Structure of zero entropy patterns of trees

David Rojas

Universitat Autònoma de Barcelona, Catalonia, Spain

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Joint work with David Juher and Francesc Mañosas

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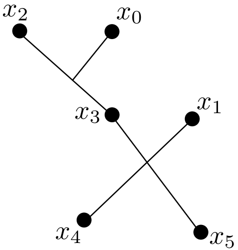
The concept of pattern



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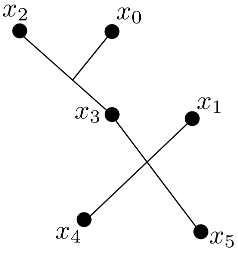

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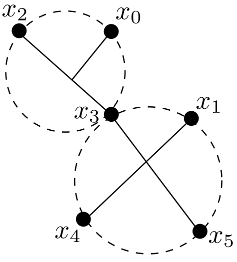

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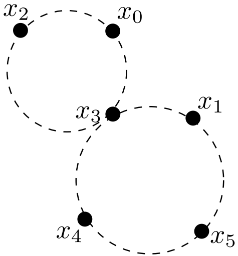

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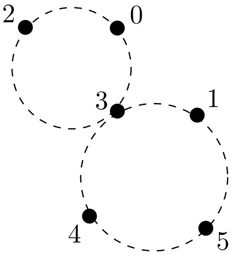


The concept of pattern

$\mathcal{X}$ a family of topological spaces (i.e. all closed intervals or all trees)

$\mathcal{F}_{\mathcal{X}}$ the family of all maps $\{f : X \to X : X \in \mathcal{X}\}$ satisfying a given restriction (i.e. continuous)

P a finite invariant set of a map $f : X \to X$ in $\mathcal{F}_{\mathcal{X}}$

The pattern of P in $\mathcal{F}_{\mathcal{X}}$ is the equivalence class $\mathcal{P}$ of all maps $g : Y \to Y$ in $\mathcal{F}_{\mathcal{X}}$ having an invariant set $Q \subset Y$ that, at combinatorial level, behaves like P.

Entropy of a pattern

The topological entropy of a pattern $\mathcal{P}$ in $\mathcal{F}_{\mathcal{X}}$ is the infimum of the topological entropies of all maps in $\mathcal{F}_{\mathcal{X}}$ exhibiting $\mathcal{P}$.
Notation: $h(\mathcal{P})$.

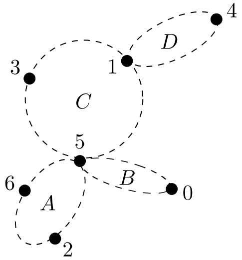
[Alsedà, Guaschi, Los, Mañosas, Mumbrú, 1997]

The canonical model of a pattern $\mathcal{P}$ is a triplet (T, P, f) that:

(a) f is essentially unique and determined by the combinatorial data of P,

(b) f minimizes the entropy.

Some techniques: openings reduce entropy



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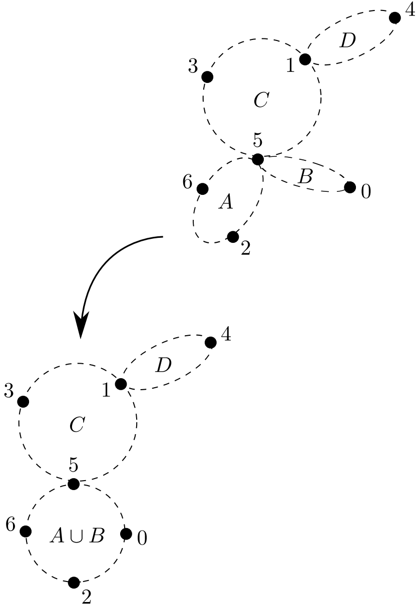

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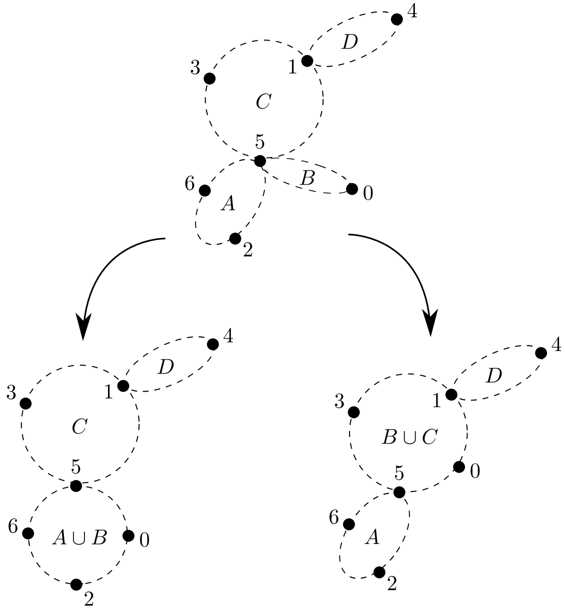
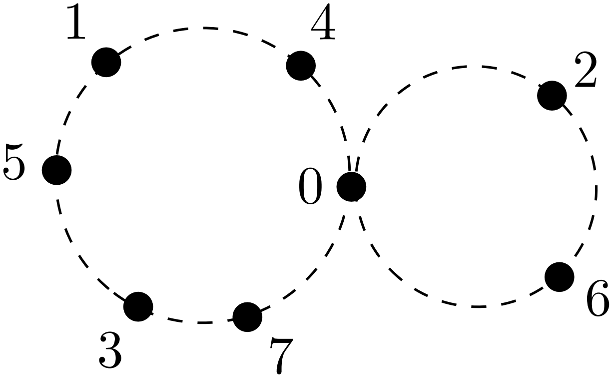


Some techniques: basic paths and π-reducibility



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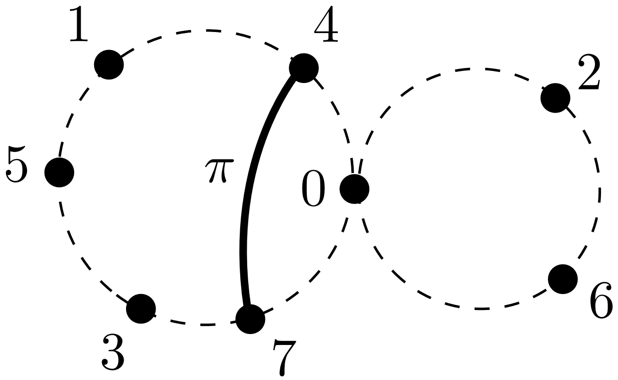

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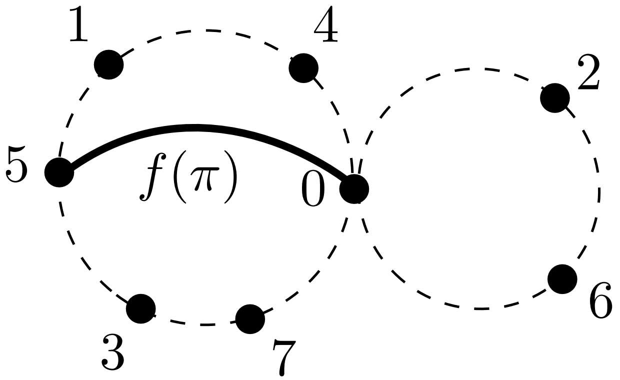

