Difference equations related to Thue-Morse, Fibonacci and Rudin-Shapiro sequences

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Thue-Morse sequence

alphabet $= \{a, b\}$ or $\{0, 1\}$ where a and b are any object.
Word, a finite sequence of consecutive symbols of the alphabet.
When contains more than one symbol, the word will be denoted by X.
Introduce a sequence of words in a recurrent way:

$$X_0 = 0$$

$$X_{n+1} = X_n \overline{X}_n$$

where $\overline{X}$ is the complementary word of X and $X_n \overline{X}_n$ denotes the concatenation of words X_n and $\overline{X}_n$
$X_0 = 0$, $X_1 = X_0\overline{X_0} = 01$, $X_2 = X_1\overline{X_1} = 0110$,
$X_3 = X_2\overline{X_2} = 01101001$, $X_4 = X_3\overline{X}_3 = 0110100110010110$.......

Thue-Morse sequence

The Thue-Morse sequence is $TM = (t_n)_{n=0}^{\infty}$ constructed as the concatenation of the infinite above words: $X_0 X_1 X_2 ......X_n.....$ which can be seen as the developping in base 2 of certain numbers which can be translated to base 10. We have,

$$X_0 = 0(2) = 0$$

,

$$X_1 = 01(2) = \frac{1}{2}$$

,

$$X_2 = 0110(2) = \frac{1}{2} + \frac{1}{2^2}$$

....

The terms of Thue-Morse sequence t_n can be described as: 0 if the number of ones in the binary representation of n is even and 1 if the number of ones in the binary representation of n is odd.

Thue-Morse sequence in base 10

By induction is not difficult to see :

$$t_{n+1} = t_n + \overline{t}_n 2^{-2^n}$$

$$t_n + \overline{t_n} = \sum_{k=1}^{2^n} \frac{1}{2^k} = 1 - 2^{-2^n}$$

$$\overline{t}_n = 1 - 2^{-2^n} - t_n$$

,

$$t_{n+1} = 2^{-2n+1}(2^{2^n} - 1)(1 + 2^{2^n} t_n)$$

Using the formula we can obtain rational approximations to the number TM, such that
$0, \frac{1}{4}, \frac{3}{8}, \frac{105}{256}, \frac{13515}{32768} \ldots$.

Properties of Thue-Morse sequence

The sequence appeared for first time in the work of Eugene Prouhet (1851) in number theory. Explicitely was introduced and studied in 1906 by Axel Thue in works of combinatory and by Marton Morse (1921) in the application to construction of geodesics.

An overlap on the set of words is a word of the form $aXaXa$ where a is a letter and X a word. A sequence is called *overlap-free* if contains no word that is an overlap.

TM is an example of infinite overlap-free word. Using it, we prove that MT does not contains any word B of the form $D\overline{D}d$ holding $D = \overline{D}$ and with d the initial symbol of D.
In the literature it is called *Thue-Morse constant* τ_{TM} to the series

$$\sum_{n=0}^{\infty} \frac{t_n}{2^{-(n+1)}} \approx 0.41245....$$

Such number is a transcendent and irrational number.

In an one-dimensional array, the Thue-Morse chain is constructed using the two numbers $d_1 = \frac{1}{4}$ and $d_2 = \frac{3}{8}$ and the rest of distances in the array will be generated by substitution. In this case $d_1 \to d_1 d_2$ and $d_2 \to d_2 d_1$.

On the problem of the scattering of a plane wave

We study the transmission of a plane wave (with wave number $k > 0$ through a one-dimensional array of n, δ-function potentials with equal strengths v located on a Thue-Morse chain x_n with distances d_1 and d_2 when $n \to \infty$.

The problem is to decide if the one dimensional array behaves as a *conductor*$(|r_n|(modulus)n \to \infty \neq 1)$ or a *insulator* $(|r_n|(modulus)\pi \to \infty = 1)$. More specifically, is there a curve in the (v, k) parameter space separating the conductor and insulator domain?.

We have the definitions of the stated coefficientes:
$r_1 = \frac{v}{2ik-v}$ and $x_1 = \frac{2ik}{2ik-v}$ holding $|r_1|^2 + |x_1|^2 = 1$ and by using induction, we obtain:

$$x_n r_n[r_n r_{n+1} - (1 - 2x_n)] = r_{n+1} - r_n$$

for $n \geq 2$

Transmission through a Thue-Morse, Fibonacci or Rudin-Shapiro chains, a structure based on any of the correesponding sequence, exhibits interesting properties concerning wave propagation and in other physical and thechnical problems.

The chains can act as a perfect reflector for certain wave numbers, meaning that no trasmission occurs. However under specific conditions, it can also exhibit conductive behaviors, where the transmission is possible. The system can also transitate between insulating and conducted states as the strenght of the potential varies, demostrating a phase transition.

Principal results (partially obtained by Avishai and Berend in Phys.Rev.(1991))
(1) If k is an multiple of $\pi/(|d_1 - d_2|)$, then there is a threshold value v_0 for v such that if $v \geq v_0$, then $r_n \leq 1$ if $n \to \infty$, whereas if $v < v_0$ then $|r_n| \to 1$.
In other words, the system exhibits a metal-insulator transition at that energy.

