Non-constant periodic solutions of the Ricker model with periodic parameters

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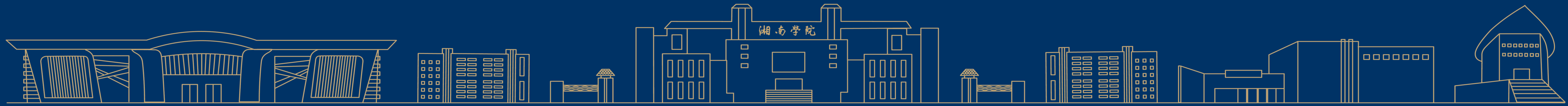
This is joint work with professors Xiaoping Wang, Fulai Chen

28th May 2025



Outlines

1. Backgrounds
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01

Backgrounds

Backgrounds

Consider the classic **discrete-time Ricker model**

$$u_{n+1} = u_n \exp\left[r\left(1 - \frac{u_n}{k}\right)\right], \qquad u_n > 0; n = 0,1,2,\ldots, \tag{1.1}$$

where u_n is the population size in generation n, r is the intrinsic growth rate, and k is the carrying capacity of the environment.

The discrete-time Ricker model exhibits complex and rich dynamics even with constant $\boldsymbol{r}$ and $\boldsymbol{k}$. Its typical dynamical feature is the periodic-doubling bifurcation to chaotic behavior.

Backgrounds

Table 1: Dynamics of a population described by the difference equation (1.1)

Dynamical behavior	Value of growth rate, r	Illustration
Globally stable equilibrium point	$2 > r > 0$	Fig. 1(a)
Stable two-point cycle	$2.526 > r > 2$	Fig. 1(b)
Stable four-point cycle	$2.656 > r > 2.526$	Fig. 1(c)
Stable cycle, period 8, giving way in turn to cycles of period 16, 32, etc. as r increases	$2.692 > r > 2.656$	
Chaos (cycles of arbitrary period, or aperiodic behavior, depending on initial condition)	$r > 2.692$	Fig. 1(d), (e), (f)

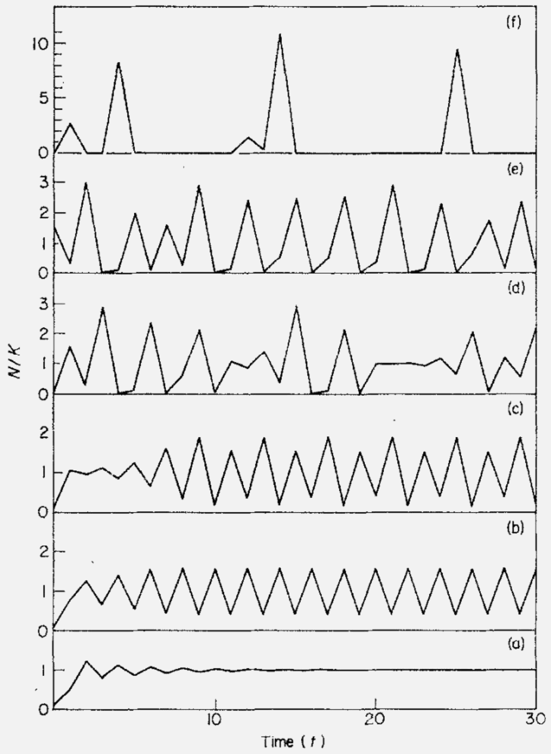

FIG. 1. Spectrum of dynamical behavior of the population density, N_t/K, as a function of time, t, as described by the difference equation (1) for various values of r. Specifically: (a) $r = 1{\cdot}8$, stable equilibrium point; (b) $r = 2{\cdot}3$, stable two-point cycle; (c) $r = 2{\cdot}6$, stable four-point cycle; (d), (e), (f) are in the chaotic regime, where the detailed character of the solution depends on the initial population value, with (d) $r = 3{\cdot}3$ ($N_0/K = 0{\cdot}075$) (e) $r = 3{\cdot}3$ ($N_0/K = 1{\cdot}5$), (f) $r = 5{\cdot}0$ ($N_0/K = 0{\cdot}02$).

- R. M. May, Biological populations obeying difference equations: stable points, stable cycles and chaos, J. Theo. Biol., 51 (1975), 511-524.

Backgrounds

The corresponding continuous model is

$$\frac{du(t)}{dt}=u(t)\left[r\left(1-\frac{u(t)}{k}\right)\right], \qquad t\geq 0 \tag{1.2}$$

whose dynamic behavior is very simple, that is to say, every positive solution of (1.2) tends to k as $t\to\infty$.