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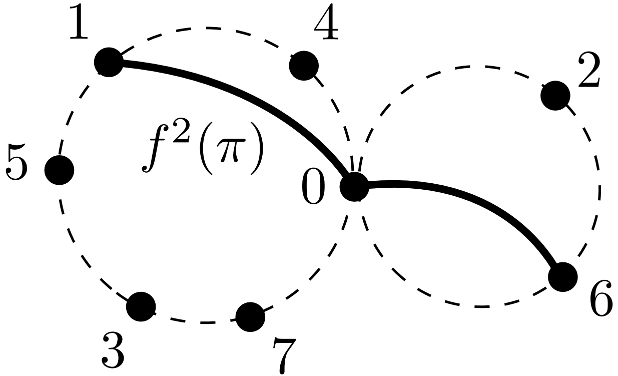

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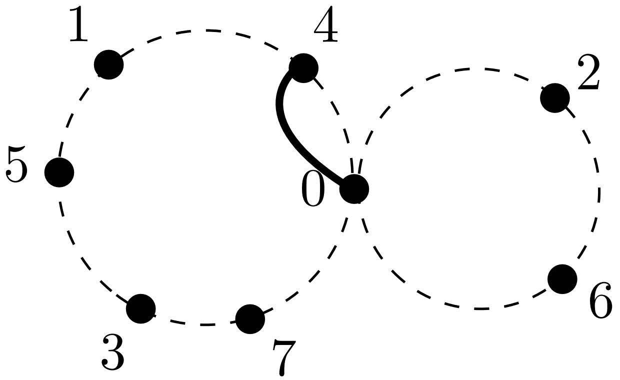

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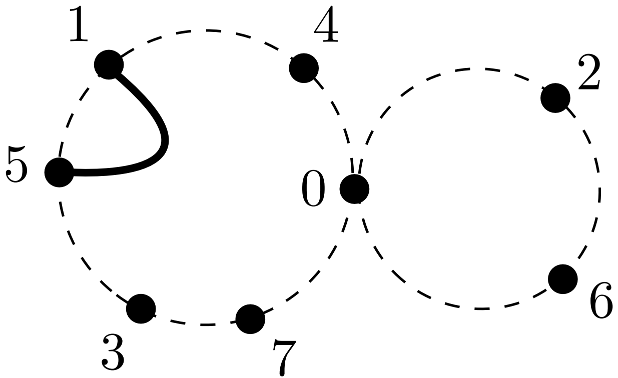

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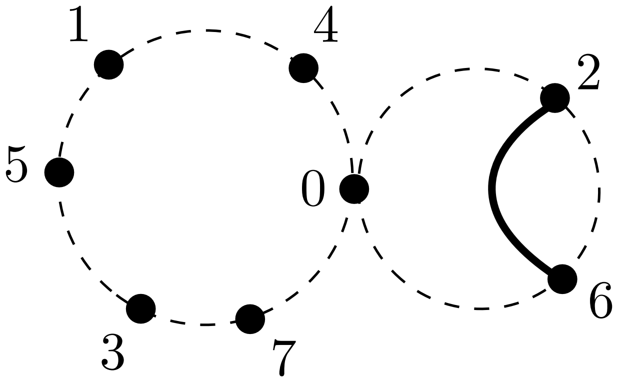

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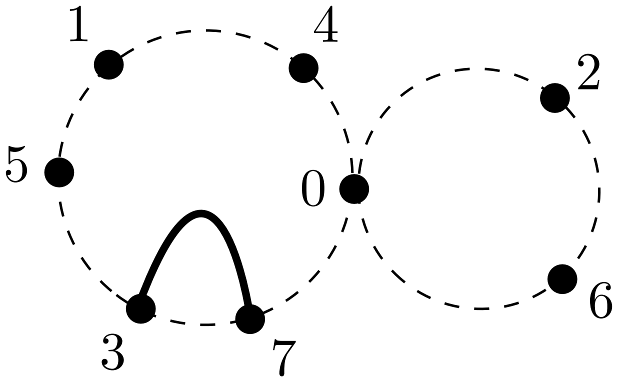

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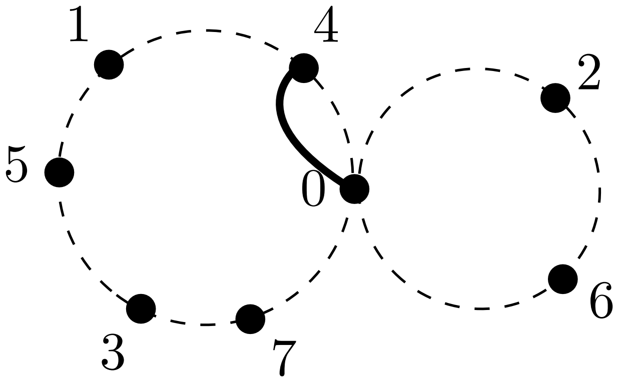
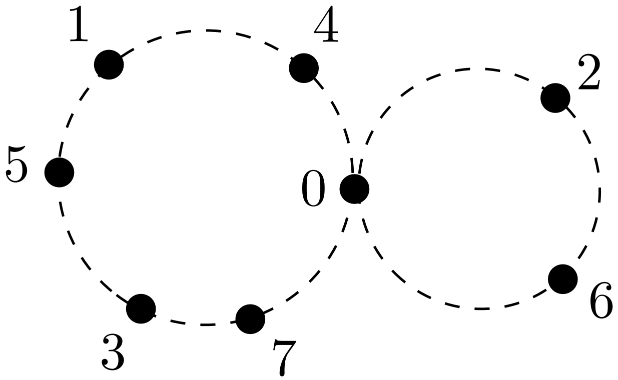


Some techniques: block structure



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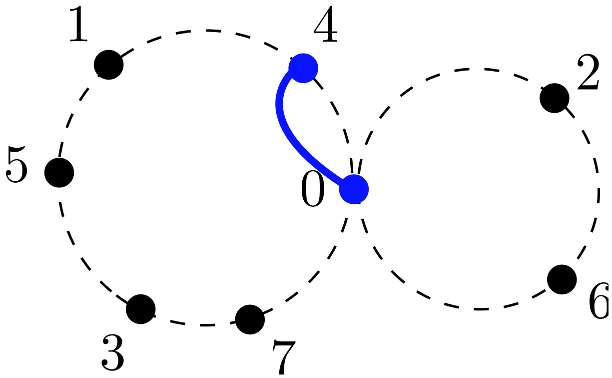

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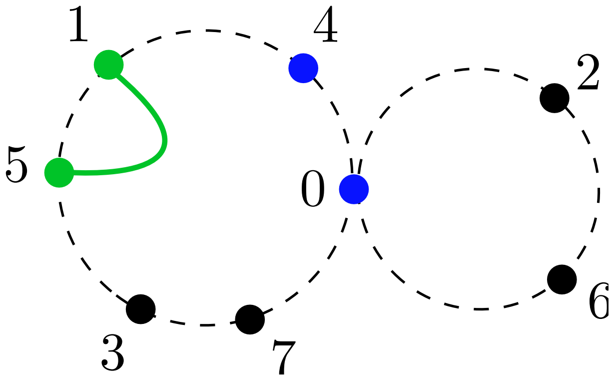

