Volume entropy of surface groups: a dynamical approach

Lluís Alsedà

Departament de Matemàtiques
Universitat Autònoma de Barcelona

http://www.mat.uab.cat/∼alseda

in collaboration with David Juher, Francesc Mañosas and Jérôme Los

Progress on Difference Equations PODE 2025, Cartagena, 28th-30th May 2025

UAB Universitat Autònoma de Barcelona

Departament de Matemàtiques

Outline

Geometric group theory

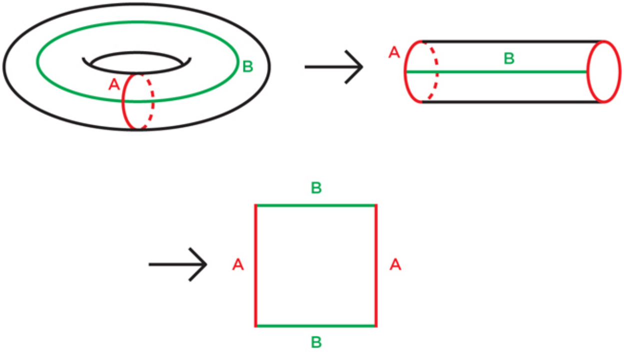
Devoted to the study of the algebraic properties of finitely generated groups via the geometric and topological properties of the spaces on which such groups act.

Often, finitely generated groups G themselves are considered as geometric objects, after endowing them with a metric (usually, the *word metric*) that assigns a distance to any pair x, y of elements of G defined as the length of the shortest word W, written in terms of the generators of G, such that $x = Wy$.

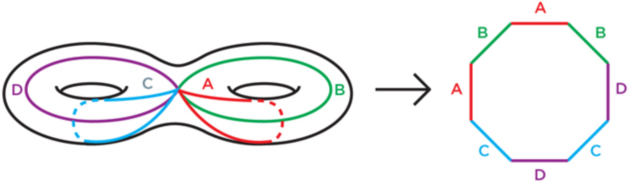
Here when we say *generators* we mean *and their inverses*.

A *presentation* $\langle X | R \rangle$ of a finitely generated group G is a set X of *generators* and a set R of *relations* (words equivalent to the identity element of G).

Example: $\langle a, b \mid ab\bar{a}\bar{b} \rangle$ Classical presentation for the fundamental group of a torus (genus $g = 1$, rank $2g = 2$).

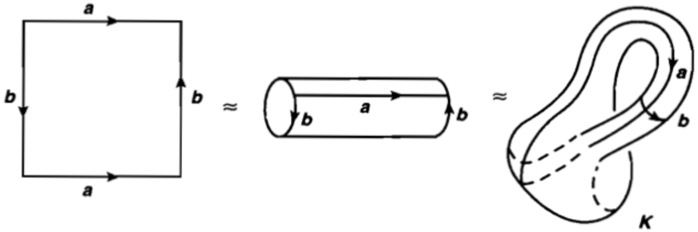


Example: $\langle a, b, c, d \mid ab\bar{a}\bar{b}cd\bar{c}\bar{d} \rangle$ Classical presentation for the fundamental group of a double torus (genus $g = 2$, rank $2g = 4$).



$\langle a, b, c, d, e \mid acde\bar{d}\bar{b}, \bar{e}\bar{c}b\bar{a} \rangle$ An exotic presentation for the same group.

$\langle a, b \mid a^2 b^2 \rangle$
$\langle a, b \mid ab\bar{a}b \rangle$ Classical presentations for the fundamental group of a Klein bottle (nonorientable surface of rank 2).



$\langle a, b, c \mid a^2 b^2 c^2 \rangle$ Classical presentation for the fundamental group of the nonorientable surface of rank 3.

$\langle a, b, c, d \mid acdb, cad\bar{b} \rangle$ An exotic presentation for the same group.

Word metric

Given a presentation $P = \langle X | R \rangle$ of G and $x \in G$, we define $\text{length}_P(x)$ as the number of symbols of a minimal word in the alphabet $X \cup \bar{X}$ representing x.

Example: $P = \langle a, b, c, d \mid acdb, cad\bar{b} \rangle$

$x = acdcad = accadd = acbd = accad\bar{c}\bar{a}\bar{b}$

$\text{length}_P(x) = 4$

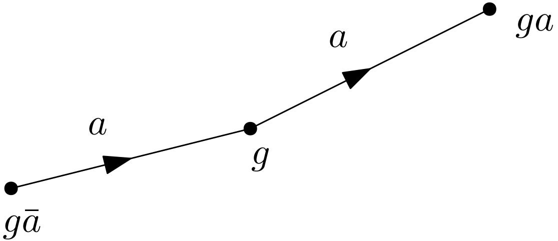
For the presentation

$$P = \langle a, b, c, d, p, q, r, t, k \mid p^{10}a = ap, p^{10}b = bp, p^{10}c = cp, \\ p^{10}d = dp, p^{10}e = ep, aq^{10} = qa, bq^{10} = qb, cq^{10} = qc, \\ dq^{10} = qd, eq^{10} = qe, pacqr = rpcaq, p^2adq^2r = rp^2daq^2, \\ p^3bcq^3r = rp^3cbq^3, p^4bdq^4r = rp^4dbq^4, p^5ceq^5r = rp^5ecaq^5, \\ p^6deq^6r = rp^6edbq^6, p^7cdcq^7r = rp^7cdceq^7, p^8ca^3q^8r = rp^8a^3q^8, \\ p^9da^3q^9r = rp^9a^3q^9, \bar{a}^3ta^3k = k\bar{a}^3ta^3, ra = ar, rb = br, rc = cr, \\ rd = dr, re = er, pt = tp, qt = tq\rangle$$

the problem of determining whether two words represent the same element of the group (*word decision problem*) is **unsolvable**.