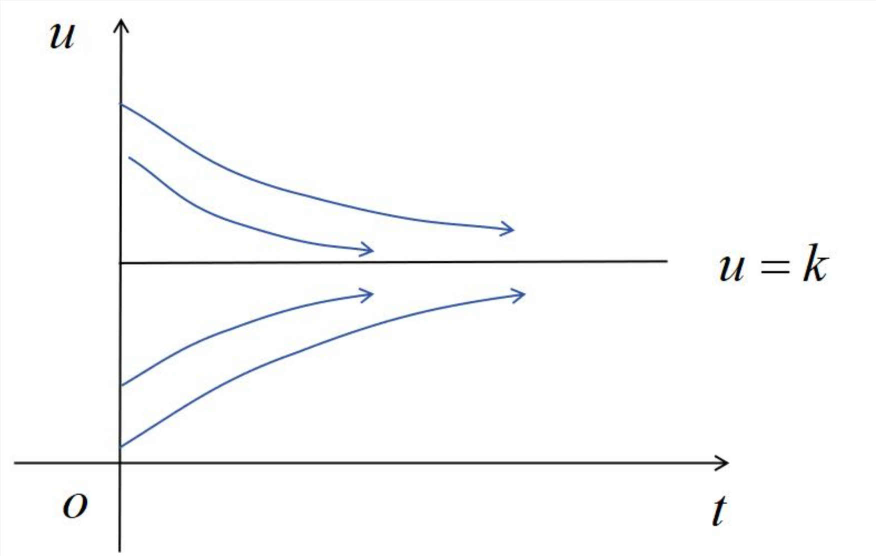

Figure 1.1

Backgrounds

In some circumstances, the intrinsic growth rate and/or the carrying capacity may vary in time, particularly periodically. For model (1.1), if both the intrinsic growth rate and the carrying capacity change periodically and simultaneously with period ω such that the model equation is

$$u_{n+1}=u_n \exp\left[r_n\left(1-\frac{u_n}{k_n}\right)\right], \qquad n=0,1,2\ldots, \tag{1.3}$$

where $r_{n+\omega}=r_n$ and $k_{n+\omega}=k_n$ have common period ω.

- S. Mohamad and K. Gopalsamy, Extreme stability and almost periodicity in a discrete logistic equation, Tohoku Math. J. 52, 107-125 (2000).
- Z. Zhou and X. Zou, Stable periodic solutions in a discrete periodic logistic equation, Appl.Math. Letters, 16(2003), 165-171.

Backgrounds

The differential equation counterpart of (1.3) give by

$$\frac{du(t)}{dt} = u(t)\left[r(t)\left(1-\frac{u(t)}{k(t)}\right)\right], \qquad t>0, \qquad (1.4)$$

r(t) and *K(t)* can be assumed to be nonconstant periodic functions with a common period ω to reflect the seasonal fluctuations. It has been shown that (1.4) has a positive ω-periodic solution $\tilde{x}(t)$ which attracts every positive solution $x(t)$ of (1.4) as t→∞.

- B.D. Coleman, Nonautonomous logistic equations as models of the adjustment of populations to environ_x0002_mental changes, Math. Biosei. 45, 159-173 (1979).
- B.D. Coleman, Y.H. Hsieh and G.P. Knowles, On the optimal choice of r for a population in a periodic environment, Math. Biosci. 46, 71-85 (1979).
- K. Gopalsamy and X.Z. He, Dynamics of an almost periodic logistic integrodifferential equation, Methods Appl. Anal. 2, 38-66 (1995).

Backgrounds

S. Mohamad et al considered equation (1.3), and obtained the following two main theorems.

Theorem A: Assume that $\{r(n)\}$ and $\{K(n)\}$ satisfy

$$0<r_*\le r(n)\le r^*, 0<K_*\le K(n)\le K^*, \quad n\in N$$

where r_*, r^*, K_*, and K^* are positive constants .Then (1.3) is extremely stable in the sense that

$$\lim_{n\to\infty}|x(n)-y(n)|=0,$$

for any two solutions $\{x(n)\}$ and $\{y(n)\}$ of (1.3).

Theorem B: Assume that $\{r(n)\}$ and $\{K(n)\}$ are almost periodic sequences satisfying $0<r_*\le r(n)\le r^*, 0<K_*\le K(n)\le K^*, \quad n\in N$ with $r^*<2$. Then (1.3) has a unique positive and globally asymptoticly stable almost periodic solution.

- S. Mohamad and K. Gopalsamy, Extreme stability and almost periodicity in a discrete logistic equation, Tohoku Math. J. 52, 107-125 (2000).

Backgrounds

Z.Zhou and X. Zou considered equation (1.3) with

$$r(3n)=1,\quad r(3n+1)=1.5,\quad r(3n+2)=1,$$
$$K(3n)=1,\quad K(3n+1)=5,\quad K(3n+2)=8,$$

for $n \in N$. Then (1.3) has a 3-periodic solution $\tilde{x}(n)$, where

$$\tilde{x}(3n)=3.2184,\quad \tilde{x}(3n+1)=0.3501,\quad \tilde{x}(3n+2)=1.4126,\quad for \quad n \in N,$$

and a 6-periodic solution $\{x^*(n)\}$ where

$$x^*(6n)=5.6940,\quad x^*(6n+1)=0.0521,\quad x^*(6n+2)=0.2299,$$
$$x^*(6n+3)=0.6072,\quad x^*(6n+4)=0.8993,\quad x^*(6n+5)=3.0774,\quad for \quad n \in N,$$

- Z. Zhou and X. Zou, Stable periodic solutions in a discrete periodic logistic equation, Appl.Math. Letters, 16(2003), 165-171.