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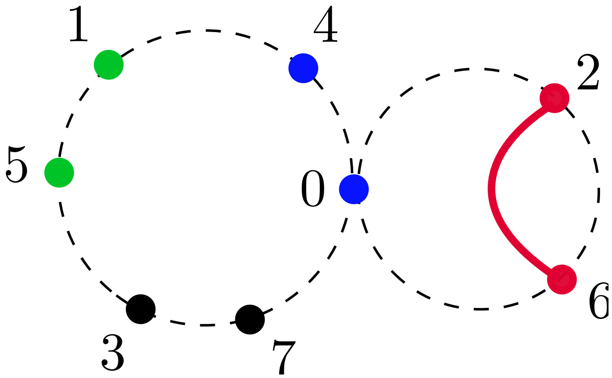

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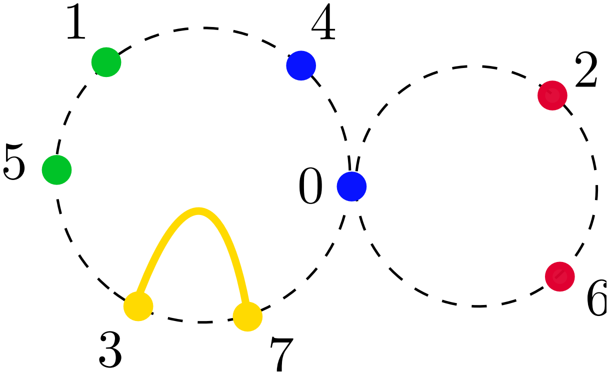

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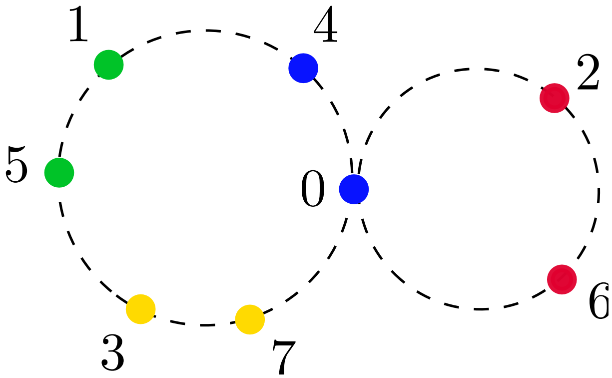

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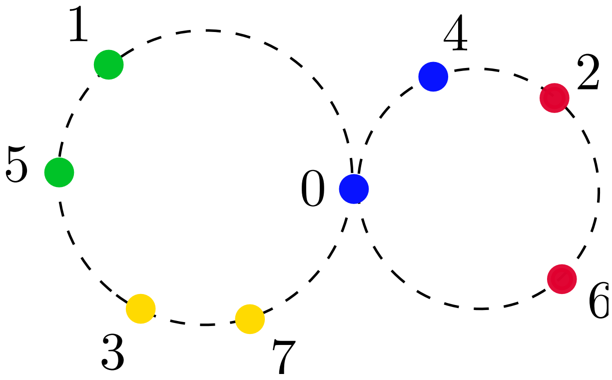

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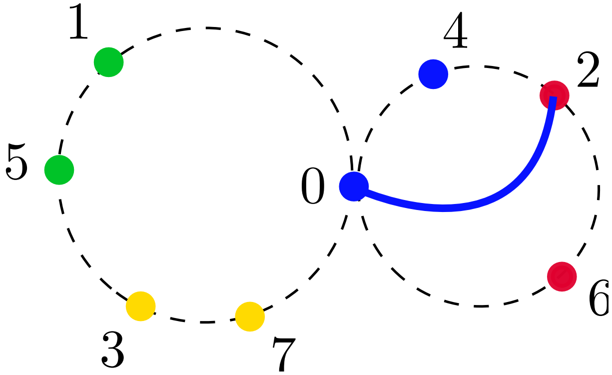

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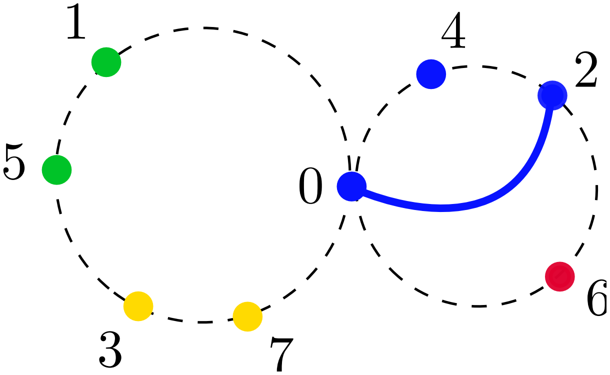

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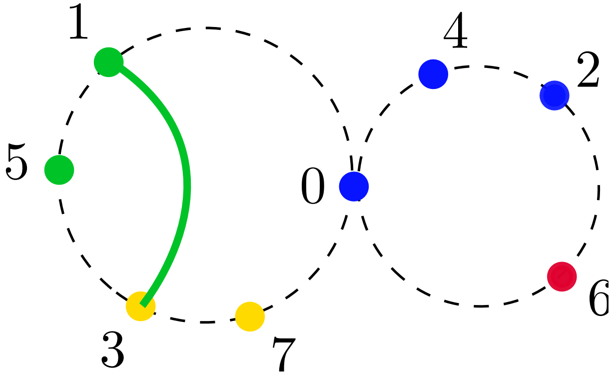

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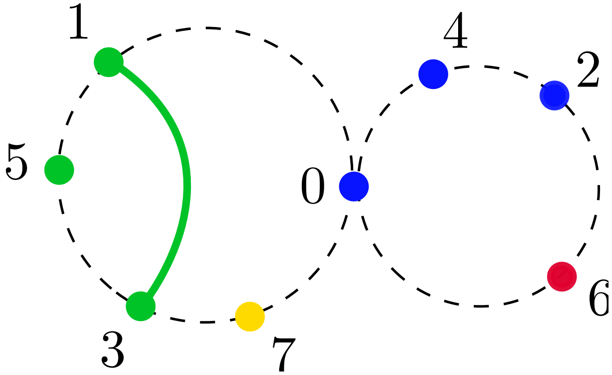

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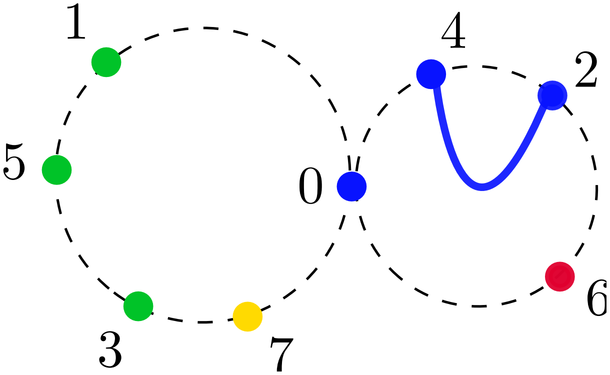