Cayley graph of (a presentation of) G

It is a directed combinatorial graph, whose vertices are identified with the elements of G. Given any vertex g and any generator a, there is an edge labeled as a going from g to ga, and an edge also labeled as a going from $g\bar{a}$ to g.

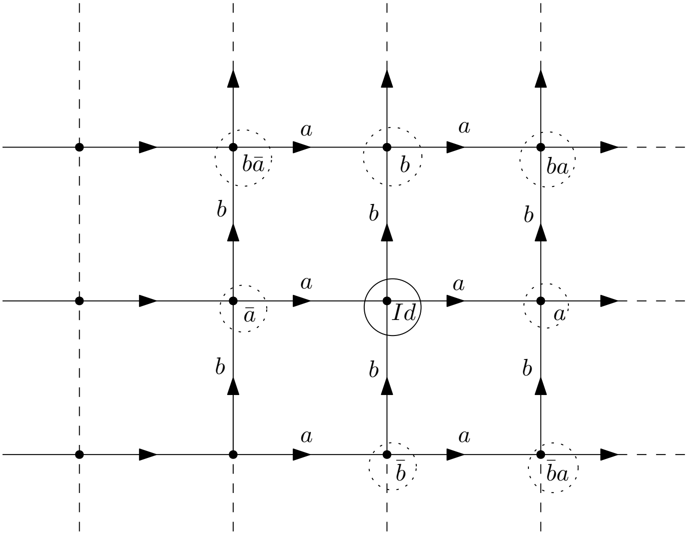


It's a regular graph since all vertices have the same degree, $2|X|$.

G acts on the Cayley graph by right product: words $=$ paths.

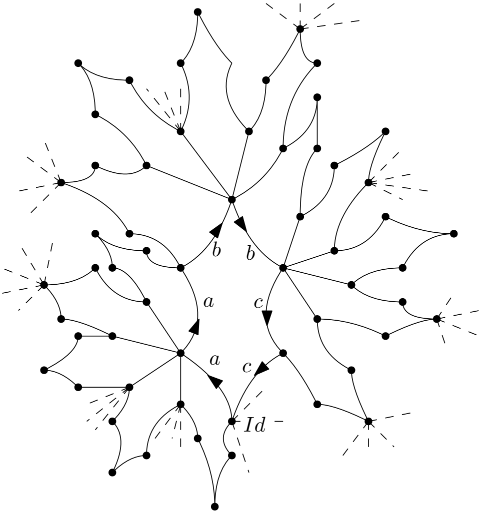
Cayley graph of (a presentation of) G

$\langle a, b \mid ab\bar{a}\bar{b} \rangle$ Classical presentation for the fundamental group of a torus



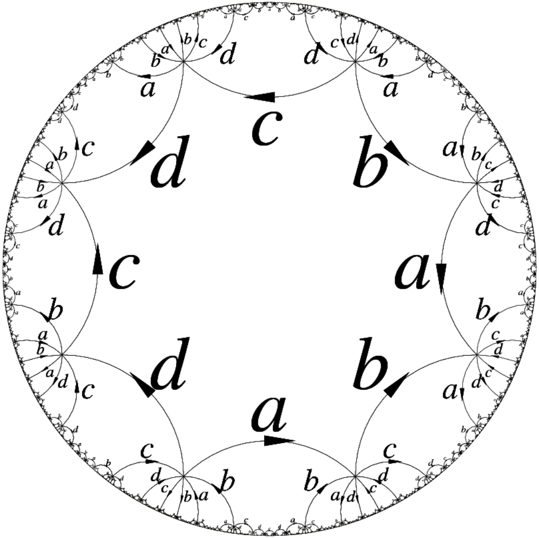
Cayley graph of (a presentation of) G

$\langle a, b, c \mid a^2 b^2 c^2 \rangle$ Classical presentation of a nonorientable surface of rank 3



Cayley graph of (a presentation of) G

$\langle a, b, c, d \mid ab\bar{a}\bar{b}cd\bar{c}\bar{d} \rangle$ Classical presentation for the fundamental group of a double torus



Volume entropy

Let G be a finitely generated group and let $P = \langle X \mid R \rangle$ be a presentation of G.

$$\sigma_m := \mathrm{Card}\{g \in G : \ \mathrm{length}_{\mathrm{P}}(g) = m\},$$

is the number of vertices at distance m from the identity in the Cayley graph.

Its exponential growth rate is called the *volume entropy*, defined as

$$\boxed{h_{\mathrm{vol}}(G, P) = \lim_{m \to \infty} \frac{1}{m} \log(\sigma_m).}$$

It is not a group invariant: it depends on the presentation.

An example: the free group $G = \langle a_1, a_2, \ldots, a_N \mid \emptyset \rangle$ of rank N.

$$\begin{aligned} m = 1: &\quad a, b, \bar{a}, \bar{b} \longrightarrow \sigma_1 = 4 \\ m = 2: &\quad aa, ab, a\bar{b}, \quad ba, bb, b\bar{a}, \quad \bar{a}b, \bar{a}\bar{a}, \bar{a}\bar{b}, \quad \bar{b}a, \bar{b}\bar{a}, \bar{b}\bar{b} \longrightarrow \sigma_2 = 12 \\ &\vdots \\ &\vdots \end{aligned}$$

$$\sigma_m = 2N(2N-1)^{m-1}$$

$$h_{\mathrm{vol}}(G, P) = \lim_{m\to\infty} \frac{1}{m} \log(\sigma_m) = \boxed{\log(2N-1)}$$

The context

We will consider **geometric presentations** of fundamental groups of (orientable and non-orientable) surfaces **of rank** $N \geq 3$. Equivalently, of negative Euler characteristic. Equivalently (for orientable surfaces), of genus $g \geq 2$.

A presentation is called **geometric** if the associated Cayley graph is **planar**.

All previously shown presentations were geometric.