Backgrounds

Z.Zhou and X. Zou proved that (1.3) has an ω-periodic solution which is globally asymptotically stable under the condition

$$\frac{\max k_n}{\min k_n} \exp(\max r_n - 1) < 2.$$

- Z. Zhou and X. Zou, Stable periodic solutions in a discrete periodic logistic equation, Appl.Math. Letters, 16(2003), 165-171.

Backgrounds

On the other hand, if the environment is relatively steady, but the individuals of some species grow periodically with period ω then $\{r_n\}_{n=0}^{\infty}$ is a positive ω-periodic sequence with $r_{n+\omega} = r_n$, and model (1.3) becomes

$$x_{n+1} = x_n \exp[r_n(1 - x_n)], \quad n = 0, 1, 2 \dots . \tag{1.5}$$

the equation (1.5) always has two constant equilibria 0 and 1. The origin 0 is always unstable.

Q.Q.Zhang et al proved that if

$$r_n \leq 2, \quad n = 0, 1, 2, \dots , \tag{1.6}$$

every positive solution of (1.5) goes to 1 as $n \to \infty$.

- Q. Q. Zhang and Z. Zhou, Global attractivity of a nonautonomous discrete logistic model, Hokkaido Math. J., 29(2000), 37-44.

Backgrounds

Some further questions:

a) Then what happens if (1.6) does not hold?

b) Does model (1.5) have non-constant periodic solutions?

c) If (1.5) have non-constant periodic solutions, how many non-constant periodic solutions does it have?

02

Main results

Main results

Existence of non-constant ω-periodic solutions

Let

$$f_n(x) = xe^{r_n(1-x)}, \quad n = 0, 1, 2, \ldots.$$

Then equation (1.5) becomes

$$x_{n+1} = f_n(x_n), \quad n = 0, 1, 2, \ldots,$$

which yields

$$x_{n+1} = f_n(f_{n-1}(\ldots f_0(x_0))), \quad n = 0, 1, 2, \ldots.$$

Set

$$f(x) = f_{\omega-1}(f_{\omega-2}(\ldots f_0(x))), \quad x \geq 0.$$

Main results

We have

$$f'(x) = \frac{f(x)}{x}[1 - r_{\omega-1} f_{\omega-2}(\dots f_0(x))][1 - r_{\omega-2} f_{\omega-3}(\dots f_0(x))] \dots (1 - r_1 f_0(x))(1 - r_0 x).$$

It is not difficult to obtained

$$f'(0) = e^{r_0 + r_1 + \cdots r_{\omega-1}} > 1$$

Hence, the origin 0 is always unstable.

Since $f_n(1) = 1, \quad n = 0, 1, 2, \dots, \omega - 1$ we have

$$f'(1) = (1 - r_0)(1 - r_1) \dots (1 - r_{\omega-1}).$$

Main results

(1) Since *f(x)* is a continuously differentiable function for $x \in [0,\infty)$ we find that $f'(0) > 1$、$f'(1) > 1$, and hence there exists a small positive number $\delta < \frac{1}{4}$, such that

$$f(x) > x, \quad \text{for} \quad x \in (0, \delta] \quad \text{and} \quad f(x) < x \quad \text{for} \quad x \in [1-\delta, 1)$$

which implies there exists an $x^* \in (\delta, 1-\delta)$ such that $f(x^*) = x^*$.

In fact, it is easy to see that $f_0(x) \in [0, \frac{1}{r_0} e^{r_0 - 1}]$,and so $f_1(f_0(x))$ is bounded for $x \in (1,\infty)$. By induction, we can prove that *f(x)* is bounded for $x \in (1,\infty)$.

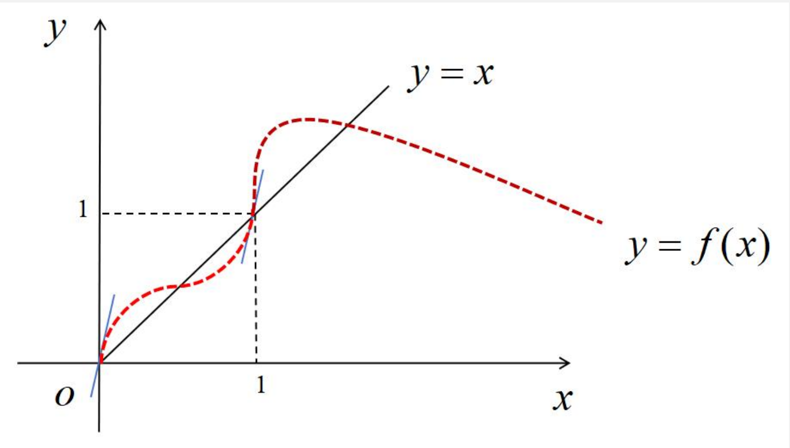

Figure 2.1

Main results

(2) Set

$$F(x) = f_{2\omega-1}(f_{2\omega-2}(\dots f_\omega(f(x)))) = f^2(x)$$

It suffices to show that *F(x)* has at least two positive fixed points except 1.