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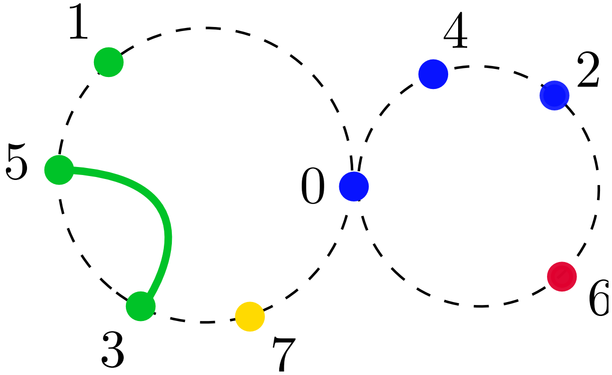

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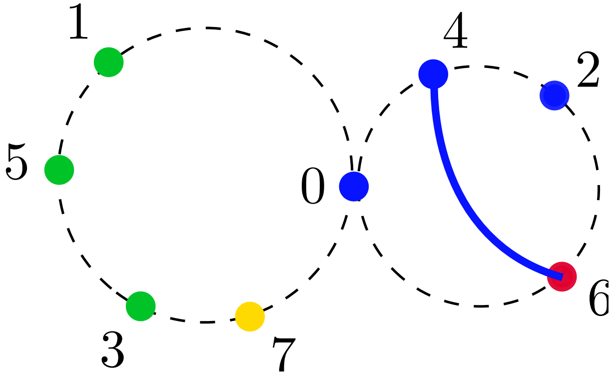

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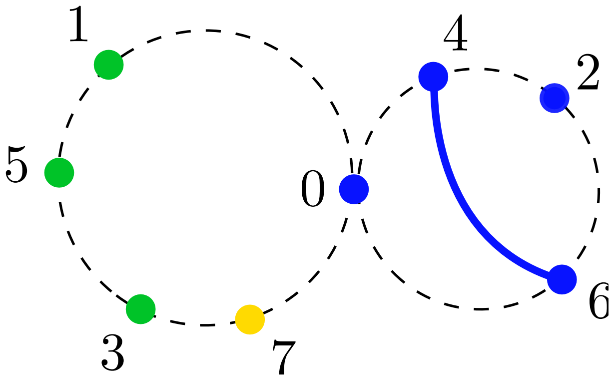

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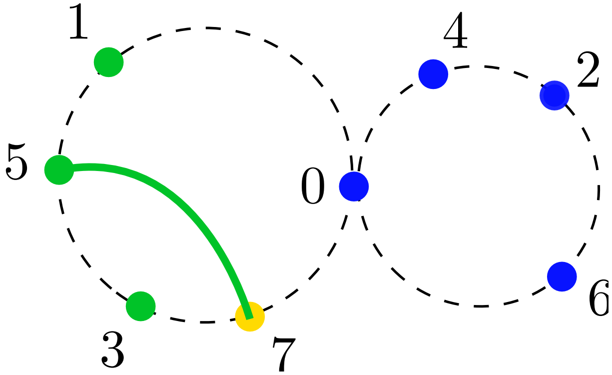

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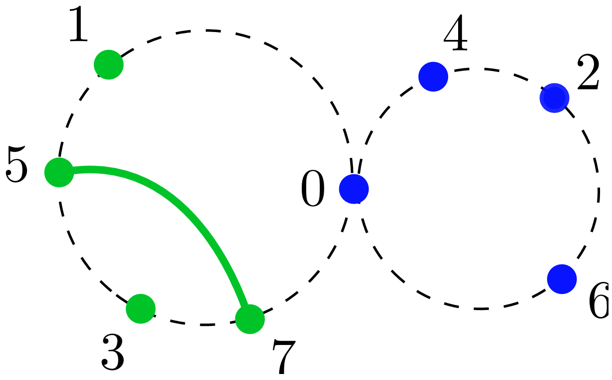

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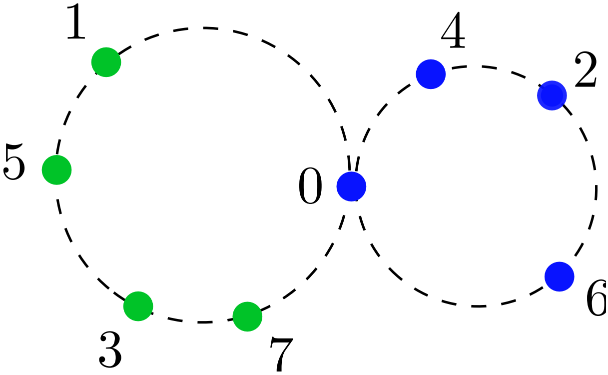

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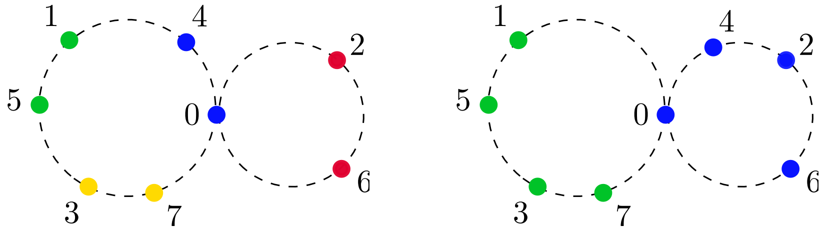

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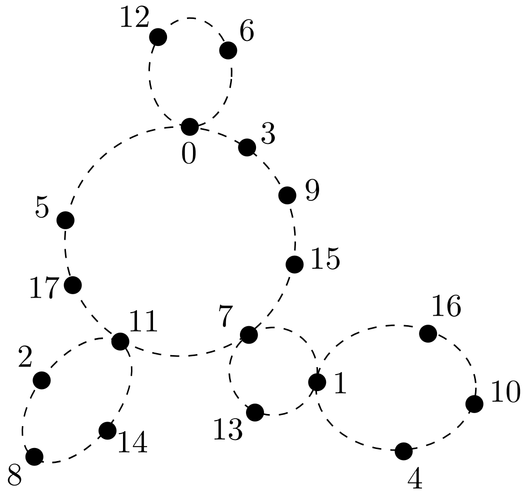

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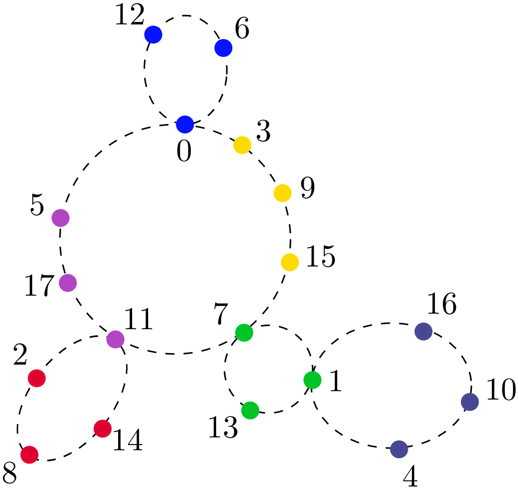