$\langle a, b, c, d \mid \bar{d}acdb, c\bar{d}ad\bar{b}\rangle$ is a non-geometric presentation for the double torus group.

The context: hyperbolicity

We note that the considered surfaces (rank $N \geq 3$) are **hyperbolic** in the geometrical sense: they can be endowed with a hyperbolic metric (each point has an open neighbourhood isometric to the hyperbolic plane).

The corresponding fundamental groups are **hyperbolic** in the geometric group theory sense [Gromov, 1980]: for the associated Cayley graph, there is a constant δ such that every geodesic triangle is δ-*thin*.

The family of all hyperbolic groups has some nice properties. For instance, the word decision problem is solvable.

Geometric presentations

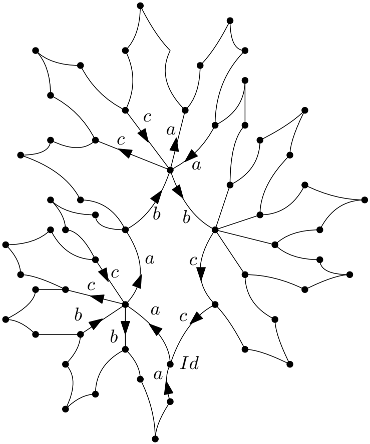
Lemma 1

Let $P = \langle X | R \rangle = \langle x_1, x_2, \dots, x_N \mid R_1, R_2, \dots, R_k \rangle$ be a geometric presentation of a surface group G. Then,

- (a) The set $\{x_1^{\pm 1}, \dots, x_N^{\pm 1}\}$ admits a cyclic ordering that is preserved by the G-action.
- (b) Each generator appears exactly twice (with plus or minus exponent) in the set R of relations.
- (c) Let a, b be a pair of adjacent generators according to the cyclic ordering given by (a). Then, there is exactly one relation R_i such that a cyclic shift of R_i contains either $b^{-1}a$ or $a^{-1}b$ as a sub-word.

The set $\{x_1^{\pm 1}, \ldots, x_N^{\pm 1}\}$ admits a cyclic ordering that is preserved by the G-action.

$$P = \langle a, b, c \mid a^2b^2c^2 \rangle$$



Cyclic ordering:

$$(a, \bar{a}, b, \bar{b}, c, \bar{c})$$

Geometricity test

The Lemma can be used to construct an algorithm that takes as input a presentation P and tests whether P is geometric.

$$P_1 = \langle a, b, c, d \mid adac, cbdb \rangle$$

$$P_2 = \langle a, b, c, d, e \mid abc, ce\bar{a}, b\bar{c}d^2 \rangle$$

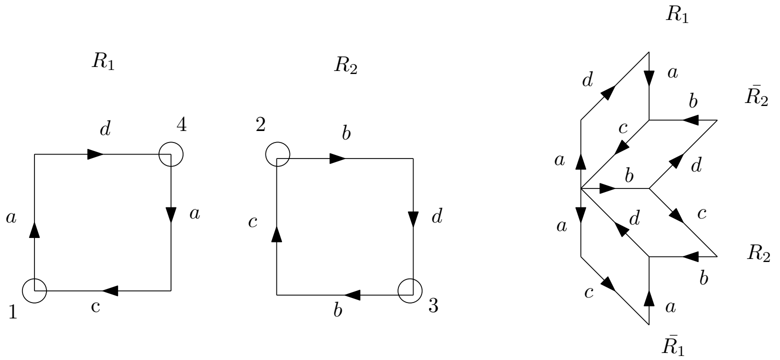
$$P_3 = \langle a, b, c, d \mid aba\bar{b}d, c^2d \rangle$$

P_2 is not geometric since it does not satisfy Lemma (b).

P_1, P_3 satisfy Lemma (b),

Geometricity test

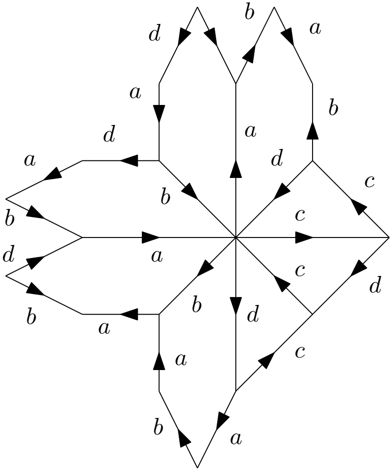
but $P_1 = \langle a, b, c, d \mid adac, cbdb \rangle$ does not satisfy (a):



The numbered circles with a number k indicate the angles used to attach the cell at step k of the algorithm. After 3 steps we cannot continue.

Geometricity test

$P_3 = \langle a, b, c, d \mid aba\bar{b}d, c^2d \rangle$



Round of 8 steps completed: P_3 is a geometric presentation.
Obtained cyclic ordering: $(a, \bar{d}, c, \bar{c}, d, b, \bar{a}, \bar{b})$.

Main goal

Construct an algorithm that takes as input a presentation P of a surface group, checks whether P is geometric and, in the affirmative, *computes the associated volume entropy*.

This problem is solved in a very general setting by manipulating word lengths uses the *cone approach*. However, it admits a beautiful dynamical approach that *relates the volume entropy with the topological entropy of an induced map at the infinity*. We believe that this dynamical approach gives insight on the word lengths growth mechanisms.

Main goal

The paper

LI. Alsedà, D. Juher, J. Los, F. Mañosas, *On families of Bowen-Series-like maps for surface groups*, Regul. Chaotic Dyn., **28(4-5)**, 659-667. 2023.

solves the problem of computing algorithmically the volume entropy of any geometric presentation of a surface group of rank $N \geq 3$ (hyperbolic groups).

The program is written in Maple and Maxima and is freely available to the scientific community.

Boundary ∂G of a hyperbolic group G [Gromov, 1980]

A *geodesic ray* is an infinite word in the alphabet $X \cup \bar{X}$ such that any finite subword is geodesic. Equivalently, an infinite unbounded path in the Cayley graph starting at *Id* such that every subsegment is geodesic.