Similar calculation , we have

$$F'(0) = e^{2(r_0+r_1+\dots+r_{\omega-1})}$$

and

$$F'(1) = (1-r_0)^2(1-r_1)^2\dots(1-r_{\omega-1})^2.$$

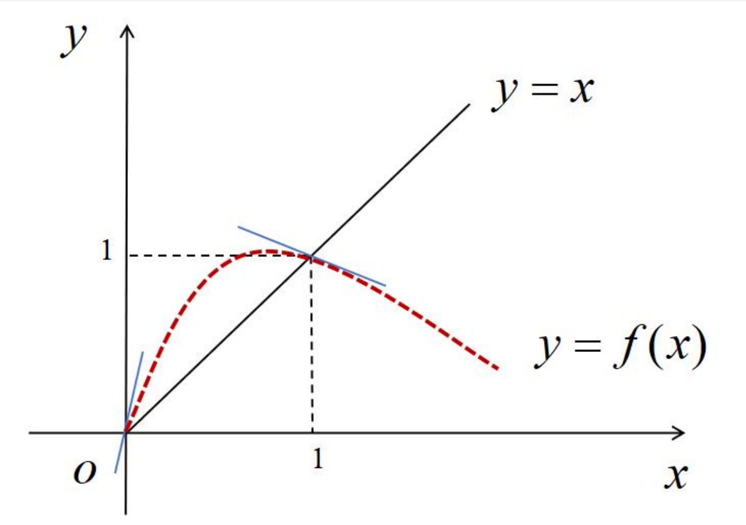

Figure 2.2

Main results

Theorem 2.1. Assume that

$$\prod_{i=0}^{\omega-1}|1-r_i|>1$$

Then the following two statements are true.

(1).Equation (1.5) has at least two non-constant ω-periodic solutions if $\prod_{i=0}^{\omega-1}(1-r_i)>1$.

(2).Equation (1.5) has at least two non-constant 2ω-periodic solutions if $\prod_{i=0}^{\omega-1}(1-r_i)<-1$.

Main results

Corollary 2.1. Assume $r_i \geq 2, i = 0,1,\ldots,\omega-1$ and $r_0 + r_1 + \cdots + r_{\omega-1} > 2\omega$. Then the following two statements are true.

(1). (1.5) has at least two non-constant ω-periodic solutions if ω is a positive even number.

(2). (1.5) has at least two non-constant 2ω-periodic solutions if ω is a positive odd number

Main results

2-periodic solutions

Clearly, the existence problem of a 2-periodic solution of (1.5) is equivalent to the existence problem of a positive fixed point except 1 of the function

$$f(x) = f_1(f_0(x)).$$

Let $y = f_0(x)$. To seek one non-constant 2-periodic solution of equation (1.5), we just need to find one positive solution except (1, 1) of the following planar system

$$\begin{cases} y = f_0(x) \\ x = f_1(y) \end{cases}$$

Main results

Which can be reduced to

$$\begin{cases} y = xe^{r_0(1-x)} \\ r_0 x + r_1 y = r_0 + r_1. \end{cases}$$

It follows that

$$H(x) = -r_0 x + r_0 + r_1 - r_1 x e^{r_0(1-x)} = 0.$$

Taking the derivative of *H(x)*, we obtain

$$H'(x) = -r_0 + r_1(r_0 x - 1)e^{r_0(1-x)}$$

Main results

Define

$$G(r_0, r_1) = H'\left(\frac{2}{r_0}\right) = -r_0 + r_1 e^{r_0 - 2}.$$

Then the parameter domain $M = \{(r_0, r_1) : r_0 > 0, r_1 > 0\}$ is divided into the following two sub-domains:

$$M^+ = \{(r_0, r_1) \in M : G(r_0, r_1) > 0\} \text{ and } M^- = \{(r_0, r_1) \in M : G(r_0, r_1) \leq 0\}.$$

It is easy to see that if $(r_0, r_1) \in M^-$, then *H(x)* is strictly decreasing for $x \in [0, \infty)$ which together with $H(0) = r_0 + r_1 > 0$, $H(1) = 0$ and $H(+\infty) = -\infty$ implies that 1 is a unique root of *H(x)* = 0 and so (1.5) has **no non-constant 2-periodic solutions.**

Main results

Similarly, define

$$Q(r_0, r_1) = -r_1 + r_0 e^{r_1 - 2}.$$

Then the domain M can be divided into

$$N^+ = \{(r_0, r_1) \in M : Q(r_0, r_1) > 0\} \text{ and } N^- = \{(r_0, r_1) \in M : Q(r_0, r_1) \leq 0\}.$$

We can easily show that if $(r_0, r_1) \in N^-$, then (1.5) **has no non-constant 2-periodic solutions** by switching the order of r_0 and r_1.