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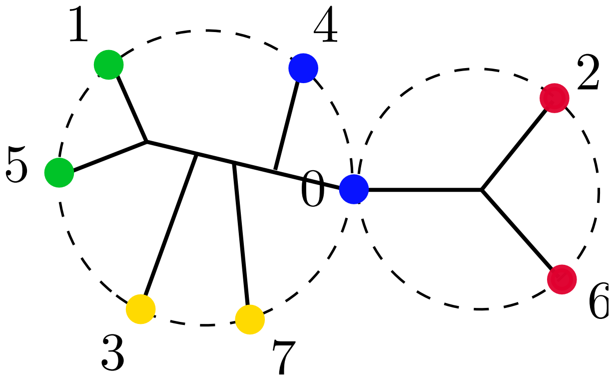

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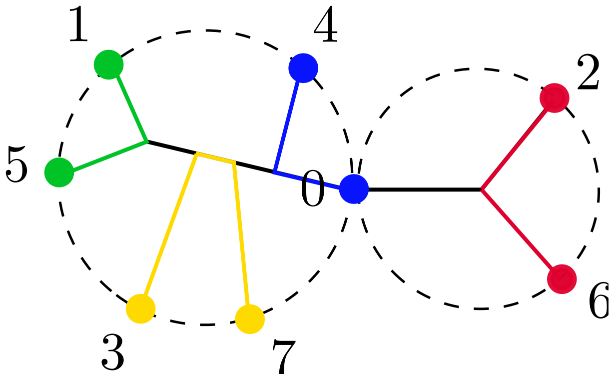

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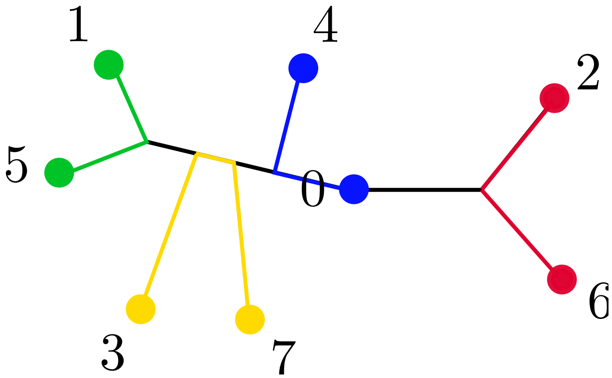

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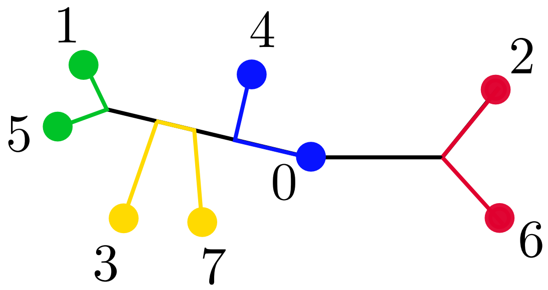

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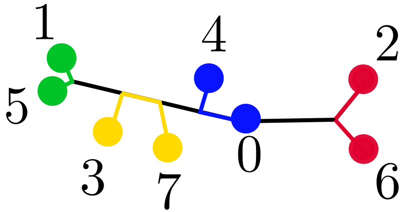

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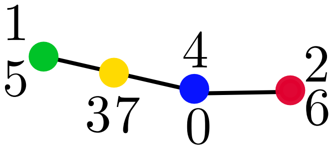

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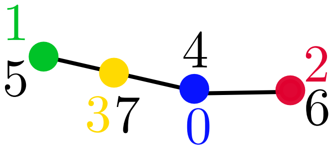

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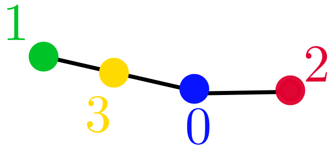

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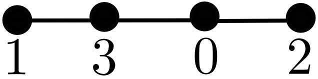

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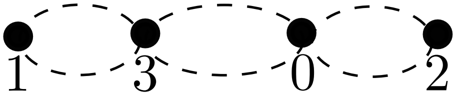

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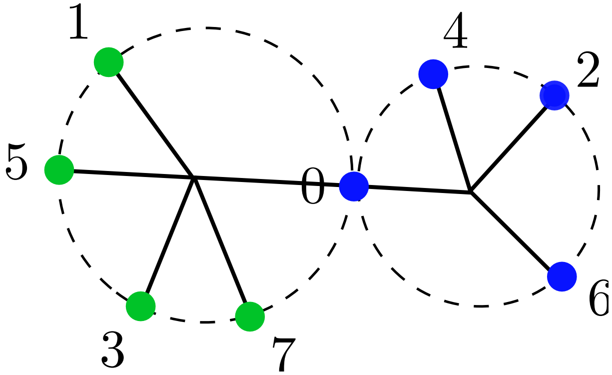

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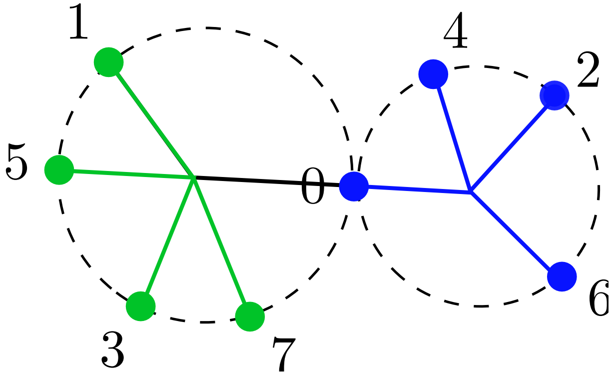

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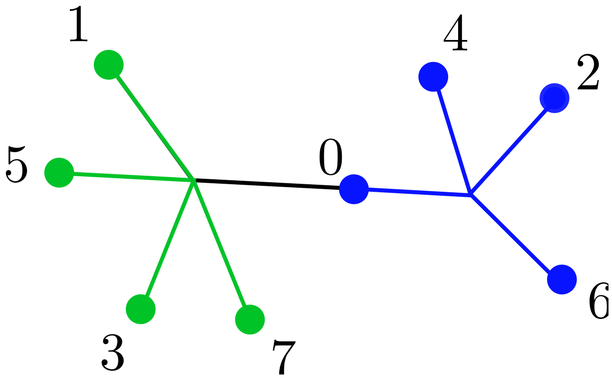

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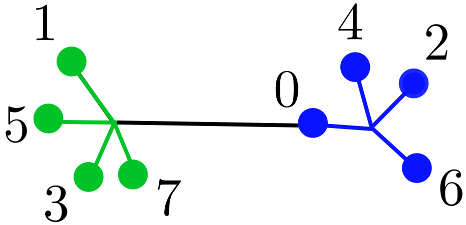

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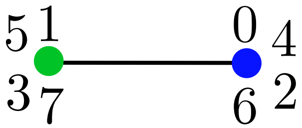

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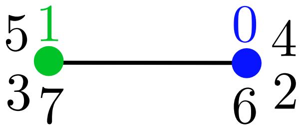

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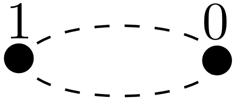


Some techniques: skeleton of a pattern

Important result

Let $\mathcal{P}$ be a pattern with a separated structure of trivial blocks. Let $\mathcal{S}$ be the corresponding skeleton. Then $h(\mathcal{P}) = h(\mathcal{S})$.

