Codimension-one and -two bifurcations and stability of certain second order rational difference equation with arbitrary parameters

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Cartagena, May, 28-30, 2025

Introduction

- We consider the second-order rational difference equation

$$x_{n+1} = C + A\frac{x_n^k}{x_{n-1}^p}, \tag{1}$$

where $k,\ p,\ A,\ C > 0$ and $x_{-1},\ x_0 > 0$.

- The change of variable

$$x_n \mapsto Cx_n \tag{2}$$

transforms equation (1) into the following topological conjugate equation

$$x_{n+1} = 1 + a\frac{x_n^k}{x_{n-1}^p}, \tag{3}$$

where $a = AC^{k-p-1} > 0$.

Motivation

- Stević proved the global stability of the positive equilibrium in the case $k = p \in (0, 1]$ and investigated the boundedness of positive solutions of the equation

$$x_{n+1} = \alpha + \frac{x_n^k}{x_{n-1}^p}, \tag{4}$$

- Khyat et al. computed the direction of the Neimark-Sacker bifurcation and gave the asymptotic approximation of the invariant curve of the equation (4) in the case when $k = p = 2$.
- Bešo et al. established boundedness, global attractivity, and Neimark-Sacker bifurcation results for the equation (3) for $k = 1, p = 2$ and gave the asymptotic approximation of the invariant curve near the unique equilibrium point.
- Berkal and Navarro performed a local stability analysis of the positive equilibrium of the equation (3) in the case when $k = p - 1$ with $k \geq 1$, and studied the existence of the Neimark-Sacker bifurcation.
- We demonstrate a complete characterization of the number and uniqueness of positive equilibrium points of equation (3) and it's dynamics in the bi-parameter space revealing various well-organized structures with both regular and chaotic oscillations.

Introduction

Equilibrium points and local stability

Behaviour of solutions in the neighborhood of a unique nonhyperbolic equilibrium point

Behaviour of solutions when there exist two positive equilibrium points

Equilibrium points

Proposition

Let a, k, p be positive real numbers and $\tilde{a} = \frac{(k-p-1)^{k-p-1}}{(k-p)^{k-p}}$.

(1) *The equation (3) has a unique positive equilibrium if one of the following conditions holds:*
   - (a) $k < p + 1$,
   - (b) $k = p+1 \quad \text{and} \quad 0 < a < 1$,
   - (c) $k > p + 1 \quad \text{and} \quad a = \tilde{a}$.

(2) *The equation (3) has two positive equilibrium points if:*

$$k > p + 1 \quad \text{and} \quad a < \tilde{a}.$$

(3) *The equation (3) has no positive equilibrium if:*
   - (a) $k > p + 1 \quad \text{and} \quad a > \tilde{a}$.
   - (b) $k = p + 1 \quad \text{and} \quad a \geq 1$.

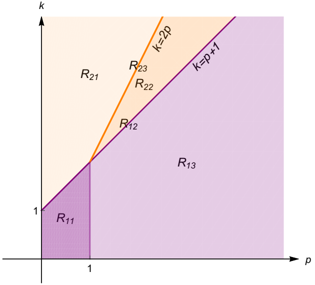
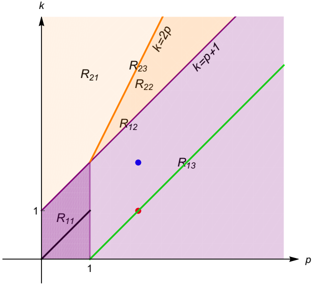


Figure: Parametric regions in (p, k)-plane

Contributions

- Stević proved:
  - The equation has positive solutions that are unbounded if $k^2 \geq 4p > 4$ or $k \geq 1 + p$ with $p \leq 1$.
  - All positive solutions to the equation are bounded if $k^2 < 4p$ or $2\sqrt{p} \leq k < 1 + p$ with $p \in (0, 1)$.
  - Positive equilibrium $\bar{x} = \alpha + 1$ is globally asymptotically stable for $k = p \in (0, 1]$.
- Khyat et al. considered the parametric point $(p, k) = (2, 2)$.
- Bešo et al. considered the parametric point $(p, k) = (2, 1)$.
- Berkal and Navarro considered the parametric values on the line $k = p - 1$ with $k \geq 1$.