The **boundary** ∂G of G is a topological, metric space. Any point in the boundary is an equivalence class of geodesic rays that remain at a uniform bounded distance from each others.

In our context, $\partial G = \mathbb{S}^1$.

Cylinders

The **cylinder** $\mathcal{C}_x$ for a generator $x \in X \cup \bar{X}$ is the subset of points $\zeta \in \partial G$ such that there exists a ray (infinite word) W converging to ζ and starting with x.

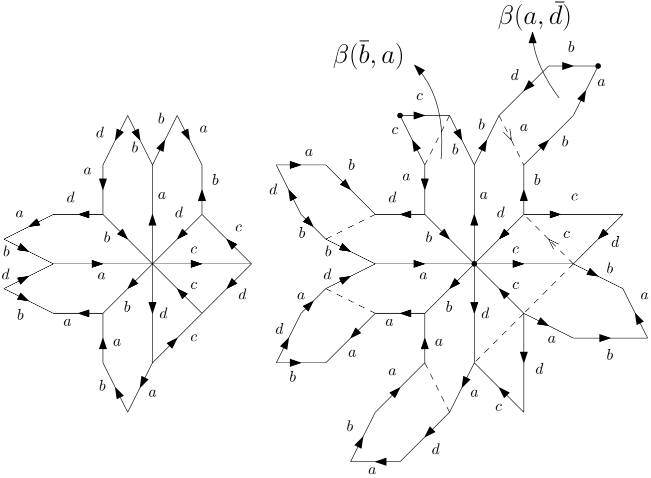
Lemma 2

The cylinders satisfy:

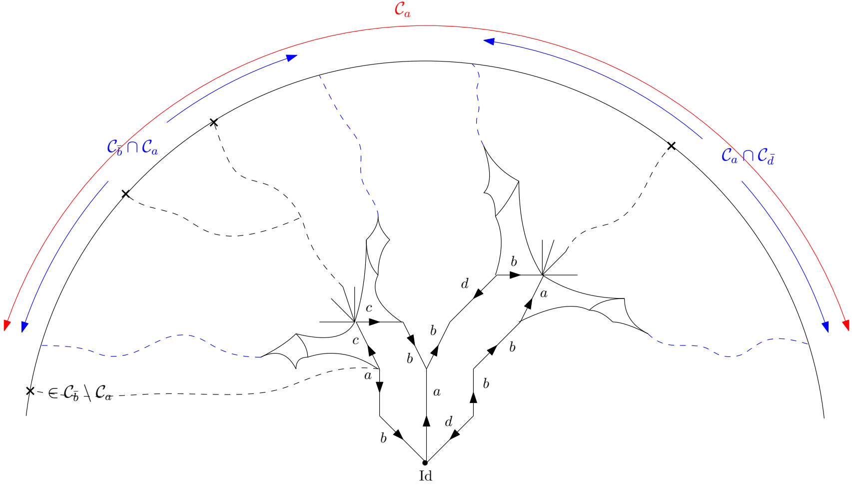
a. $\mathcal{C}_x$ is connected and $\mathcal{C}_x \cap \mathcal{C}_y \neq \emptyset$ if and only if x and y are adjacent generators in the cyclic ordering. In this case it is an interval.
b. For any $\theta \in \mathcal{C}_x \cap \mathcal{C}_y$, there is an infinite word W such that $\theta \in \partial G$ has two geodesic ray expressions $L_x W$ and $L_y W$, where $\{L_x, L_y\}$ are the two geodesic segments of the minimal bigon $\beta(x, y)$.

Minimal bigons

If we complete the cells adjacent to *Id* up to the closest vertices for which there is geodesic ambiguity, we get what we call the *minimal bigons*. It turns out that this is all we need to compute the volume entropy. $P_3 = \langle a, b, c, d \mid aba\bar{b}d, c^2 d \rangle$



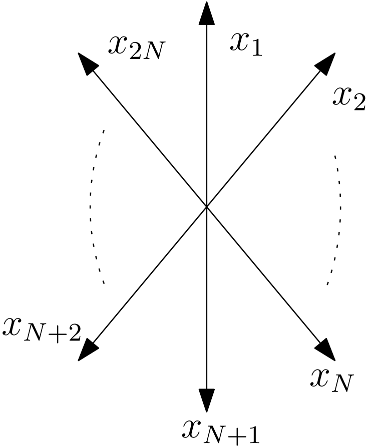
Cylinders



$\mathcal{C}_a$
$\mathcal{C}_{\bar{b}} \cap \mathcal{C}_a$
$\mathcal{C}_a \cap \mathcal{C}_{\bar{d}}$
$\in \mathcal{C}_{\bar{b}} \setminus \mathcal{C}_a$
c
c
b
b
a
a
d
b
a
b
b
d
b
Id

Notation

The elements of $X \cup \bar{X}$ will be denoted by $x_1, x_2, \ldots, x_{2N}$, where the indices are defined modulo $2N$, in such a way that x_j is adjacent to $x_{j\pm1}$ in the cyclic ordering given by Lemma 1(a).