Main results

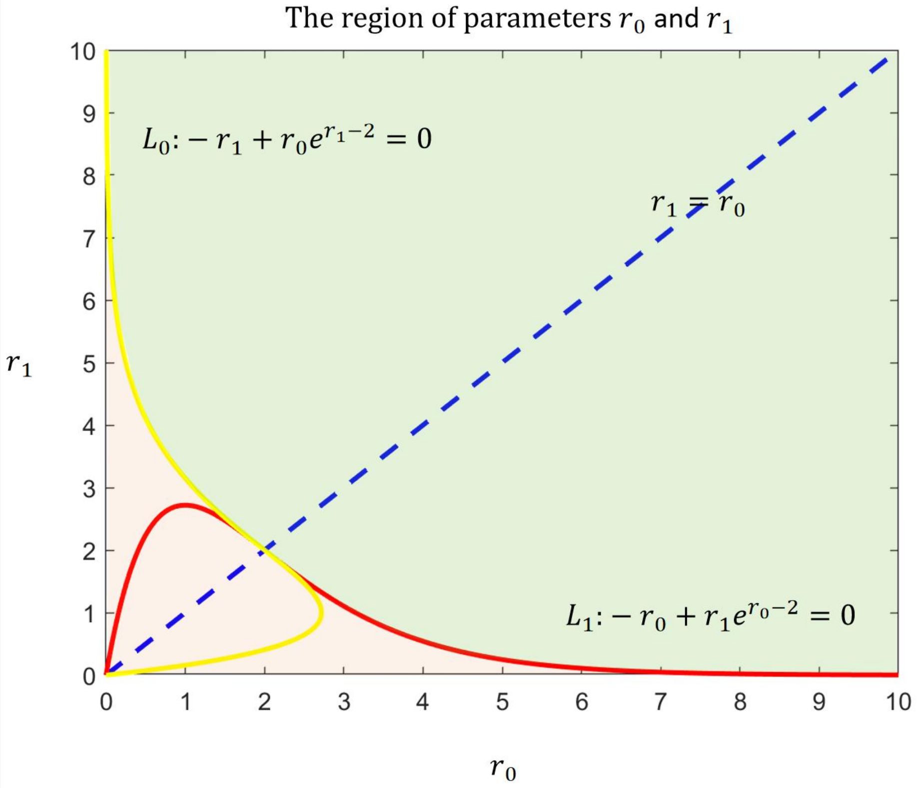

Figure 2.3

The curve L_1 (in red): $-r_0 + r_1 e^{r_0 - 2} = 0$

The curve L_2(in yellow): $-r_1 + r_0 e^{r_1 - 2} = 0$

The domain M into two sub-domains $M^+$and M^-, and N^+ and N^-, respectively.

Theorem 2.2. Assume that $(r_0, r_1) \in M^- \cup N^-$. Then (1.5) has no non-constant 2-periodic solutions.

Main results

For the case when $(r_0, r_1) \in M^+$.

It is easy to see that $H'(x) = 0$ has exactly two positive roots denoted by $h_i, i = 1,2$ which satisfy $0 < h_1 < \frac{2}{r_0} < h_2$.

Set

$$H_i = H(h_i) = -r_0 h_i + r_0 + r_1 - r_1 h_i e^{r_0(1-h_i)}, \quad i = 1, 2.$$

It is clear that $H_2 > H_1$. Therefore, there are only three possible cases to consider.

Case 1. $H_1 > 0$ **or** $H_2 < 0$.

Case 2. $H_1 = 0$ **or** $H_2 = 0$.

Case 3. $H_1 < 0$ **and** $H_2 > 0$.

Main results

For the case when $(r_0, r_1) \in N^+$.

It is easy to see that $F'(y) = 0$ has exactly two positive roots denoted by $f_i, i = 1,2$ which satisfy $0 < f_1 < \frac{2}{r_0} < f_2$. Set

$$F_i = F(f_i) = -r_1 f_i + r_1 + r_0 - r_0 f_i e^{r_1(1-f_i)}, \quad i = 1, 2.$$

It is clear that $F_2 > F_1$.Therefore, there are only three possible cases to consider.

Case 4. $F_1 > 0$ **or** $F_2 < 0$.

Case 5. $F_1 = 0$ **or** $F_2 = 0$.

Case 6. $F_1 < 0$ **and** $F_2 > 0$.

Main results

Theorem 2.3 Assume that $(r_0, r_1) \in M^+ \cup N^+$. Then the following three statements are true.

(1) If Case 1or Case 4 is true, then (1.5) has no non-constant 2-periodic solutions.

(2) If Case 2 or Case 5 is true, then (1.5) has a unique non-constant 2-periodic solution. And further more, $r_0 r_1 = r_0 + r_1$ means that the equilibrium 1 is semi-stable, and $r_0 r_1 \neq r_0 + r_1$ implies that the 2-periodic solution is semi-stable.

(3) If Case 3 or Case 6 is true, then (1.5) has exact two non-constant 2-periodic solutions.

03

Numerical simulations

Numerical simulations

Example 3.1. We set ω = 5. For the periodic sequence $\{r_i\}$ given by

$$r_0 = 0.5, \quad r_1 = 1.5, \quad r_2 = 5.01, \quad r_3 = 2.29, \quad r_4 = 2.21, \tag{3.1}$$

we have

$$\prod_{i=0}^{4}(1 - r_i) = 1.5648 > 1.$$

According to Theorem 2.1, equation (1.5) has at least two 5-periodic solutions.

For $\{r_i\}$ given by

$$r_0 = 2.12, \quad r_1 = 2.43, \quad r_2 = 2.72, \quad r_3 = 2.39, \quad r_4 = 2.21, \tag{3.2}$$

we have

$$\prod_{i=0}^{4}(1 - r_i) = -4.6332 < -1.$$

It follows from Theorem 2.1 that equation (1.5) has at least two 10-periodic solutions.