(2) For any k, if v is large enough, then the sequence of reflection coefficients $|r_n|$ has a subsequence $|(r_2)_n|$, which tends exponentially to 1.
(3) Theoretical considerations are presented giving some evidence to the claim that if k is not a multiple of $\pi/(|d_1 - d_2|)$, we have $|(r_2)_n| \to 1$ for any $v > 0$ except for a "small" set (say, of zero measure). However, this exceptional set is in general nonempty. Numerical calculations we have carried out seem to claim that the behavior of the subsequence $|(r_2)_n|$ is not special, but rather typical of that of the whole sequence r_n
(4) An instructive example we have constracted shows that it is possible to have $|r_n| \to 1$ for some strength v while $|r_n|$ different to 1 for a larger value of v.

On the Thue-Morse nonlinear difference equation

We have the following difference equation:

$$x_{n+2} = x^2(x_{n+1} - 2) + 2,\ \ n \geq 1$$

The unfolding of such equation leads to the discrete two dimensional dynamical system $(\mathbb{R}^2, TM)$

$$TM(x, y) = (x(4 - x - y), xy)$$

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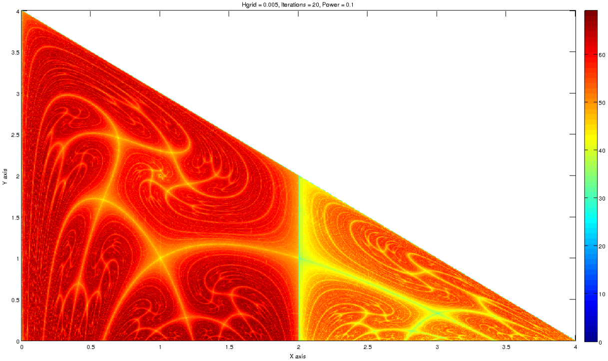

Fibonacci sequence and chain

Using the substitution rules $0 \to 01, 1 \to 0$, the words associated are $X_0 = 0$, $X_1 = 01$, $X_2 = 010$,....and $X_{n+1} = X_n X_n$ for all $n \geq 0$

The Fibonacci sequence is

$$(00101001001....)$$

The Fibonacci chain

We concentrate on the scattering from an infinite system of δ-potential functions located on one dimensional array of points with Fibonacci numbers F_n and corresponding Fibonacci chain as taken as examples of distances of points chosen in the arry.

We study transmission and reflection of a plane wave (with a positive wave number k) through a one-dimensional array of n δ-potential function with equal strengths v located on numbers given by the numbers given by

$$x_n = n + u[\frac{n}{\Phi}]$$

where u is an auxiliar irrational number, $\Phi = \frac{1+\sqrt{5}}{2}$ and [.] the integer part.
Using analytical and number theoretical methods, we arrive to the results:

(1) For a positive k, integer multiple of $\pi\Phi$ (a countable dense set on the positive part of the $k$axis), the system is a *perfect reflector*; namely, the reflection coefficient equals 1 (physically, the system behaves as an *insulator*).

(2) If $= \frac{\pi}{2(2n+1)}$,(for $n = 0, 12...$) and cos$\phi - 1 > 0$ where $\Phi = \text{arctg}\frac{v}{k}$, the system may conduct (it means that the reflection coefficient is strictly smaller than 1).

(3) For $k = \frac{\pi}{2(2n+1)}$ and 3cos$\phi - 1 < 0$, the system is an insolator.

(4) For k a rational non integer multiple of π, the system is a conductor for small values of $\frac{v}{k}$

(2) And (3) are physically remarkable since they imply for a fixed $k = \frac{1}{2(2n+1)}\pi$ for
(n = 0,1,2...) that there is a phase transition between to be a conductor and an insolater as the strengh v varies continuously near $k\sqrt{8}$.

Result (4) means that at least one phase transition of this kind occurs at any k which is a rational non-integer multiple of π once $\frac{v}{k}$ is large enough.

The Fibonacci chain

$$x_{n+3} = x_{n+1}x_{n+2} - x_n$$

is a Fibonacci equation
(Fibonacci difference equation) and applying its unfolding we have the two dimensional discrete nonlinear dynamical system $F : \mathbb{R}^3 \to \mathbb{R}^3$:

$$F(x, y, z) = (y, z, yz - x)$$

We claim that there is a compact subset $K \subset \mathbb{R}^3$ such that the orbits of all points outside are converging to ∞ and inside it, the dynamics is complicate.

Rudin-Shapiro sequence

Let the following functional equation $\alpha : \mathbb{N} \to \mathbb{N}$:

$$\alpha(2n) = \alpha(n)$$

$$\alpha(2n+1) = (-1)^n \alpha(n)$$

with $n \geq 0$

$$\alpha(0) = 1$$

In 1970, J.Brillhart and L.Carliz got

$$\alpha(n) = (-1)^{e_0 e_1 + \ldots\ldots + e_{k-1} e_k} \qquad (1)$$

where $n = \sum_{r=0}^{k} e_r 2^r$
and $e_r = 0$ or 1 which is clearly a sequence of $\pm 1's$

The exponent on -1 in equation (1) counts the number of pairs of consecutive $1's$ in the binary representation of n. For example if $n = 115_{10} = 1110011_2$ we have $\alpha(115) = (-1)^3 = -1$.

In the problem of a *step potential* we can use the Rudin-Shapiro. We have that after some routinary calculus:

$$\alpha_{n+1} = 2\alpha_n\alpha_{n-1} - \alpha_{n-2}$$

with its unfolding

$$RS(x, y, z) = 2zy - x$$

$\alpha_1 = \frac{1}{2}(E - V_1)$, $\alpha_0 = \frac{1}{2}(E - V_0)$ and $\alpha_{-1} = 0$.