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Multifractal dimensions of invariant subsets for piecewise monotonic maps

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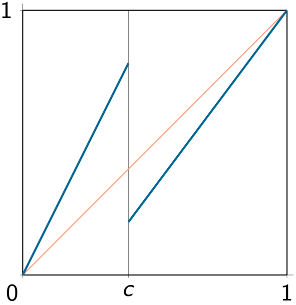
Consider a *piecewise monotonic map* $T : [0,1] \to [0,1]$, this means that there exists a finite family $\mathcal{Z}$ of pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$ such that $T\big|_Z$ is continuous and strictly monotonic for all $Z \in \mathcal{Z}$. Moreover, assume that for every $Z \in \mathcal{Z}$ the map $T\big|_Z$ is differentiable and its derivative can be extended to a continuous function on the closure of Z. We call T *expanding*, if $\inf |T'| > 1$. Although T need not continuous one can use a standard doubling points construction we can make T to a continuous map on a compact metric space.

For a finite union of open intervals U set

$$A(U) := [0,1] \setminus \bigcup_{j=0}^{\infty} T^{-j} U \,.$$

It is the set of all points whose orbit never enters U. Throughout this talk we will always assume that $A \neq \emptyset$.

Next we briefly describe the standard doubling points construction. Let E be the set of endpoints of intervals of monotonicity (except 0 and 1), and define $C(T) := \bigcup_{n=0}^{\infty} T^{-n}E \setminus \{0,1\}$. Replace each $c \in C(T)$ by two points c^- and c^+ with $c^- < c^+$. We can extend T to a continuous map $\widehat{T} : \widehat{X} \to \widehat{X}$ on this new compact metric space $\widehat{X}$.

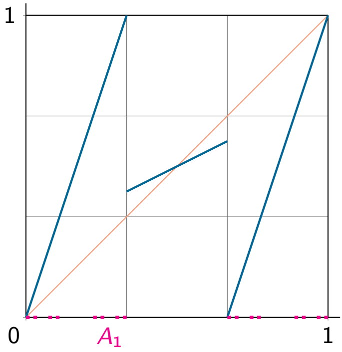


Our goal is to investigate the “size” of $A(U)$. Different notions of fractal dimensions are known for this purpose, in particular the Hausdorff dimension. However only a number is assigned to a set, we do not have any information if the set is “symmetric” or “asymmetric”. For this purpose one can use “multifractal dimensions”. Before we define “multifractal Hausdorff dimensions” we will give two examples in order to motivate our goal. Note that in both examples the sets A_j consists exactly of those points not satisfying $\lim_{n\to\infty} T_j^n x = \frac{1}{2}$. Calculating the Hausdorff dimension we obtain
$\dim_{\mathrm{H}}(A_1) = \frac{\log 2}{\log 3} = 0.630929753571457437099527114343$ and
$\dim_{\mathrm{H}}(A_2) = 0.1154857721125173134499955642356865179888$. It shows the obvious fact that A_2 is “smaller” than A_1, but it does not give us any information that A_2 is “less symmetric” than A_1.

Define

$$T_1 x := \begin{cases} 3x\,, & \text{if } x \in \left[0, \frac{1}{3}\right], \\ \frac{x}{2} + \frac{1}{4}\,, & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ 3x - 2\,, & \text{if } x \in \left[\frac{2}{3}, 1\right], \end{cases}$$

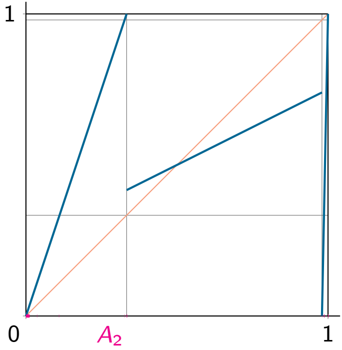
$U_1 := \left(\frac{1}{3}, \frac{2}{3}\right)$ and $A_1 := A(U_1)$, which is the usual Cantor set.



Let

$$T_2 x := \begin{cases} 3x\,, & \text{if } x \in \left[0, \frac{1}{3}\right], \\ \frac{x}{2} + \frac{1}{4}\,, & \text{if } x \in \left(\frac{1}{3}, 1 - 10^{-8}\right), \\ 10^8 x - 10^8 + 1\,, & \text{if } x \in \left[1 - 10^{-8}, 1\right], \end{cases}$$

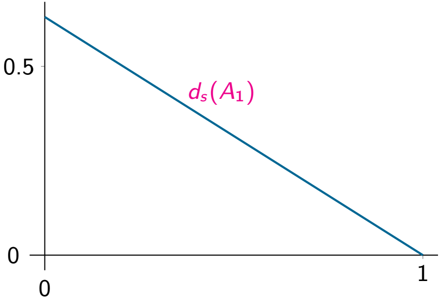
$U_2 := \left(\frac{1}{3}, 1 - 10^{-8}\right)$ and $A_2 := A(U_2)$,a "much more asymmetric" Cantor set.