Numerical simulations

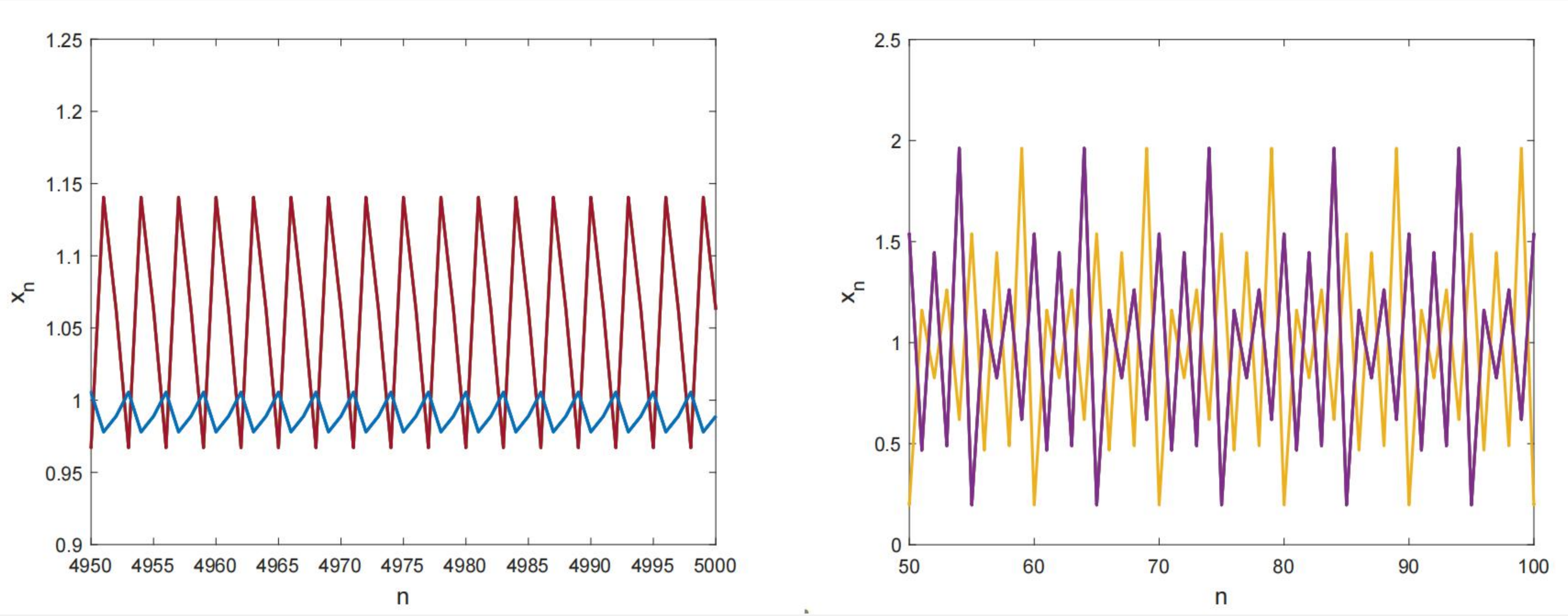

Figure 3.1: With $\omega = 5$, equation (1.5) is a 5-periodic equation. When the periodic sequence $\{r_i\}$ is given in (3.1), there exist two locally asymptotically stable 5-periodic solutions as shown in the left figure. When sequence $\{r_i\}$ is given in (3.2), it shows that there exist two locally asymptotically stable 10-periodic solutions in the right figure. The unique positive fixed point is unstable.

Numerical simulations

Example 3.2. Let

$$r_0 = r_1 = \cdots = r_{\omega-2} = 2, r_{\omega-1} = 2 + \epsilon, r_{n+\omega} = r_n,$$

for $n = 0, 1, \ldots,$ where ϵ is an arbitrary positive constant and ω is a positive integer. Then by Corollary 2.1, equation (1.5) has at least two ω-periodic non-constant periodic solutions if ω is even, and at least two 2ω-periodic non-constant periodic solutions if ω is odd.

Numerically, we consider two different cases. When $\omega = 4$ and

$$r_0 = 2, \quad r_1 = 2, \quad r_2 = 2, \quad r_3 = 2.5, \tag{3.3}$$

equation (1.5) has two 4-periodic equations.

When $\omega = 5$ and

$$r_0 = 2, \quad r_1 = 2, \quad r_2 = 2, \quad r_3 = 2, \quad r_4 = 2.5, \tag{3.4}$$

equation (1.5) has two 10-periodic equations.

