in \mathbb{Q}$ these ω-limit sets are periodic orbits.

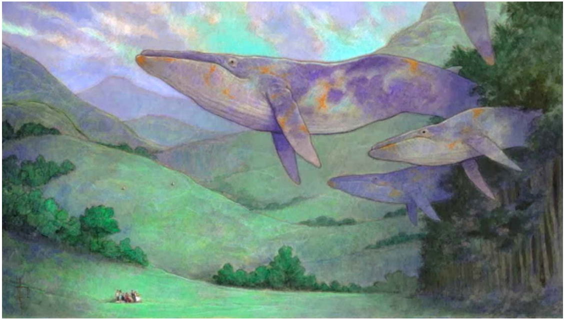
When $a < 0$ and $b/a \in \mathbb{Q}$, **generically**, the ω-limit sets are periodic orbits.

However: genericity does not explains numerics: the generic set in the above result must have full Lebesgue measure in Γ.

Theorem

Set $a < 0$. For $-b/a < -2$ and for $-b/a \in [-112/137, -13/16] \cup [603/874, 563/816]$ $\exists$ a full Lebesgue measure set of initial conditions in Γ, such that there are at most three possible ω-limit sets. Moreover, if $b/a \in \mathbb{Q}$ these ω-limit sets are periodic orbits.

Thank you very much!



I really like the Cartagena's whale tail... here there are also other whales, drawn by Miguel Anxo Prado (*Ardalén*, 2014).