Cutting points

By Lemma 2(b) there are $2N$ disjoint intervals

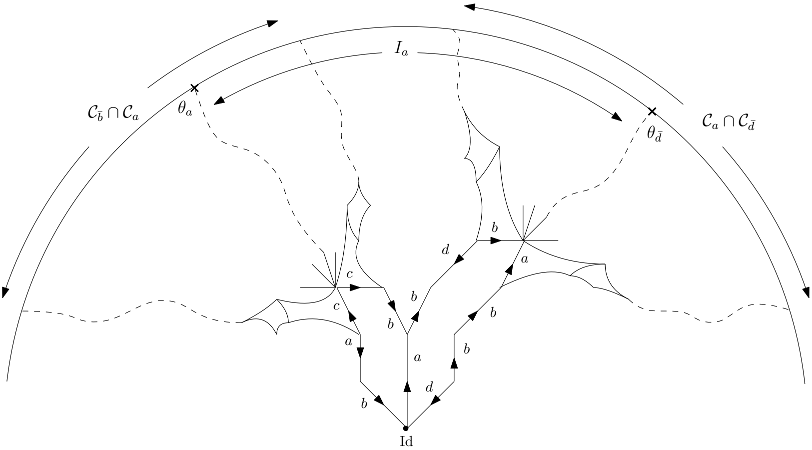
$$J_j := \mathcal{C}_{x_{j-1}} \cap \mathcal{C}_{x_j} \subset \mathbb{S}^1.$$

For each $\Theta := (\theta_1, \theta_2, \dots, \theta_{2N}) \in J_1 \times J_2 \times \dots \times J_{2N}$ we consider the finite partition of $\mathbb{S}^1$ given by the intervals

$$I_j := [\theta_j, \theta_{j+1}) \subset \mathbb{S}^1.$$

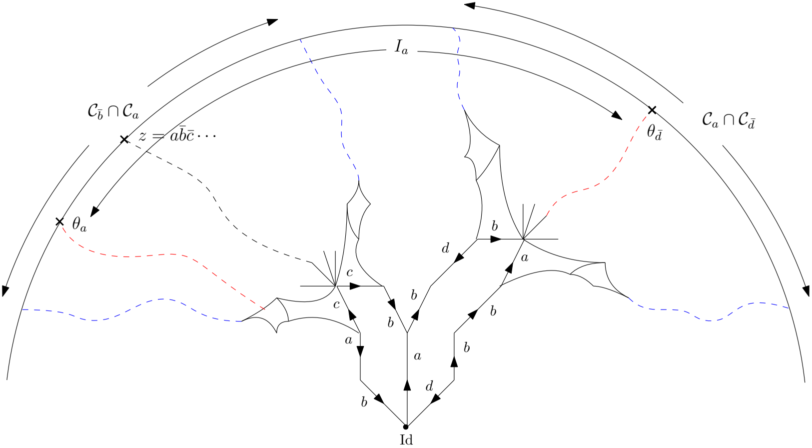
The points θ_j are called **cutting points** and Θ is called the **cutting parameter**.

Cutting points

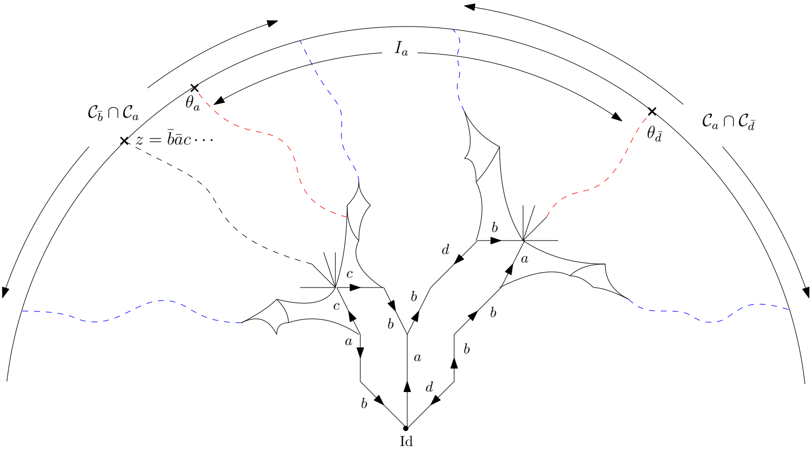


Cutting points and codings

When choosing a particular $\Theta = (\theta_1, \theta_2, \ldots, \theta_{2N})$, we are fixing the coding of any point $z \in \mathbb{S}^1$ as an infinite word in the alphabet $X \cup \bar{X}$.



Compare with



Bowen-Series-like maps

For each cutting parameter $\Theta := (\theta_1, \theta_2, \dots, \theta_{2N})$ we consider the map

$$\Phi_\Theta \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \text{ such that } \Phi_\Theta(z) = x_j^{-1}(z) \text{ if } z \in I_j.$$

Such a map is called a **Bowen-Series-like map**.

From the combinatorial point of view (points $=$ infinite words), we are simply deleting the first symbol: $\Phi_\Theta(x_j abcd \cdots) = abcd \cdots$
So, Φ_Θ is nothing but the standard **shift map**.

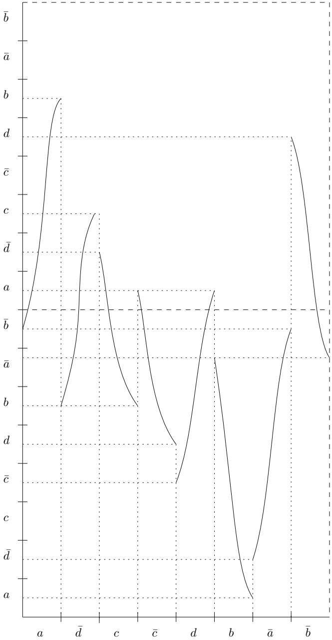
$$\text{Parameter } \Theta \Leftrightarrow \text{Partition } \mathbb{S}^1 = \bigcup_{j=1}^{2N} I_j$$
$$\Leftrightarrow \text{Fixed word for each } z \in \mathbb{S}^1 \Leftrightarrow \text{Shift map } \Phi_\Theta$$

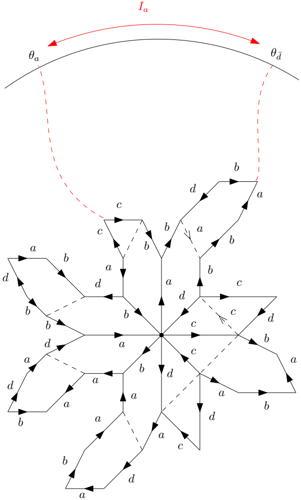
We have thus a family of maps Φ_Θ indexed by $\Theta = (\theta_1, \theta_2, \ldots, \theta_{2N})$, the cutting parameter.