Numerical simulations

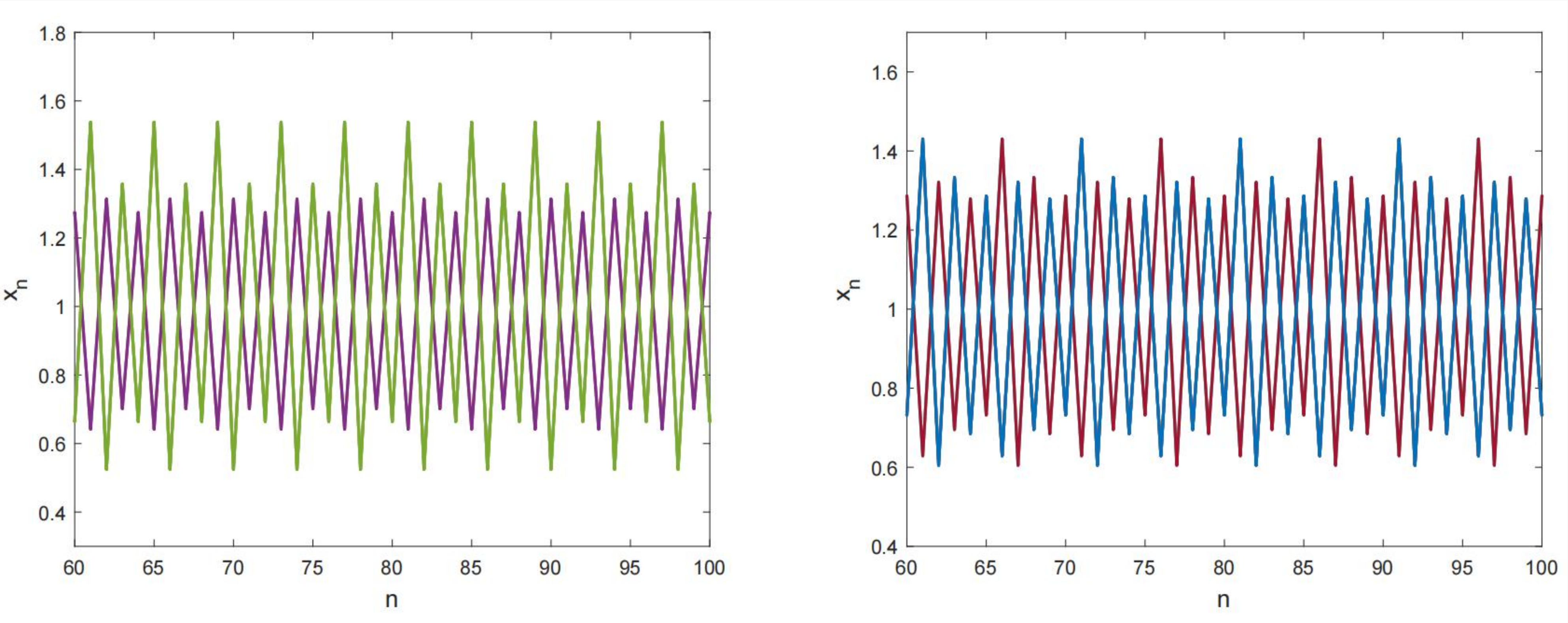

Figure 3.2: With ω = 4 and the periodic sequence $\{r_i\}$ is given in (3.3), equation (1.5) is a 4-periodic equation.The numerical result shows that there exist two a locally asymptotically stable 4-periodic solutions as in the left figure. With ω = 5 and sequence $\{r_i\}$ is given in (3.4), equation(1.5) is a 5-periodic equation. The numerical result shows that there exist two a locally asymptotically stable 10-periodic solutions as in the right figure.The unique positive fixed point is unstable.

Numerical simulations

Example 3.3. Let

$$\omega = 2, r_0 = 1, r_1 = r > 0, r_{n+2} = r_n, \quad \text{for} \quad n = 0, 1, \ldots.$$

then

$$H(x) = -x + 1 + r(1 - xe^{1-x}) \text{ and } H'(x) = -1 + r(x-1)e^{1-x}.$$

Since $f'(1) = (1-r_0)(1-r_1) = 0$,1 is locally stable.

When $r \le e$, we have $G(r_0, r_1) = -r_0 + r_1 e^{r_0 - 2} = -1 + re^{-1} \le 0$. By Theorem 2.3, equation (1.5) has no non-constant 2-periodic solutions when $r \le e$.

Now we assume $r > e$. Then $G(r_0, r_1) = -r_0 + r_1 e^{r_0-2} = -1 + re^{-1} > 0$. Let $h_1(r)$ and $h_2(r)$ be two positive roots of $H'(x) = 0$ with $h_1(r) < h_2(r)$. Since $H''(x) = r(2-x)e^{1-x}$,it is easy to see that $1 < h_1(r) < 2 < h_2(r)$, for $r > e$, and $h_1(e) = h_2(e) = 2$. By a direct calculation, we have

$$\frac{dh_i(r)}{dr} = \frac{1 - h_i(r)}{r(2 - h_i(r))}, i = 1, 2,$$

Numerical simulations

Which implies that $h_1(r) \in (1,2)$ is strictly decreasing for $r \in (e,\infty)$ and $\lim_{r\to\infty} h_1(r) = 1$, and that $h_2(r) \in (2,\infty)$ is strictly increasing for $r \in (e,\infty)$ and $\lim_{r\to\infty} h_2(r) = \infty$. By using the fact that $r = \frac{e^{h_i(r)-1}}{h_i(r)-1}$, for $i = 1,2,$ we have

$$H_i(r) = H(h_i(r)) = -h_i(r)+1+r(1-h_i(r)e^{1-h_i(r)}) = -h_i(r)+1+\frac{e^{h_i(r)-1}-h_i(r)}{h_i(r)-1}, \quad \text{for} \quad i = 1,2.$$