Local stability analysis of the unique equilibrium point

Theorem

The unique equilibrium point $\bar{x}$ of equation (3) is

(1) *locally asymptotically stable if*

$$k \leq p+1 \wedge \left(p \leq 1 \vee \left(p > 1 \wedge \bar{x} < \frac{p}{p-1}\right)\right),$$

(2) *repeller if*

$$k \leq p+1 \wedge p > 1 \wedge \bar{x} > \frac{p}{p-1},$$

(3) *nonhyperbolic if one of the following is true*

(a) $k \leq p+1 \wedge p > 1 \wedge a = a_0$, $a_0 = \frac{(p-1)^{k-p-1}}{p^{k-p}}$,

(b) $k > p+1 \wedge a = \tilde{a}$, $\tilde{a} = \frac{(k-p-1)^{k-p-1}}{(k-p)^{k-p}}$.

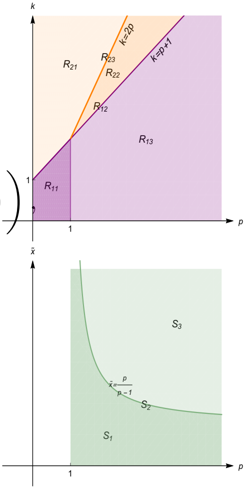


Figure: Regions of local stability scenarios in parametric spaces (p, k) and $(p, \bar{x})$

Introduction

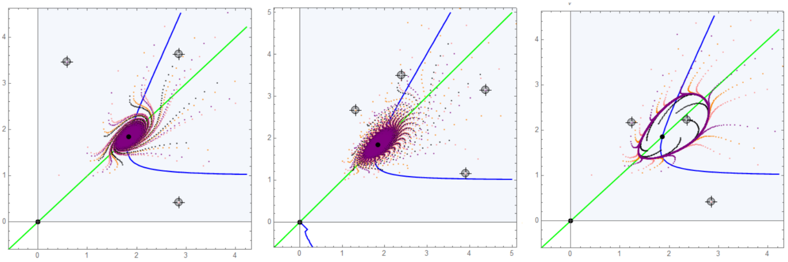
Equilibrium points and local stability

Behaviour of solutions in the neighborhood of a unique nonhyperbolic equilibrium point

Behaviour of solutions when there exist two positive equilibrium points

The unique nonhyperbolic equilibrium point in the parametric region R_{13}

- If (p, k) belongs to the region R_{13}, then there is a unique equilibrium point $\bar{x}$ of (3) regardless of the value of $a > 0$.
- For parametric value $a = a_0 = \frac{(p-1)^{k-p-1}}{p^{k-p}}$ the equilibrium point $\bar{x} = \frac{p}{p-1}$ is nonhyperbolic.

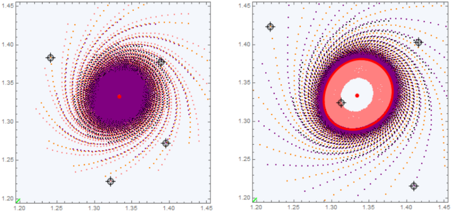


(a) $(p, \bar{x}) \in S_1$, (b) $(p, \bar{x}) \in S_2$ (c) $(p, \bar{x}) \in S_3$

Figure: Numerical simulations for parametric values $(p, k) = (2.1901, 2.6901) \in R_{13}$

Existence of Neimark-Sacker bifurcation in the region R_{13}

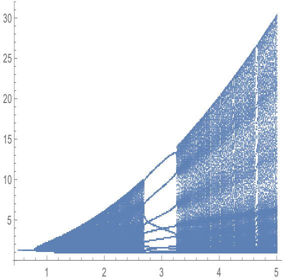
- If $(p,k) \in R_{13}$ the supercritical Neimark-Sacker bifurcation appears when a passes through the bifurcation value a_0.
- There exists a neighborhood $\mathcal{U}$ of the equilibrium point $(\bar{x}, \bar{x})$ such that the ω-limit set of solutions with initial conditions $x_0,\ x_{-1} \in \mathcal{U}$ is $(\bar{x}, \bar{x})$ if $a < a_0$, and belongs to a closed invariant C^1 curve $\Gamma(a)$ encircling the equilibrium point $(\bar{x}, \bar{x})$ if $a > a_0$.