Observe

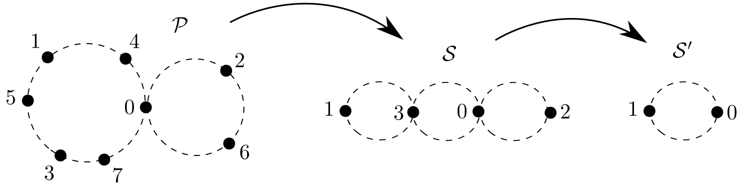
The period of the skeleton $\mathcal{S}$ strictly divides the period of the pattern $\mathcal{P}$.

Characterization of entropy zero patterns

[Alsedà, Juher, Mañosas. 2015]

$h(\mathcal{P}) = 0$ if and only if either $\mathcal{P}$ is trivial or has a maximal separated structure of trivial blocks such that the associated skeleton $\mathcal{S}$ has entropy 0.

We can use it recursively since $h(\mathcal{S}) = 0$ as well



Not all is fun in necromancy...

Problematic

Although the existence of reducible paths and so trivial block structure is fully combinatorial (only depends on the relative position of the points - the pattern - and ignores the topology of the tree), the construction of the skeleton of a pattern needs from the topology of the tree in the canonical model.

What one wants to do

When a trivial block structure is found in a pattern, one would like to join all points of each block together. However, where we put the resulting point?

From topological to combinatorial

The combinatorial collapse

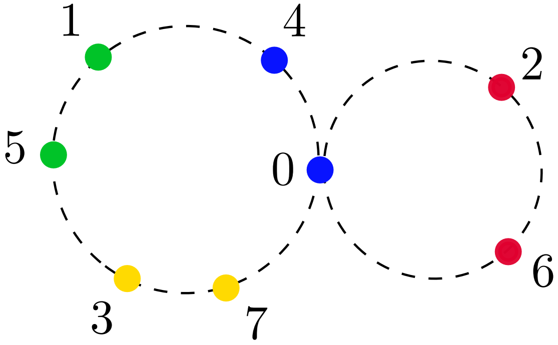
Let $\mathcal{P} = (T, P, f)$ be a zero entropy pattern with maximal separated structure of trivial blocks denoted by $P_0, P_1, \ldots, P_{p-1}$. A p-periodic pattern $\mathcal{C} = (S, Q, g)$ is called the combinatorial collapse of $\mathcal{P}$ if the following properties are satisfied:

(a) $g(i) = j$ if and only if $f(P_i) = f(P_j)$.

(b) for any $0 \leq i < j \leq p-1$, there is a discrete component of $\mathcal{P}$ intersecting the blocks P_i, P_j if and only if there is a discrete component of $\mathcal{C}$ containing the points i, j.

The combinatorial collapse

Ok... better with pictures



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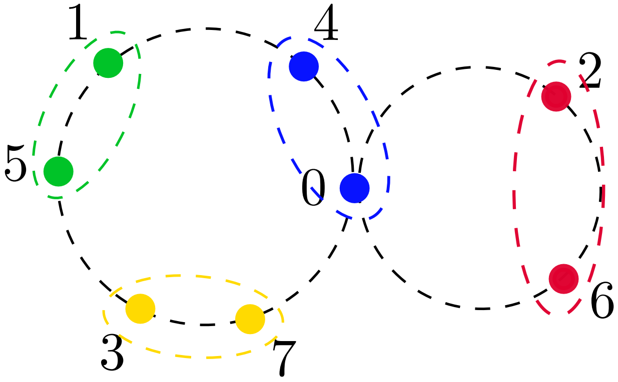

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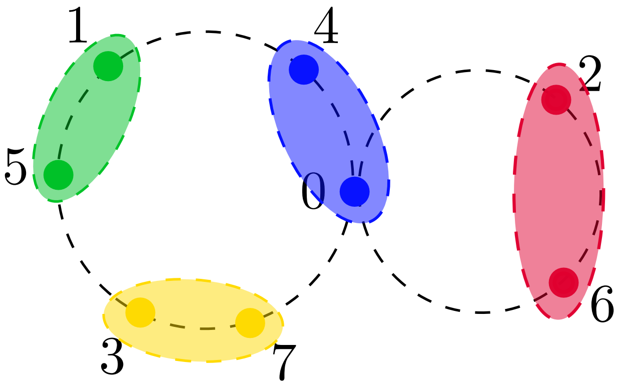

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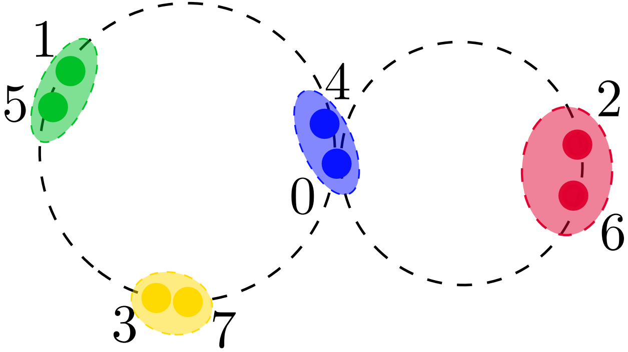

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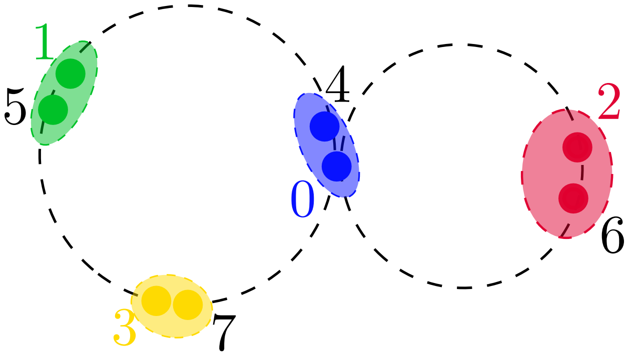

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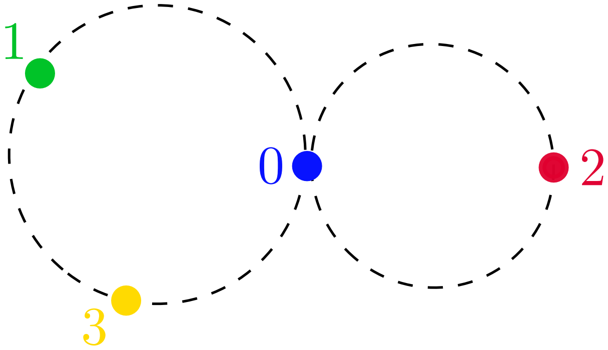