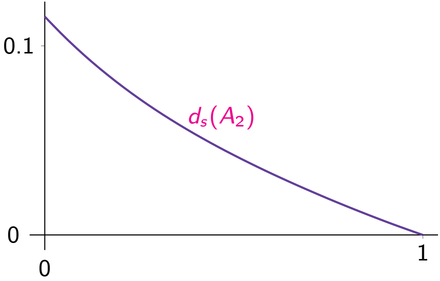


As usual denote by $|I|$ the length of an interval I. If $n \in \mathbb{N}$, let $\mathcal{Z}_n$ be the collection of all nonempty sets of the form $\bigcap_{j=0}^{n-1} T^{-j} Z_j$ with $Z_0, Z_1, \ldots, Z_{n-1} \in \mathcal{Z}$. We define $m(Z)$ for $Z \in \mathcal{Z}_n$ by $m(Z) := \left(\frac{1}{2}\right)^n$ (observe that soon we will consider a more general situation). For $s \in \mathbb{R}$ we define the "multifractal Hausdorff dimension $d_s(A)$" by

$$d_s(A) := \sup \left\{ t : \lim_{n \to \infty} \sum_{\substack{Z \in \mathcal{Z}_n \\ Z \cap A \neq \emptyset}} m(Z)^s |Z|^t = \infty \right\} .$$

Since $|Z| = \left(\frac{1}{3}\right)^n$ for $Z \in \mathcal{Z}_n$ in our first example we get $d_s(A_1) = (1-s)\frac{\log 2}{\log 3}$. Our next figures show the graphs of $s \mapsto d_s(A_j)$ for $s \in [0,1]$. Here we see that these graphs have a different form (however, note that we use a different scaling on the y-axis).


0.5
0
0
1
$d_s(A_1)$


0.1
0
0
1
$d_s(A_2)$

Given a point $x \in A$ we define the *local dimension* $\dim_{\mathrm{loc}}(x)$ as

$$\dim_{\mathrm{loc}}(x) := \lim_{r \to 0^+} \frac{\log m((x-r, x+r))}{\log(2r)},$$

where $m(B) := \lim_{n\to\infty} \sum_{\substack{Z \in \mathcal{Z}_n \\ Z \cap A \cap B \neq \emptyset}} m(Z)$ if $B \subseteq [0,1]$ (note that $2r$ is the length of the interval $(x-r, x+r)$). If $\alpha \in [0,1]$ define $L(\alpha) := \{x \in A : \dim_{\mathrm{loc}}(x) = \alpha\}$. Then the map

$$\sigma : \alpha \mapsto \dim_{\mathrm{H}}(L(\alpha))$$

is called the *multifractal spectrum* of A.
Because of a result by D. Rand σ is the Legendre-transform of $s \mapsto d_s(A)$, this means $\sigma(\alpha) = \inf \{d_s(A) + \alpha s : s \in \mathbb{R}\}$.

We will need the following definitions in order to give a more general definition of multifractal Hausdorff dimensions. Denote by C the family of all functions f such that for every $Z \in \mathcal{Z}$ the restriction $f\big|_Z$ can be extended to continuous function $f_Z : \overline{Z} \to \mathbb{R}$. Observe that a function $f \in C$ may have discontinuities at the endpoints of the intervals of monotonicity. Moreover let C_∞ be the family of all functions f such that for every $Z \in \mathcal{Z}$ the restriction $f\big|_Z : Z \to \mathbb{R}$ is continuous and can be extended to continuous function $f_Z : \overline{Z} \to \mathbb{R} \cup \{+\infty\}$. From now on we will always assume that $\varphi := -\log|T'| \in C_\infty$. Suppose that $\psi \in C$. We call a Borel probability measure m on $[0,1]$ an $e^{-\psi}$-*conformal measure*, if $m(TI) = \int_I e^{-\psi}\, dm$ holds for every interval I contained in an element of $\mathcal{Z}$.

Results on multifractal dimensions or multifractal spectrum are usually obtained only for Markov maps. Unfortunately our map T need not be a Markov map.

