Controlling Chaos and Mixed Mode Oscillations in a Bertrand Duopoly Game

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Progress on Difference Equations International Conference **(PODE 2025)**, Cartagena, 28th–30th May 2025

Outline

Introduction

- **Problem**: Bertrand duopoly model exhibits chaos under bounded rationality.[(1)]
- **Key Findings**:
  - Flip bifurcations and mixed-mode oscillations (MMOs).
  - Instability driven by adjustment speed (k).
- **Solution**: State feedback control stabilizes Nash equilibrium.

[(1)] Named from the article: Bertrand, J. (1883) "Book review of *théorie mathématique de la richesse sociale* and of *recherches sur les principes mathématiques de la théorie des richesses*", *Journal de Savants* 67: 499–508.

What is a Duopoly?

- A duopoly is a market with only two major firms competing.
- Example: Coca-Cola vs. Pepsi – both sell similar products, and adjust prices based on each other.
- Their decisions influence the entire market—competition can be stable or chaotic.

Figure: Real-world example of a duopoly

Duopoly Model Description

Market Setup: Two firms produce differentiated products (q_1, q_2) in discrete time $(t = 0, 1, 2, \dots)$ and their Prices (p_1, p_2).

Key Equations:

1. Consumer Utility: $U(q_1, q_2) = \alpha(q_1 + q_2) - \frac{1}{2}(q_1^2 + 2dq_1q_2 + q_2^2)$
   - α: Market size
   - $d \in (-1, 1)$: Differentiation ($d \to 0$: independent; $d < 0$: complements)
2. Inverse Demand: $p_i = \alpha - q_i - dq_j$
3. Direct Demand: $q_i = \frac{a(1-d)-bp_i+dp_j}{b}, \quad i = 1, 2 \quad \text{where } b = 1 - d^2$
4. Quadratic Costs:
   $C_1(q_1) = (c_1q_1 + c_2)q_1 + c_3 \quad \text{and} \quad C_2(q_2) = (c_4q_2 + c_5)q_2 + c_6$
5. Profit Functions: $\pi_i = p_iq_i - C_i(q_i) \quad i = 1, 2.$

Duopoly Model Description

Bounded Rationality Adjustment:

$$p_i(t+1) = p_i(t) + k_i p_i(t) \cdot \phi_i, \quad i = 1, 2.$$

- The firms' marginal profits are expressed as:

$$\phi_1 = \frac{\partial \pi_1}{\partial p_1} \quad and \quad \phi_2 = \frac{\partial \pi_2}{\partial p_2} \tag{1}$$

- $k_i > 0$: Speed of adjustment

Dynamical System:

$$\begin{cases} p_1(t+1) = p_1(t) + k_1 p_1(t) \cdot \phi_1(p_1, p_2) \\ p_2(t+1) = p_2(t) + k_2 p_2(t) \cdot \phi_2(p_1, p_2) \end{cases} \tag{2}$$

Both firms have similar adjustment speeds $k_1 = k_2 = k$. The system dynamics are discussed in subsequent sections.

Equilibrium and Stability

The equilibrium points are found by solving the condition $p_i(t+1) = p_i(t)$ in system (2). The following fixed points are obtained:

$$E_0 = (0,0),$$
$$E_1 = \left(0, \frac{a(1-d)(b+2c_4)+c_5 b}{b^2}\right),$$
$$E_2 = \left(\frac{a(1-d)(b+2c_1)+c_2 b}{b^2}, 0\right),$$
$$E_3 = (p_1^*, p_2^*) \quad \text{(Bertrand–Nash equilibrium)},$$

- **Thm 1:** The equilibrium point E_0 is always unstable (a repelling node).
- **Thm 2:** The equilibrium points E_1 and E_2 are unstable: Saddle type if $k < \frac{(b+c_i)}{p_j b}$, and Repelling node if $k > \frac{b^2}{(b+c_i)p_j}$.

Economic interpretation: The Bertrand–Nash equilibrium E_3 is the only economically relevant and potentially stable state, representing a sustained duopoly where both firms remain active.