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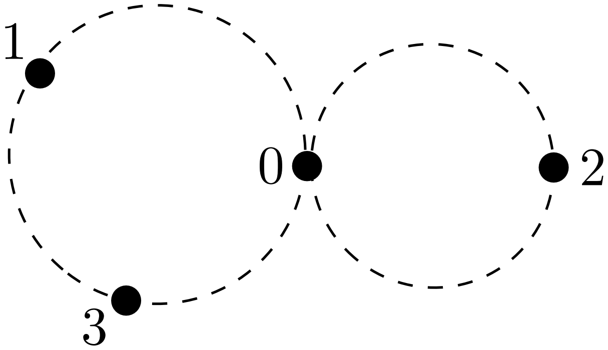

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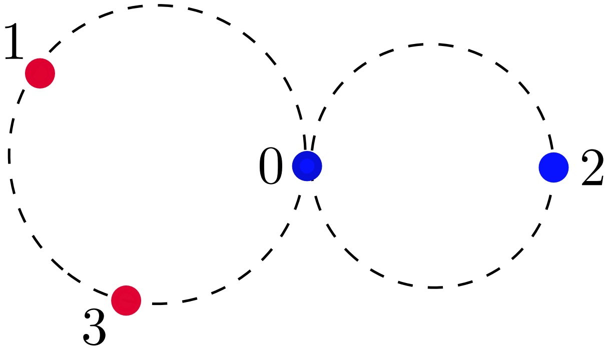
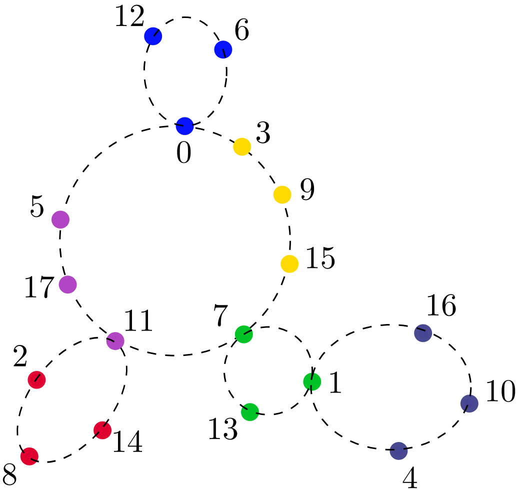


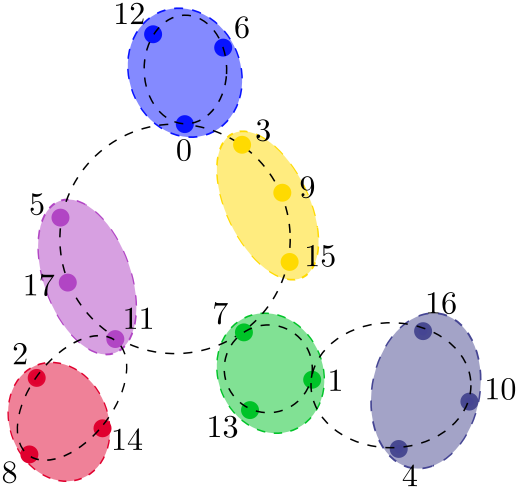
The combinatorial collapse

Ok... better with pictures



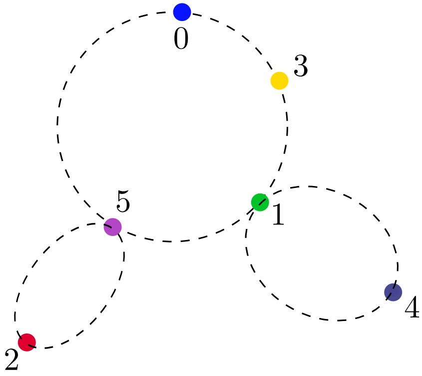
The combinatorial collapse

Ok... better with pictures



The combinatorial collapse

Ok... better with pictures



The combinatorial collapse

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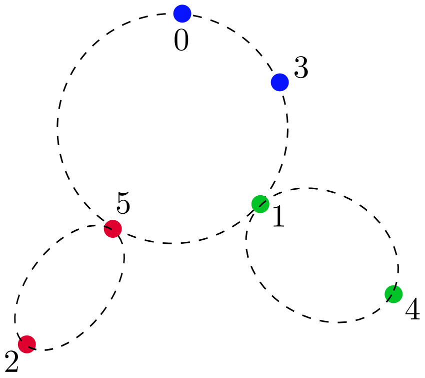
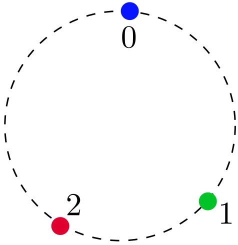


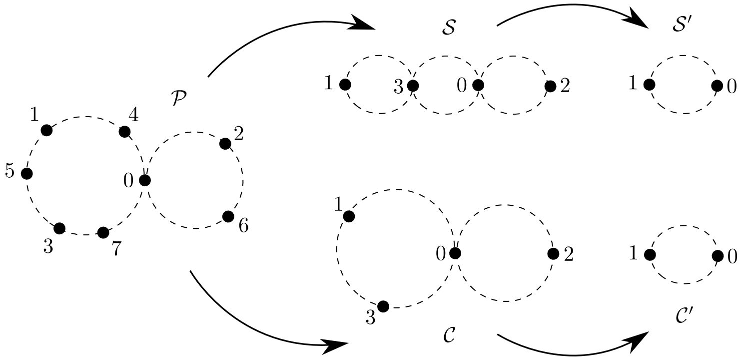
The combinatorial collapse

Ok... better with pictures



The combinatorial collapse

Collapsing is not (always) the same as the skeleton



Observe

The combinatorial collapse is an opening of the skeleton! Since the skeleton of zero-entropy patterns has entropy zero and the opening does not increase entropy...

Do we have a characterization? YES!

[Juher, Mañosas, Rojas. Now]

$h(\mathcal{P}) = 0$ if and only if either $\mathcal{P}$ is trivial or has a maximal separated structure of trivial blocks such that the associated combinatorial collapse $\mathcal{C}$ has entropy 0.

Use it recursively

Sequence of collapses

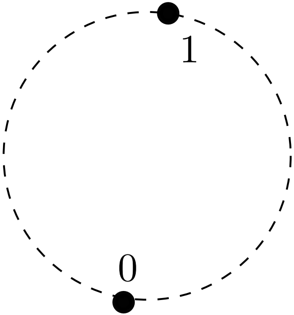
A zero entropy pattern has associated a sequence of patterns $\{\mathcal{P}_i\}_{i=0}^{r}$ and a sequence of integers $\{p_i\}_{i=0}^{r}$ for some $r \geq 0$ such that:

(a) $\mathcal{P}_r = \mathcal{P}$.

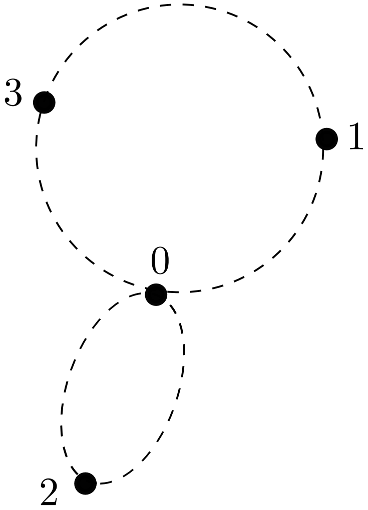
(b) $\mathcal{P}_0$ is a trivial p_0-periodic pattern.