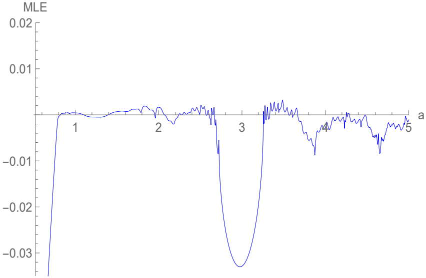
(a) $a = 0.79 < a_0$ (b) $a = 0.792 > a_0$

Figure: Trajectories and invariant curve (red) for parametric values $(p,k) = (4,1) \in R_{13}$ and $a_0 = 0.790123$

Bifurcation diagram and Maximum Lyapunov exponents for parametric values in the region R_{13}



(a)

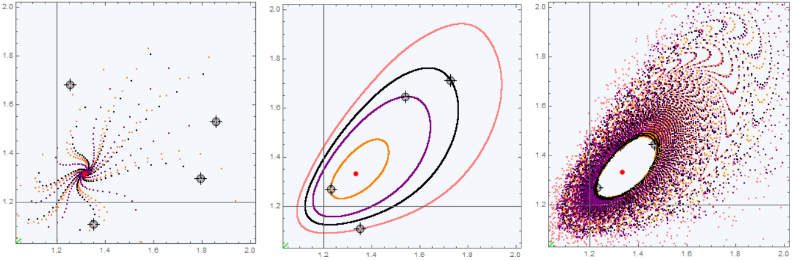


(b)

Figure: $a \in (0.5, 5)$ and $(x_0, y_0) = (1.4, 1.2)$
$(p, k) = (4, 1) \in R_{13}, a_0 = 0.790123$

The unique nonhyperbolic equilibrium point in the parametric region R_{12}

- If (p, k) lies on the line R_{12} and $0 < a < 1$, then equation (3) has a unique equilibrium point $\bar{x}$.
- For parametric value $a = a_0 = \frac{1}{p}$ and $p > 1$ the unique equilibrium point $\bar{x} = \frac{p}{p-1}$ is nonhyperbolic.



(a) $(p, \bar{x}) \in S_1$, (b) $(p, \bar{x}) \in S_2$ (c) $(p, \bar{x}) \in S_3$

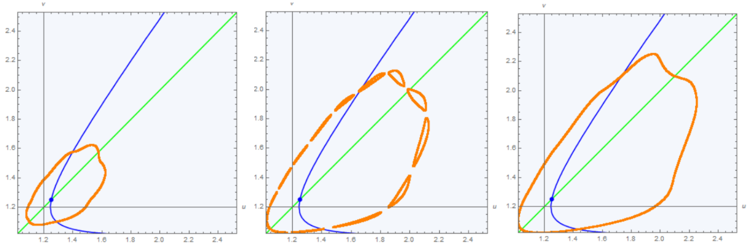
Figure: Numerical simulations for parametric values $(p, k) = (4, 5) \in R_{12}$

The unique nonhyperbolic equilibrium point in the parametric region R_{12}

- Using *Wolfram Mathematica*, we obtain that the first and the second Lyapunov coefficients are zero,

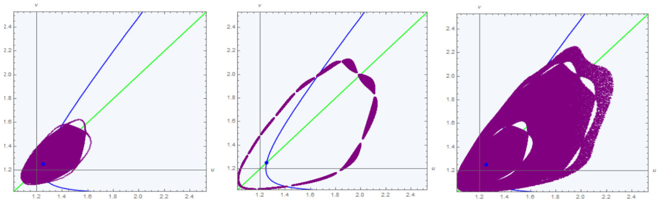
$$\ell_1(0) = 0, \quad \ell_2(0) = 0.$$

- Unique equilibrium point is a Hopf point of codimension 3.