Consider a nonempty closed set $B \subseteq A$. Then B is called a *Markov subset* of A, if there exists a finite partition $\mathcal{Y}$ refining $\mathcal{Z}$ (this means $\forall Y \in \mathcal{Y}\, \exists Z \in \mathcal{Z}$ with $Y \subseteq Z$) such that $T(B \cap Y) \subseteq B$ for all $Y \in \mathcal{Y}$ and for any $Y_1, Y_2 \in \mathcal{Y}$ one has either $B \cap Y_2 \subseteq T(B \cap Y_1)$ or $Y_2 \cap T(B \cap Y_1) = \emptyset$. Denote the collection of all Markov subsets of A by $\mathcal{M}(A)$. If $h_{\text{top}}(A, T) > 0$ then $\mathcal{M}(A)$ is not empty. From now on we always assume that $h_{\text{top}}(A, T) > 0$. By $p(A, T, f)$ we denote the topological pressure. Given $f \in C_\infty$ we define

$$q(A, T, f) := \sup_{B \in \mathcal{M}(A)} \sup_{\substack{g \in C \\ g \le f}} p(B, T, g)\,.$$

According to a result of F. Hofbauer we get that $q(A, T, f) = p(A, T, f)$ if $f \in C$ and $p(A, T, f) > \sup_{x \in A} f(x)$.

Suppose that $\psi \in C$ satisfies $q([0,1], T, \psi) = 0$. Then a result by F. Hofbauer and M. Urbański implies the existence of an $e^{-\psi}$-conformal measure m. From now on we always assume that $\psi \in C$, $q(A, T, \psi) = 0$, m is an $e^{-\psi}$-conformal measure and $A \subseteq \operatorname{supp} m$.
Let $E, F \subseteq [0,1]$, $s \in \mathbb{R}$ and $\varepsilon > 0$. We denote by $\mathcal{U}_\varepsilon(F)$ the collection of all at most countable covers of F by intervals of length at most ε centered at an element of F, and by $\mathcal{V}_\varepsilon(F)$ the collection of all at most countable packings (pairwise disjoint) of F by intervals of length at most ε centered at an element of F.
Moreover, we call $\mathcal{F}$ a cover of E if it is a family of at most countably many subsets with $E \subseteq \bigcup_{F \in \mathcal{F}} F$. Denote the collection of all covers of E by $\mathcal{F}(E)$. Now we are able to define the multifractal Hausdorff dimension and the multifractal packing dimension.

To this end we first define

$$\nu_{s,t}(E) := \sup_{F \subseteq E} \lim_{\varepsilon \to 0^+} \inf_{\mathcal{C} \in \mathcal{U}_\varepsilon(F)} \sum_{C \in \mathcal{C}} m(C)^s |C|^t \quad \text{and}$$

$$\pi_{s,t}(E) := \inf_{\mathcal{F} \in \mathcal{F}(E)} \sum_{F \in \mathcal{F}} \left(\lim_{\varepsilon \to 0^+} \sup_{\mathcal{C} \in \mathcal{V}_\varepsilon(F)} \sum_{C \in \mathcal{C}} m(C)^s |C|^t \right) ,$$

where the value ∞ is allowed. Both, $\nu_{s,t}$ and $\pi_{s,t}$ are Borel measures. We define the *multifractal Hausdorff dimension* $d_s(E)$ and the *multifractal packing dimension* $D_s(E)$ by

$$d_s(E) := \sup\{t \in \mathbb{R} : \nu_{s,t}(E) = \infty\} \quad \text{and}$$
$$D_s(E) := \sup\{t \in \mathbb{R} : \pi_{s,t}(E) = \infty\} .$$

Obviously we have always $d_s(E) \leq D_s(E)$.

Define $s_A := \inf\{s \in \mathbb{R} : q(A, T, s\psi) = 0\}$. For $s \in \mathbb{R}$ set $z_s(A) := \sup\{t \in \mathbb{R} : q(A, T, t\varphi + s\psi) > 0\}$. Let $c_s(A)$ be the infimum of all $t \in \mathbb{R}$ such that there exists an $e^{-t\varphi - s\psi}$-conformal measure with support A.

Now we present some results which have been obtained together with F. Hofbauer and T. Steinberger.

Theorem 1

For $s \in [0, s_A)$ we have $z_s(A) = c_s(A)$.

In the case of expanding maps Theorem 1 implies also results on multifractal Hausdorff dimensions of A. Observe that the condition used in Theorem 2 below is a slight generalization of “expanding” (only expanding on A). This condition obviously implies that $\varphi \in C$. Moreover, it also implies that the function $t \mapsto p(A, T, t\varphi + s\psi)$ has a unique zero for any $s \in \mathbb{R}$.

Theorem 2

Suppose that $\sup_{x\in A} \varphi(x) < 0$ *and* $\sup_{x\in A} \psi(x) < 0$*. Then for* $s \in [0, s_A)$ *the unique zero of the function* $t \mapsto p(A, T, t\varphi + s\psi)$ *equals* $z_s(A)$*, and we have*

$$d_s(A) = z_s(A) = c_s(A) .$$

Thank you very much

¡Muchas gracias!