Properties:

1. $\Phi_\Theta\big|_{I_j}$ is a homeomorphism onto its image.
2. At the cutting points the map is not continuous.

Φ_Θ is, thus, a *piecewise homeomorphism* of $\partial G = \mathbb{S}^1$.


$\bar{b}$
$\bar{a}$
b
d
$\bar{c}$
c
$\bar{d}$
a
$\bar{b}$
$\bar{a}$
b
d
$\bar{c}$
c
$\bar{d}$
a
a $\bar{d}$ c $\bar{c}$ d b $\bar{a}$ $\bar{b}$


I_a
θ_a
$\theta_{\bar{d}}$

Topological entropy

Defined for continuous (Adler, Konheim, McAndrew) and discontinuous (Bowen) self-maps maps of compact spaces. For piecewise continuous piecewise monotone maps $\Phi\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of the circle, it can be defined as follows (Misiurewicz, Ziemian).

Let $\mathbb{S}^1 = \bigcup_{j=1}^{2N} I_j$ be a partition of $\mathbb{S}^1$ by intervals such that Φ restricted to each I_j is a homeomorphism.

For $m \in \mathbb{N}$, the *itinerary intervals of level m* are defined as

$$I_{j_0,j_1,\dots,j_{m-1}} := I_{j_0} \cap \Phi^{-1}(I_{j_1}) \cap \dots \cap \Phi^{-(m-1)}(I_{j_{m-1}})$$

$$X_m := \text{ number of non-empty intervals } I_{j_0,j_1,\dots,j_{m-1}} \text{ of level } m.$$

$$\boxed{h_{\text{top}}(\Phi) = \lim_{m\to\infty} \frac{1}{m} \log(X_m)}$$

The main theorem

$$h_{\mathrm{vol}}(G, P) = \lim_{m\to\infty} \frac{1}{m} \log(\sigma_m)$$

$$h_{\mathrm{top}}(\Phi_\Theta) = \lim_{m\to\infty} \frac{1}{m} \log(X_m)$$

Proposition

The following inequalities are satisfied for each parameter Θ:

$$\sigma_m \leq X_m \leq m\sigma_m.$$

Main Theorem

Let G be a surface group of rank larger than 2 and let P be any geometric presentation of G. Then, for any cutting parameter Θ, $h_{top}(\Phi_\Theta) = h_{vol}(G, P) = \log(\lambda)$, where $1/\lambda$ is the smallest root in $(0, 1)$ of an integer polynomial $Q_P(t)$ that can be explicitly computed from P.

Comments

The entropy stability property inside the family of Bowen-Series-like maps Φ_Θ is remarkable, since the dynamics of two different maps in the family are quite different, in particular they are not pairwise topologically conjugate or even semi-conjugate. For some choices of the parameters Θ the map Φ_Θ is Markov, unlike for other choices.

The theorem states that the volume entropy of the group presentation P can be computed as the inverse of a real root of an integer polynomial that can be **algorithmically** obtained from P, by using the Milnor-Thurston theory of kneading invariants.

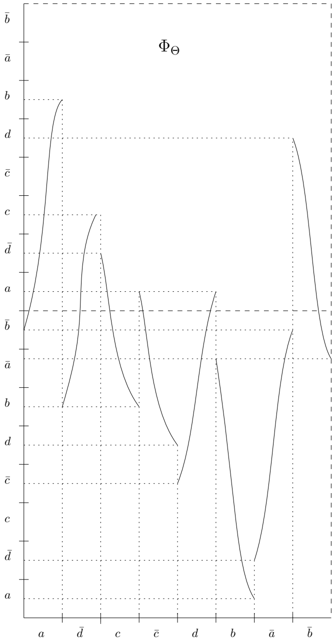
The theory was originally stated for continuous piecewise monotone maps f of the interval. It states that the entropy of f can be computed knowing the itineraries of the **turning points** (points separating maximal intervals of monotonicity of f).

It can be adapted (Alsedà, Mañosas) to our context (piecewise continuous, piecewise monotone maps Φ_Θ of the circle) by considering the interval map $\widehat{\Phi}_\Theta \colon [0,1] \longrightarrow [0,1]$ defined as

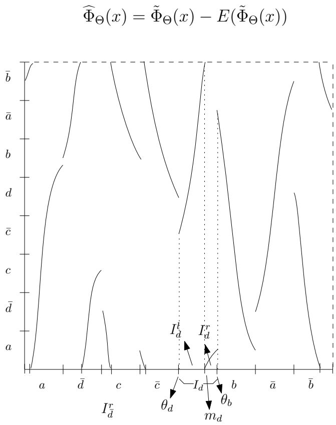
$$\widehat{\Phi}_\Theta(x) = \tilde{\Phi}_\Theta(x) - E(\tilde{\Phi}_\Theta(x)),$$

where $\tilde{\Phi}_\Theta$ is the lifting of Φ_Θ and $E(y)$ is the integer part of y.

It is necessary to consider the discontinuity points as turning points.



Φ_Θ
$\bar b$
$\bar a$
b
d
$\bar c$
c
$\bar d$
a
$\bar b$
$\bar a$
b
d
$\bar c$
c
$\bar d$
a
a $\bar d$ c $\bar c$ d b $\bar a$ $\bar b$



$\widehat{\Phi}_\Theta(x) = \tilde{\Phi}_\Theta(x) - E(\tilde{\Phi}_\Theta(x))$
$\bar b$
$\bar a$
b
d
$\bar c$
c
$\bar d$
a
I_d^l I_d^r
a $\bar d$ c $\bar c$ I_d b $\bar a$ $\bar b$
$I_{\bar d}^r$
θ_d
m_d
θ_b

Set of turning points:

$$\theta_a < m_a < \theta_{\bar{d}} < m_{\bar{d}} < \theta_c < m_c < \theta_{\bar{c}} < m_{\bar{c}} < \theta_d < m_d < \theta_b < \theta_{\bar{a}} < \theta_{\bar{b}} < m_{\bar{b}}.$$

The number and ordering of the intervals in the partition is independent of the particular choice of the cutting points θ_{x_i}.