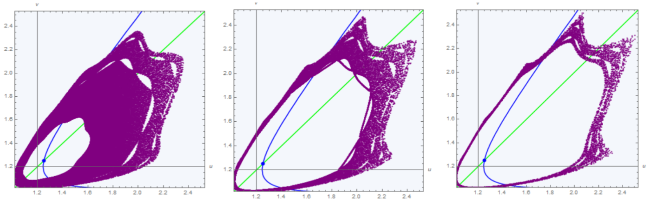
(a) $(x_0, y_0) = (1.6, 1.5)$ (b) $(x_0, y_0) = (2, 1.4)$ (c) $(x_0, y_0) = (2.1, 2.1)$

Figure: $(p, k) = (5, 6)$, $a = a_0 = 0.2$, $n = 1100000$



(a) $(x_0, y_0) = (1.6, 1.5)$ (b) $(x_0, y_0) = (2, 1.4)$ (c) $(x_0, y_0) = (2.1, 2.1)$

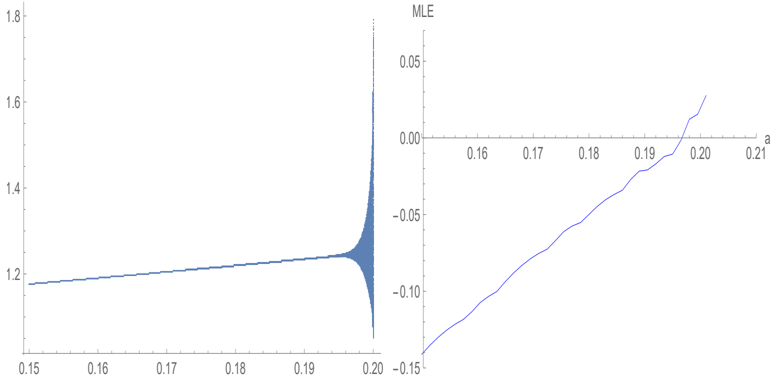
Figure: $(p, k) = (5, 6)$, $a = 0.199999 < a_0$, $n = 1100000$



(a) $(x_0, y_0) = (1.6, 1.5)$, $n = 180000$ (b) $(x_0, y_0) = (2, 1.4)$, $n = 47000$ (c) $(x_0, y_0) = (2.1, 2.1)$, $n = 12000$

Figure: $(p, k) = (5, 6)$, $a = 0.200001 > a_0$

Bifurcation diagram and Maximum Lyapunov exponents

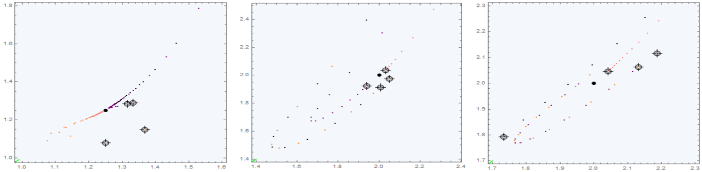


(a) (b)

Figure: $(p, k) = (5, 6)$, $a \in (0.15, 0.22)$, $a_0 = 0.2$ and $(x_0, y_0) = (1.6, 1.5)$.

The unique nonhyperbolic equilibrium point in parametric regions R_{21}, R_{22} and R_{23}

- For parameter values (p, k) in the regions R_{21}, R_{22}, or R_{23}, if $a = \tilde{a} = \frac{(k-p-1)^{k-p-1}}{(k-p)^{k-p}}$, there is unique equilibrium point $\bar{x} = \frac{k-p}{k-p-1}$ and it is nonhyperbolic.
- If $(p, k) \in R_{22}$, the equilibrium point $\bar{x}$ is unstable since one of the eigenvalues of the associated map is greater then one.
- If $(p, k) \in R_{21}$, by using the center manifold theory we prove that the unique equilibrium point $\bar{x}$ is semi-stable.