The Bertrand–Nash equilibrium point is given by $E_3 = (p_1^*, p_2^*)$, where:

$$p_1^* = \frac{2(b+c_4)\left[a(1-d)(b+2c_1)+c_2 b\right] + d(b+2c_1)\left[a(1-d)(b+2c_4)+c_5 b\right]}{4(b+c_1)(b+c_4) - d^2(b+2c_1)(b+2c_4)}, \qquad (3)$$

and

$$p_2^* = \frac{2(b+c_1)\left[a(1-d)(b+2c_4)+c_5 b\right] + d(b+2c_4)\left[a(1-d)(b+2c_1)+c_2 b\right]}{4(b+c_1)(b+c_4) - d^2(b+2c_1)(b+2c_4)}. \qquad (4)$$

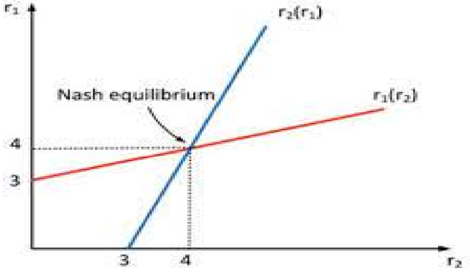


Figure: Graphical representation of the Bertrand–Nash equilibrium

The following Jacobian matrix is used to assess the stability of the system's Nash equilibrium point E_3.

$$J(p_1^*, p_2^*) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} = \begin{bmatrix} 1 + k \cdot p_2^* \cdot \frac{\partial^2 \Pi_1}{\partial p_1 \partial p_2} & k \cdot p_2^* \cdot \frac{\partial^2 \Pi_1}{\partial p_2^2} \\ k \cdot p_1^* \cdot \frac{\partial^2 \Pi_2}{\partial p_1^2} & 1 + k \cdot p_1^* \cdot \frac{\partial^2 \Pi_2}{\partial p_1 \partial p_2} \end{bmatrix}$$

To analyze the stability of **the Nash equilibrium point**, three simultaneous conditions must be satisfied for it to be locally asymptotically stable.

$$\begin{aligned} &\text{(i)} \quad 1 - \det J > 0 \\ &\text{(ii)} \quad 1 - \operatorname{Tr} J + \det J > 0 \\ &\text{(iii)} \quad 1 + \operatorname{Tr} J + \det J > 0 \end{aligned}$$

The initial condition (i) transforms into this:

$$k \cdot \left[4(b + c_1)(b + c_4) - d^2 \cdot (b + 2c_1)(b + c_4)\right] - 2b^2 \cdot \left(\frac{b + c_1}{p_2^*} + \frac{b + c_4}{p_1^*}\right) < 0$$

Bifurcation and Chaos

The second condition (ii) transforms into this:

$$\frac{k^2 p_1^* p_2^*}{b^4} \cdot \left[4(b+c_1)(b+c_4) - d^2 \cdot (b+2c_1)(b+c_4)\right] > 0$$

Finally, the third condition (iii) transforms into this:

$$F(k) = k^2 \cdot \left[4(b+c_1)(b+c_4) - d^2 \cdot (b+2c_1)(b+c_4)\right] - 4kb^2 \cdot \left(\frac{b+c_1}{p_2^*} + \frac{b+c_4}{p_1^*}\right) + \frac{4b^4}{p_1^* p_2^*} > 0.$$

Theorem

The Bertrand–Nash equilibrium point E_3 is asymptotically stable if $k < k_1^f$, where k_1^f is the solution of $F(k) = 0$.
Furthermore, the system (2) undergoes a flip bifurcation at $k = k_1^f$.
Period-2 points bifurcate from E_3, and chaotic dynamics may emerge through a period-doubling cascade for $k > k_1^f$.

Numerical Simulations:

We fix parameters as $\alpha = 3.1$, $c_1 = 0.5$, $c_2 = 0.6$, $c_3 = 1.1$, $c_4 = 0.8$, $c_5 = 0.6$, $c_6 = 0.2$, and $d = 0.2$ to explore the system's dynamics with respect to the adjustment speed k. Figure 3a displays the bifurcation diagrams for prices p_1 and p_2 as k varies. For $k < 0.218$, the system remains stable at the Bertrand–Nash equilibrium. Beyond this critical value, period-doubling bifurcations arise, visible as branching in the diagrams, leading to chaotic attractors that clearly illustrate the transition from stable duopoly pricing to complex chaotic behavior as the firms adjust their strategies more aggressively.