Now we must find the dynamical itinerary of each turning point from the left and from the right. So, now we need to **precise** the map Θ_Φ. In other words, we need to **choose the cutting points** $\Theta = (\theta_1, \theta_2, \ldots, \theta_{2N})$. Recall that **any** choice leads to the same entropy! So, we are free.

Milnor-Thurston kneading invariants

The cutting points θ_i have geodesic ambiguity

$$\theta_i = L \cdots = R \cdots$$

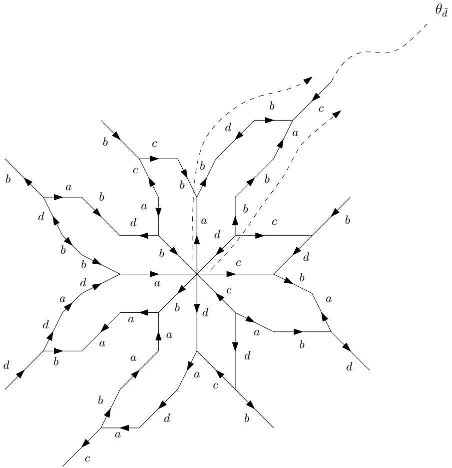
up to the top vertex v of the bigon $\beta(x_{i-1}, x_i) = \{L, R\}$.

Choice: we choose the cutting point θ_i in such a way that there is no geodesic ambiguity from v:

$$\theta_i = LW = RW$$

for a unique infinite word W. Equivalently, the word W corresponds to a point that does not belong to the intersection of cylinders. In particular, is not a cutting point.

Milnor-Thurston kneading invariants



$$\theta_{\bar{d}(+)} \in I^{l}_{\bar{d}},\ \Phi(\theta_{\bar{d}})_{(+)} \in I_b,\ \Phi^2(\theta_{\bar{d}})_{(+)} \in I_b,\ \Phi^3(\theta_{\bar{d}})_{(+)} \in I^{r}_{a}.$$
$$\theta_{\bar{d}(-)} \in I^{r}_{a},\ \Phi(\theta_{\bar{d}})_{(-)} \in I_b,\ \Phi^2(\theta_{\bar{d}})_{(-)} \in I^{l}_{\bar{d}}, \Phi^3(\theta_{\bar{d}})_{(-)} \in I_b$$

$$\theta_{\bar{d}(+)} \in I^l_{\bar{d}},\ \Phi(\theta_{\bar{d}})_{(+)} \in I_b,\ \Phi^2(\theta_{\bar{d}})_{(+)} \in I_b,\ \Phi^3(\theta_{\bar{d}})_{(+)} \in I^r_a.$$
$$\theta_{\bar{d}(-)} \in I^r_a,\ \Phi(\theta_{\bar{d}})_{(-)} \in I_b,\ \Phi^2(\theta_{\bar{d}})_{(-)} \in I^l_{\bar{d}}, \Phi^3(\theta_{\bar{d}})_{(-)} \in I_b$$

Now we consider the formal symbols

$$\omega_0(\theta^+_{\bar{d}}) = I^l_{\bar{d}},\ \omega_1(\theta^+_{\bar{d}}) = I_b,\ \omega_2(\theta^+_{\bar{d}}) = -I_b,\ \omega_3(\theta^+_{\bar{d}}) = I^r_a,$$

$$\omega_0(\theta^-_{\bar{d}}) = I^r_a,\ \omega_1(\theta^-_{\bar{d}}) = I_b,\ \omega_2(\theta^-_{\bar{d}}) = -I^l_{\bar{d}},\ \omega_3(\theta^-_{\bar{d}}) = -I_b,$$

where the signs $+/-$ correspond to the increasing/decreasing character of the corresponding iterate of the map.

Milnor-Thurston kneading invariants

Finally we construct the *jump series* for $\theta_{\bar{d}}$, a formal power series in the alphabet of the intervals $\{I_a^l, I_a^r, \ldots\}$:

$$\nu_j(\theta_{\bar{d}}) = \Omega_{v_j}(t) = \sum_{i=0}^{\infty} \left(\omega_i(\theta_{\bar{d}}^+) - \omega_i(\theta_{\bar{d}}^-) \right) t^i.$$

By the choice of the cutting point $\theta_{\bar{d}}$, the jump series vanishes beyond the length of the minimal bigon. So, it reduces to a polynomial:

$$\nu_{\theta_d}(t) = (I_d^l - I_a^r) + (-I_b + I_d^l)t^2 + (I_a^r + I_b)t^3$$

List of kneading invariants (I_x, I_x^l, I_x^r replaced by x, x_l, x_r):