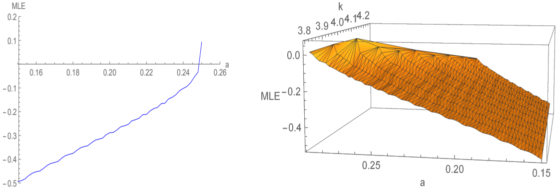


(a) $(p, k) \in R_{21}$, (b) $(p, k) \in R_{22}$ (c) $(p, k) \in R_{23}$

Figure: Trajectories of the initial points near the unique nonhyperbolic equilibrium point

The unique nonhyperbolic equilibrium point in R_{23}

- To investigate stability of the unique equilibrium point $\bar{x}$ in the region R_{23} we observe intriguing dynamical phenomena while varying two parameters, the point (a, k) in a sufficiently small neighborhood of the point $(a_0, k_0) = \left(\frac{(p-1)^{p-1}}{p^p}, 2p\right)$.
- We discuss the $1:1$ resonance and the existence of a Bogdanov-Takens codimension-2 bifurcation revealing the occurrence of fold and Neimark-Sacker bifurcations with codimension 1.



(a) $(p, k) = (2, 4)$, $a \in (0.15, 0.26)$, $a_0 = 0.25$

(b) $p = 2$, $k \in (3.8, 4.25)$, $a \in (0.15, 0.35)$, $(a_0, k_0) = (0.25, 4)$

Figure: Maximum Lyapunov exponents for $(x_0, y_0) = (1.993, 1.993)$

Local stability analysis in the case of two equilibrias

Theorem

If $k > p+1$ and $a < \tilde{a} = \frac{(k-p-1)^{k-p-1}}{(k-p)^{k-p}}$, then equation (3) has two real positive equilibrium points $\bar{x}_1 < \bar{x}_2$, where $\bar{x}_2$ is a saddle point, and one of the following holds:

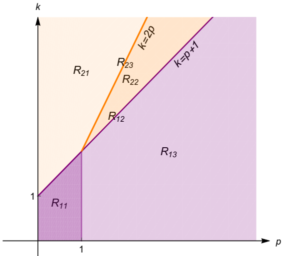
(1) *If $p \leq 1 \vee (p > 1 \wedge k \geq 2p)$, then $\bar{x}_1$ is locally asymptotically stable.*

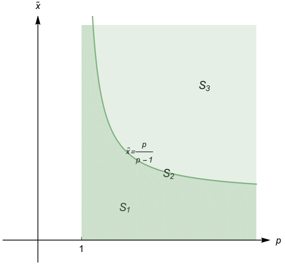
(2) *If $p > 1 \wedge k < 2p$, then*

   (a) *if $\bar{x}_1 < \frac{p}{p-1}$, then $\bar{x}_1$ is locally asymptotically stable,*

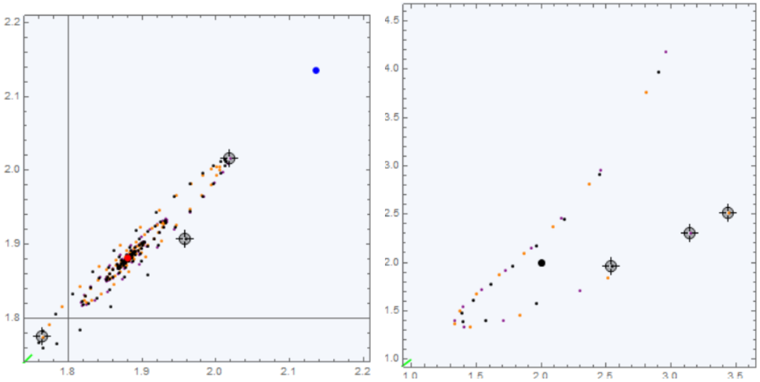
   (b) *if $\bar{x}_1 = \frac{p}{p-1}$, then $\bar{x}_1$ is nonhyperbolic,*

   (c) *if $\bar{x}_1 > \frac{p}{p-1}$, then $\bar{x}_1$ is a repeller.*





Collision of the locally asymptotically stable and a saddle equilibrium point in the region R_{23}