Bifurcation and Chaos

For $k > 0.28$ (beyond the stability region), the system exhibits chaotic dynamics characterized by strange attractors (Fig. 3b).

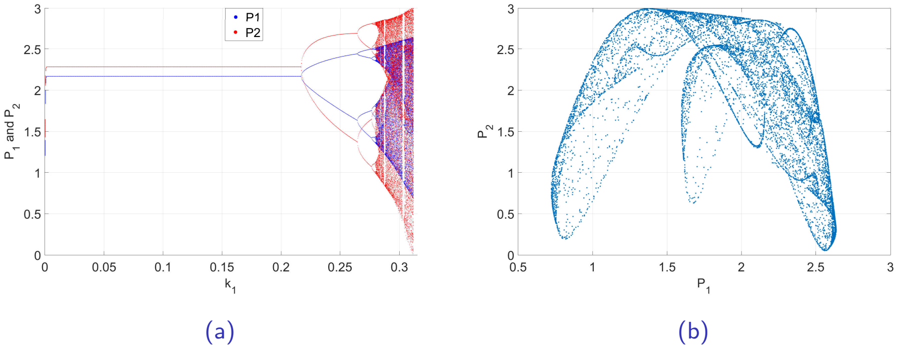


Figure: Chaotic dynamics: (a) Transition to chaos via bifurcations; (b) Resulting strange attractor geometry.

Mixed Mode Oscillations

Mixed-mode oscillations (MMOs) exhibit alternating large and small amplitude peaks, transitioning to chaos via bifurcations near the Nash equilibrium. Their irregular patterns (e.g., $1^2 1^2 1^2 1^3$) demonstrate complex, non-periodic dynamics, visualized in Fig. 4.

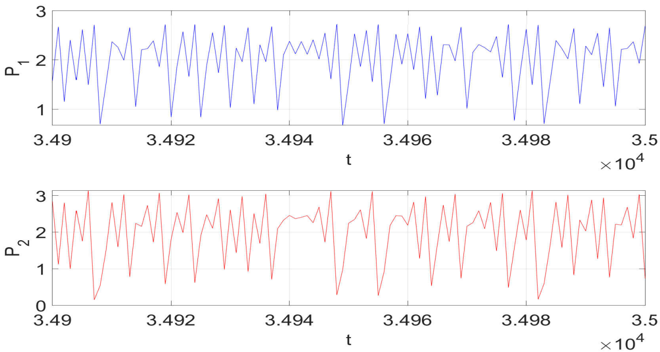


Figure: Time series of oscillator, patterns of Mixed-Mode Oscillations

Controlled System: To mitigate chaotic behavior that emerges when $k > 0.28$, we introduce a control force $u(t)$ into system (2) using state feedback control [1]. Specifically, Firm 1 updates its price $p_1(t+1)$ based on its own previous price $p_1(t)$, the competitor's previous price $p_2(t)$, and the coordinates of the Bertrand–Nash equilibrium. This dynamic adjustment enables stabilization by correcting deviations through small, continuous interventions.

The proposed method is both practical and cost-effective. Firms can implement it using real-time analytics for minor price corrections, or it may be enforced through regulatory policies. This minimally invasive strategy effectively controls chaos and promotes stable, predictable market behavior.