(c) for $1 \leq i \leq r$, $\mathcal{P}_i$ has a maximal separated structure of $\prod_{j=0}^{i-1} p_j$ trivial blocks of cardinality p_i and $\mathcal{P}_{i-1}$ is the corresponding combinatorial collapse.

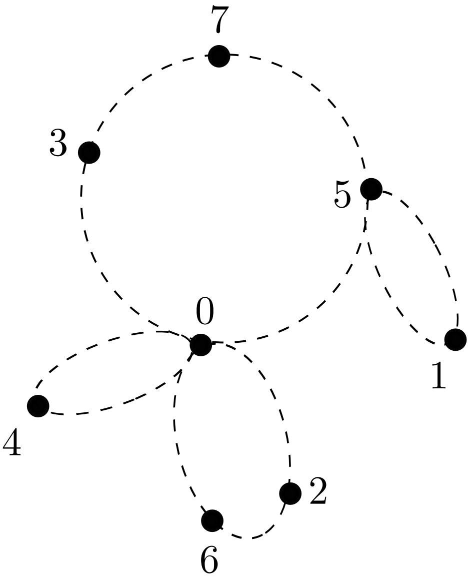
Better than collapse... explode!



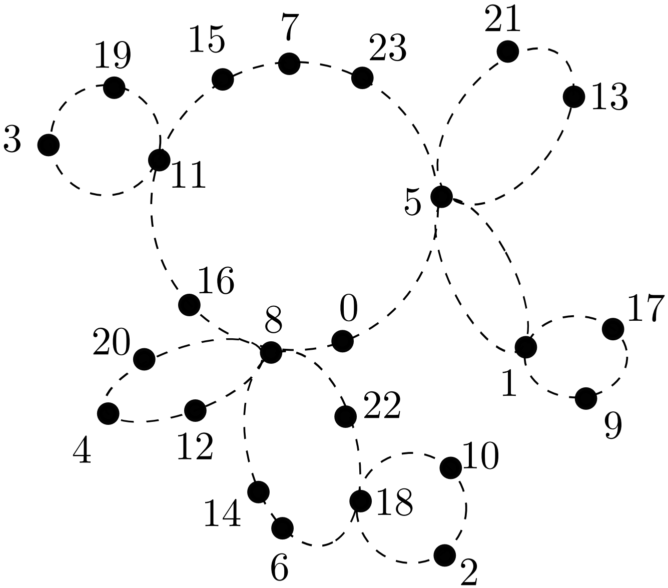
Better than collapse... explode!



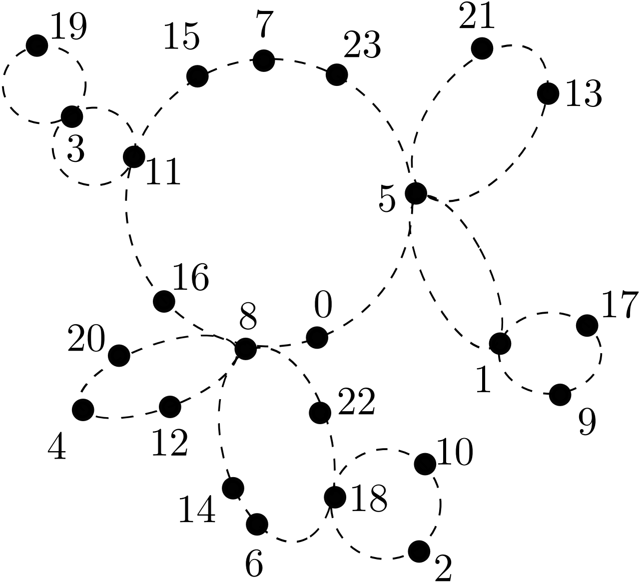
Better than collapse... explode!



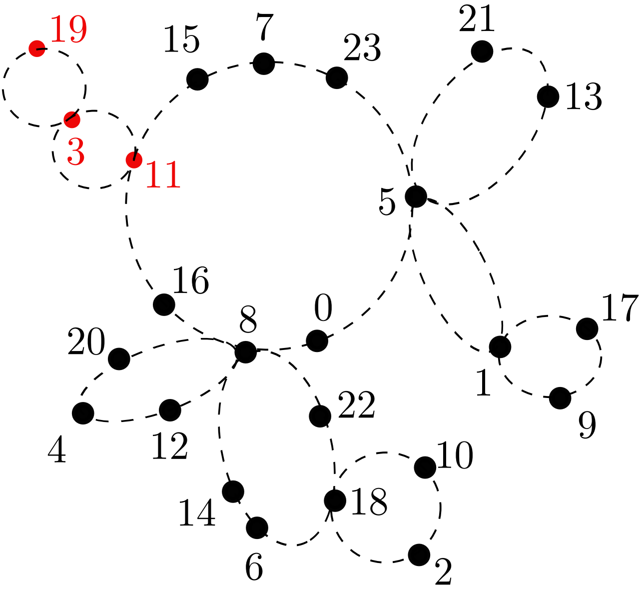
Better than collapse... explode!



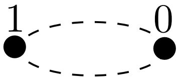
Better than collapse... explode!



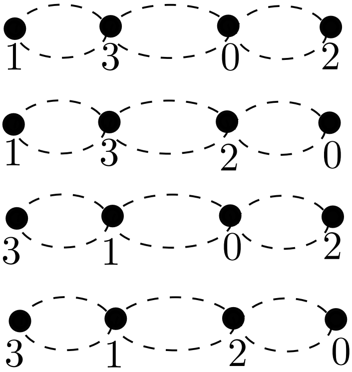
Better than collapse... explode!



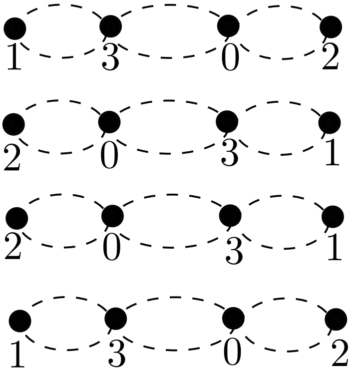
The interval case



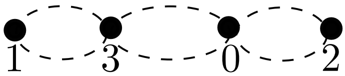
The interval case



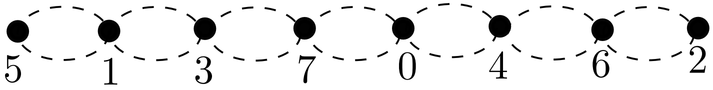
The interval case



The interval case



The interval case



Many thanks for your attention

References

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Ll. Alsedà, J. Guaschi, J. Los, F. Mañosas, P. Mumbrú. Canonical representatives for patterns of tree maps. *Topology* **36** (1997) 1123-1153.

Ll. Alsedà, D. Juher, F. Mañosas. On the minimum positive entropy for cycles on trees. *Transactions Amer. Math. Soc.* **369** (2017) 187–221.