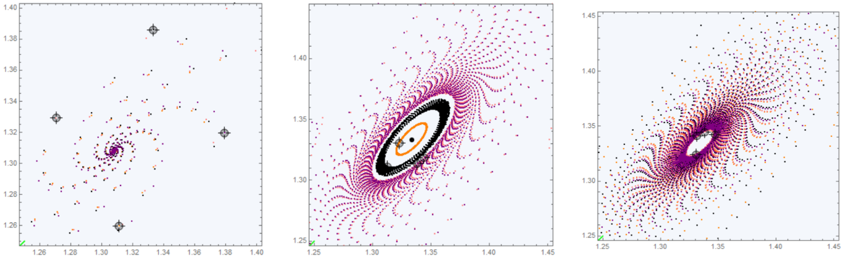
(a) Locally asymptotically stable $\bar{x}_1$ (red) and a saddle $\bar{x}_2$ (blue) equilibrium points $a = 0.249 < \tilde{a}$, $(p, \bar{x}_1) \in S_1$

(b) Nonhyperbolic equilibrium point $a = \tilde{a} = 0.25$, $(p, \bar{x}_1) = (p, \bar{x}_2) \in S_2$

Figure: $(p, k) = (2, 4) \in R_{23}$

Local stability of the equilibrium $\bar{x}_1$ in the parametric region R_{22}

- For $(p,k) \in R_{22}$ and $a = a_0 = \frac{(p-1)^{k-p-1}}{p^{k-p}}$, the equilibrium point $\bar{x}_1 = \frac{p}{p-1}$ is nonhyperbolic.



(a) $(p, \bar{x}_1) \in S_1$, $a = 0.18 < a_0$

(b) $(p, \bar{x}_1) \in S_2$, $a = 0.1875 = a_0$

(c) $(p, \bar{x}_1) \in S_3$, $a = a_0 > 0.1877$

Figure: Local stability of the equilibrium $\bar{x}_1$ for parametric values $(p,k) = (4,6) \in R_{22}$.

The nonhyperbolic equilibrium point $\bar{x}_1$ and Neimark-Sacker bifurcation in the region R_{22}

- The subcritical Neimark-Sacker bifurcation occurs at $\bar{x}_1$.
- In the small neighborhood of a_0, such that $a < a_0$, closed invariant curve $\Gamma(a)$ encircling $\bar{x}_1$ appears.
- All solutions that start in the interior of this curve, converge to $\bar{x}_1$, and this curve is repelling for all solutions that start in the exterior of this curve.

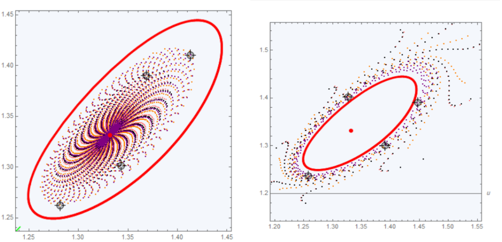
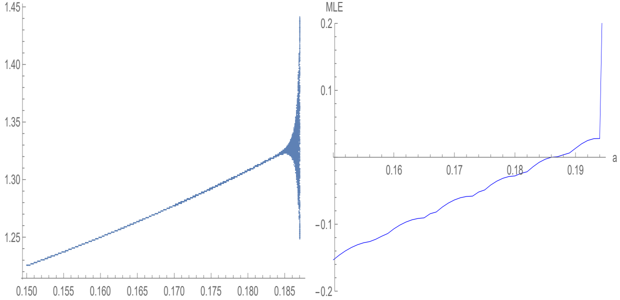


Figure: Trajectories and invariant curve (red) for parametric values $(p, k) = (4, 6)$, $a_0 = 0.1875$, $a = 0.187$

Bifurcation diagram and Maximum Lyapunov exponents



(a) (b)

Figure: Bifurcation diagram and Maximum Lyapunov exponents for parametric values $(p, k) = (4, 6) \in R_{22},\ a \in (0.15, 0.187)$ and $(x_0, y_0) = (1.35, 1.28)$.

THANK YOU

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