Chaos Control of the Model

The controlled system is defined as follows:

$$\begin{cases} p_1(t+1) = p_1(t) + k \cdot \dfrac{\partial \Pi_1}{\partial p_1} + u(t), \\ p_2(t+1) = p_2(t) + k \cdot \dfrac{\partial \Pi_2}{\partial p_2}, \end{cases} \tag{5}$$

where $u(t) = -v_1(p_1(t) - p_1^*) - v_2(p_2(t) - p_2^*)$,
with v_1 and v_2 being feedback gains designed to stabilize the system near the Bertrand–Nash equilibrium.
The Jacobian matrix of the controlled system (5), evaluated at the equilibrium point (p_1^*, p_2^*), is given by:

$$J_c(p_1^*, p_2^*) = \begin{bmatrix} j_{11} - v_1 & j_{12} \\ j_{21} & j_{22} - v_2 \end{bmatrix}.$$

The trace and determinant of the Jacobian matrix J_c are computed as:

$$T_c = j_{11} + j_{22} - v_1, \quad D_c = j_{22}(j_{11} - v_1) - j_{21}(j_{12} - v_2).$$

Let us define:

$$\begin{cases} \alpha_1 = \frac{J_{22}}{J_{21}}, \quad \beta_1 = \frac{-J_{11}J_{22}+J_{12}J_{21}+1}{J_{21}}, \\ \alpha_2 = \frac{-1+J_{22}}{J_{21}}, \quad \beta_2 = \frac{J_{11}+J_{22}-1-J_{11}J_{22}+J_{12}J_{21}}{J_{21}}, \\ \alpha_3 = \frac{1+J_{22}}{J_{21}}, \quad \beta_3 = \frac{-J_{11}-J_{22}-1+J_{11}J_{22}-J_{12}J_{21}}{J_{21}}. \end{cases}$$

Theorem (2)

The system (5) can be stabilized toward its unstable Bertrand–Nash equilibrium point $E_3 = (p_1^*, p_2^*)$ using the control law (19) if:

$$\max\{\alpha_2 v_1 + \beta_2,\ \alpha_3 v_1 + \beta_3\} < v_2 < \alpha_1 v_1 + \beta_1. \quad (6)$$

The condition (6), as stated in Theorem 2, defines three linear inequalities that form a triangular region in the (v_1, v_2) plane, as shown in Fig. 5(a) for $k = 0.31$. This region corresponds to the set of control parameters for which the Jacobian matrix of the controlled system has eigenvalues strictly inside the unit disc, ensuring local asymptotic stability.

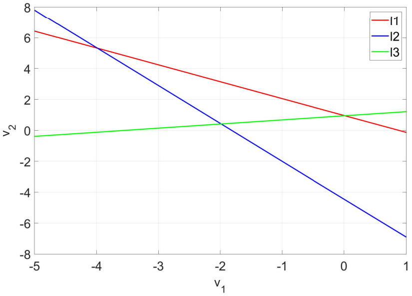
Mathematically, this implies that small perturbations decay over time.

Economically, the interior of the triangle represents stable price adjustment strategies around the Bertrand–Nash equilibrium, while points outside or on the boundary may lead to instability and erratic pricing behavior.

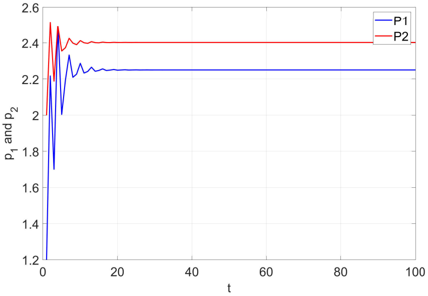
Chaos Control of the Model

We simulated the feedback control method's efficacy in stabilizing chaos around an unstable state. The fixed parameters were: $(v_1, v_2) = (-2, 2)$,

$\alpha = 3.1,\ c_1 = 0.5,\ c_2 = 0.6,\ c_3 = 1.1,\ c_4 = 0.8,\ c_5 = 0.6,\ d = 0.2\ k = 0.31.$



(a)



(b)

Figure: (a) Stability region; (b) Time series of the controlled system.

Conclusion

This paper analyzes a Bertrand duopoly model where boundedly rational firms adjust prices strategically, leading to complex dynamics such as flip bifurcations, chaos, and mixed-mode oscillations. We demonstrate that market stability depends critically on adjustment speeds and product differentiation. To counteract chaotic behavior, we propose a state feedback control method, which numerical simulations confirm effectively restores equilibrium. Our results provide both theoretical insights into oligopoly dynamics and a practical tool for stabilizing unstable markets.

References

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Thank you for your attention.