$$
\begin{aligned}
\nu_{\theta_a}(t) &= (a_l - \bar{b}_r) + (\bar{b}_l + \bar{a})t + (-\bar{c}_r + c_r)t^2 \\
\nu_{m_a}(t) &= (a_r - a_l) + t\nu_{\theta_a}(t) \\
\nu_{\theta_{\bar{d}}}(t) &= (\bar{d}_l - a_r) + (-b + \bar{d}_l)t^2 + (a_r + b)t^3 \\
\nu_{m_{\bar{d}}}(t) &= (\bar{d}_r - \bar{d}_l) + t\nu_{\theta_a}(t) \\
\nu_{\theta_c}(t) &= (c_l - \bar{d}_r) + (-\bar{d}_l - c_r)t \\
\nu_{m_c}(t) &= (c_r - c_l) + t\nu_{\theta_a}(t) \\
\nu_{\theta_{\bar{c}}}(t) &= (\bar{c}_l - c_r) + (-a_r + b)t + (-b - \bar{a})t^2 \\
\nu_{m_{\bar{c}}}(t) &= (\bar{c}_r - \bar{c}_l) + t\nu_{\theta_a}(t) \\
\nu_{\theta_d}(t) &= (d_l - \bar{c}_r) + (\bar{c}_r + d_l)t \\
\nu_{m_d}(t) &= (d_r - d_l) + t\nu_{\theta_a}(t) \\
\nu_{\theta_b}(t) &= (b - d_r) + (-\bar{a} - a_r)t + (-\bar{a} - d_r)t^2 + (-\bar{b}_l - a_r)t^3 \\
\nu_{\theta_{\bar{a}}}(t) &= (\bar{a} - b) + (\bar{d}_l + a_r)t + (\bar{a} + a_l)t^2 + (\bar{d}_r + \bar{b}_l)t^3 \\
\nu_{\theta_{\bar{b}}}(t) &= (\bar{b}_l - \bar{a}) + (-d_l - \bar{b}_r)t + (-\bar{b}_r + \bar{b}_l)t^2 + (\bar{a} - d_l)t^3 \\
\nu_{m_{\bar{b}}}(t) &= (\bar{b}_r - \bar{b}_l) + t\nu_{\theta_a}(t)
\end{aligned}
$$

Finally, we formally write the above kneading invariants as a linear combination of the base

$$(a_l, a_r, \bar{d}_l, \bar{d}_r, c_l, c_r, \bar{c}_l, \bar{c}_r, d_l, d_r, b, \bar{a}, b_l, b_r)$$

and organize the coefficients of all invariants but the first one in matrix form, obtaining the following 13×14 *kneading matrix*:

Milnor-Thurston kneading invariants

$$\begin{pmatrix}
-1+t & 1 & 0 & 0 & 0 & t^3 & 0 & -t^3 & 0 & 0 & 0 & t^2 & t^2 & -t \\
0 & -1+t^3 & 1+t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2+t^3 & 0 & 0 & 0 \\
t & 0 & -1 & 1 & 0 & t^3 & 0 & -t^3 & 0 & 0 & 0 & t^2 & t^2 & -t \\
0 & 0 & -t & -1 & 1 & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & -1 & 1+t^3 & 0 & -t^3 & 0 & 0 & 0 & t^2 & t^2 & -t \\
0 & -t & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & t-t^2 & -t^2 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & t^3 & -1 & 1-t^3 & 0 & 0 & 0 & t^2 & t^2 & -t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+t & 1+t & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & t^3 & 0 & -t^3 & -1 & 1 & 0 & t^2 & t^2 & -t \\
0 & -t-t^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1-t^2 & 1 & -t-t^2 & -t^3 & 0 \\
t^2 & t & t & t^3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1+t^2 & t^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t-t^3 & 0 & 0 & -1+t^3 & 1+t^2 & -t-t^2 \\
t & 0 & 0 & 0 & 0 & t^3 & 0 & -t^3 & 0 & 0 & 0 & t^2 & -1+t^2 & 1-t
\end{pmatrix}$$

Milnor-Thurston kneading invariants

Now we delete any column (for instance, the first one) and compute the determinant D of the obtained 13×13 matrix. The only factor of D containing real roots in $[0,1)$ is

$$t^{10} - 3t^9 - 14t^8 - 13t^7 - 17t^6 - 12t^5 - 17t^4 - 13t^3 - 14t^2 - 3t + 1,$$

and the smallest root is $\lambda \approx 0.170554162$.

The volume entropy of the presentation P_3 is then

$$\boxed{\log(1/\lambda) \approx \log(5.86324007)}.$$

It all depends on the presentation

Analyzing carefully all steps, one realizes that all the information used (graph of the map, minimal bigons, itineraries, kneading invariants) depends, at the end, only on the presentation:

$$P_3 = \langle a, b, c, d \mid aba\bar{b}d, c^2d \rangle$$

Examples

Presentation (relations)	Program output	Polynomial
$[acded\bar{d}\bar{b}, \bar{e}\bar{c}\bar{b}\bar{a}]$	$\log(8.50591006)$	$t^4 - 7t^3 - 12t^2 - 7t + 1$
$[acde\bar{b}, \bar{d}\bar{e}\bar{c}\bar{b}\bar{a}]$	$\log(8.78515105)$	$t^4 - 8t^3 - 6t^2 - 8t + 1$
$[aba\bar{c}\bar{d}, ce^2, dbf^2]$	$\log(9.91984307)$	$t^{20} - 4t^{19} - 44t^{18} - 122t^{17}$ $-206t^{16} - 280t^{15} - 381t^{14} - 484t^{13}$ $-579t^{12} - 606t^{11} - 606t^{10} - 606t^9$ $-579t^8 - 484t^7 - 381t^6 - 280t^5$ $-206t^4 - 122t^3 - 44t^2 - 4t + 1$
$[aihlk\bar{c}a, \bar{c}e^2,$ $dbf^2k, g\bar{h}j^2, idgb\bar{l}]$	Non geometric	–
$[aia\bar{c}h, ce^2, dbf^2,$ $g\bar{h}j^2, idgb]$	$\log(17.9527833)$	$t^{20} - 13t^{19} - 80t^{18} - 149t^{17}$ $-187t^{16} - 196t^{15} - 252t^{14} - 348t^{13}$ $-370t^{12} - 426t^{11} - 312t^{10} - 426t^9$ $-370t^8 - 348t^7 - 252t^6 - 196t^5$ $-187t^4 - 149t^3 - 80t^2 - 13t + 1$

TABLE 1. Some outputs of the algorithm.