It is easy to have $H_i(e) = e-3 < 0, i = 1,2$, and $\lim_{r\to\infty} H_1(r) = 0$ and $\lim_{r\to\infty} H_2(r) = \infty$.. Therefore, there must be some $r^* > e$ such that $H_2(r^*) = 0$. Thus, we know that

$$H_1(r) \in (e-3,0), \quad \text{for} \quad r > e,$$

and

$$H_2(r) \in (e-3,0), \quad \text{for} \quad r \in (e,r^*) \quad \text{and} \quad H_2(r) \in (0,\infty) \quad \text{for} \quad r > r^*.$$

By Theorem 2.3 again, we see that $r \in (e,r^*)$ implies that equation (1.5) has no non-constant 2-periodic solutions, $r = r^*$ means that equation (1.5) has a unique non-constant 2-periodicsolution, and $r > r^*$ indicates that equation (1.5) has exactly two non-constant 2-periodic solutions.

Numerical simulations

Example 3.4. Let

$$\omega = 2, r_0 = 3, r_1 = \frac{3}{2}, r_{n+2} = r_n, \quad \text{for} \quad n = 0, 1, \ldots.$$

then

$$H(x) = -3x + \tfrac{9}{2} - \tfrac{3}{2}xe^{3(1-x)} \text{ and } H'(x) = -3 + \tfrac{3}{2}(3x-1)e^{3(1-x)}.$$

Since $r_0 r_1 = r_0 + r_1 = \frac{9}{2}$, we see that the positive equilibrium 1 is semi-stable, and equation (1.5) has a unique non constant 2-periodic solution.

Numerical simulations

Example 3.5. We let the periodic sequence r_i given by

$$r_0 = 2.12, \quad r_1 = 2.43, \quad r_2 = 0.72, \quad r_3 = 2.39, \quad r_4 = 2.21, \tag{3.5}$$

such that the following product is positive but

$$\prod_{i=0}^{4} (1 - r_i) = 0.7542 < 1.$$

Nevertheless, the unique positive fixed point $x^* = 1$ is locally asymptotically stable and, numerically, we have one 5-periodic solutions which is also locally asymptotically stable. Solutions approach either of them with different initial values.

Numerical simulations

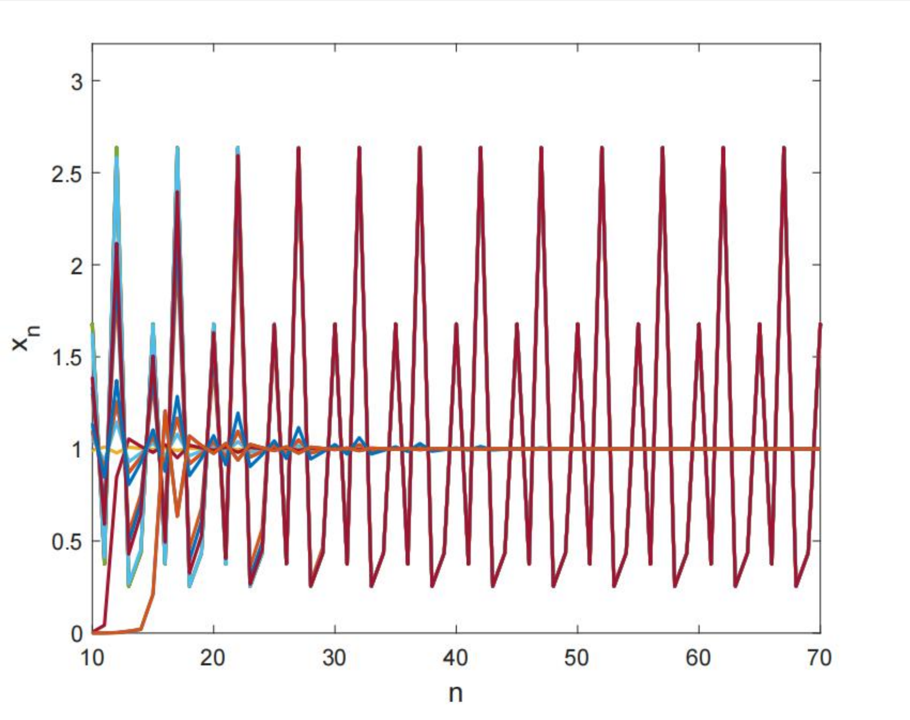

Figure3.3: With the periodic sequence r_i given in (3.5), the unique positive fixed point $x^* = 1$ is locally asymptotically stable and there is a 5-periodic solution which is also locally asymptotically stable . Solutions approach the fixed point or this 5-periodic solution depending on their initial values.

04

Further questions

Further questions

Question 1： Does equation (1.5) have more than two non-constant ω-periodic solutions if $\prod_{i=0}^{\omega-1}(1-r_i)>1$.

Question 2： Does equation (1.5) ave more than two non-constant 2ω-periodic solutions if $\prod_{i=0}^{\omega-1}(1-r_i)<-1$.

Thanks for your attention!