Connecting stationary fronts of Nagumo lattice differential equation with a functional equation

Progress on Difference Equations International Conference

Jakub Hesoun

University of West Bohemia
Departement of Mathematics

hesounj@kma.zcu.cz

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Overview

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   - From PDE to LDE
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2. Our results
   - Mirroring technique
   - Functional mirroring
   - Functional equation

Overview

Motivated by study of traveling wave solutions of LDE, we study the second-order difference equation

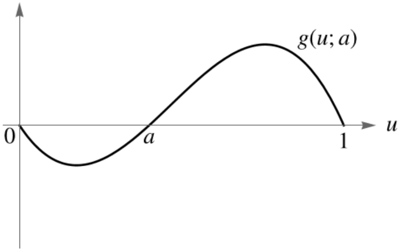
$$0 = D\left(u_{i-1} - 2u_i + u_{i+1}\right) + g(u_i; a), \quad i \in \mathbb{Z},$$

with the bistable nonlinearity g. For example

$$g(u; a) = u(1-u)(u-a), \quad a \in (0,1).$$

We seek a monotone solutions which satisfy limit boundary conditions

$$\lim_{i \to -\infty} u_i = 0, \quad \lim_{i \to +\infty} u_i = 1$$



From PDE to LDE

Nagumo PDE:

$$u_t = Du_{xx} + g(u;a), \quad u = u(x,t), \quad x \in \mathbb{R},\ t > 0.$$

Nagumo LDE:

$$u_i'(t) = D\left(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)\right) + g(u_i(t);a), \quad i \in \mathbb{Z},$$

with the bistable nonlinearity g.

Traveling wave solution

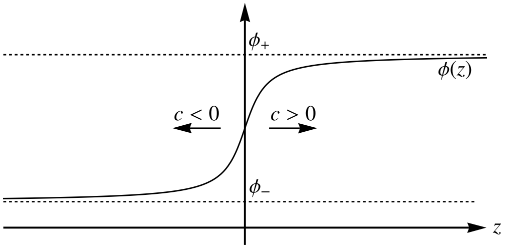
Let $u_{1,2} = \phi_\pm$ be two stationary solutions of the PDE (or LDE). If there exists a solution

$$u(x,t) = \phi(z), \quad (\text{ or } u_i(t) = \phi(z) \text{ }),$$

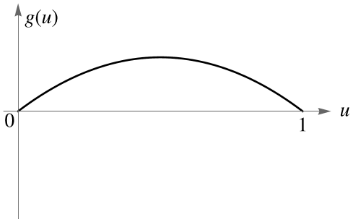
where $c \in \mathbb{R} \setminus \{0\}$ and $z = x - ct$ (or $z = i - ct$), which satisfies

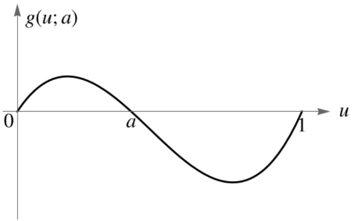
$$\lim_{z \to -\infty} \phi(z) = \phi_- \quad \text{and} \quad \lim_{z \to +\infty} \phi(z) = \phi_+,$$

then we call it a traveling wave solution.



Continuous x Discrete - monostable case





Continuous space

- Aronson, Weinberger(1975)

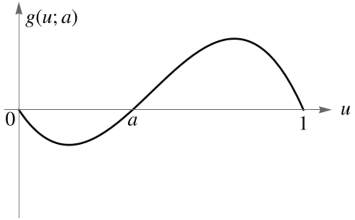
There are no heterogeneous stationary solutions.
A non-zero solution converges to stable stationary solution.
Analysis of phase portrait proves existence of a wave solution.

Discrete space

- Britton(1984), Bell(1984), Keener(1987)

Just discrete counterparts.
There are no heterogeneous stationary solutions.
A non-zero solution converges to stable stationary solution.

Continuous x Discrete - bistable case



Continuous space

- Aronson, Weinberger(1975)

If $\int_0^1 g(u)du \neq 0$ there are no heterogeneous stationary solutions.
A non-zero solution converges to stable solution.
Analysis of phase portrait proves existence of a wave solution.

Discrete space

- Bell(1984), Keener(1987)

Even if $\int_0^1 g(u)du \neq 0$, for $D \ll 1$ there exist a large number of stationary solutions (topological chaos), which are stable to ℓ^2 perturbation.

Stationary solutions of Nagumo LDE

Stationary solutions of LDE: $\{u_i\}_{i\in\mathbf{Z}}$

$$0 = D\left(u_{i-1} - 2u_i + u_{i+1}\right) + g(u_i; a), \quad i \in \mathbb{Z}.$$

Keener's approach

$$\begin{aligned} u_{i+1} &= 2u_i - v_i + \tfrac{g(u_i;a)}{D} \\ v_{i+1} &= u_i. \end{aligned}$$

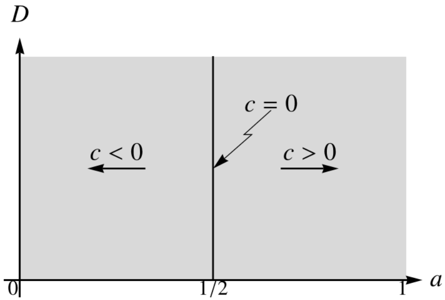
Morse's theorem (Smale's horseshoe theorem) leads to

Topological chaos

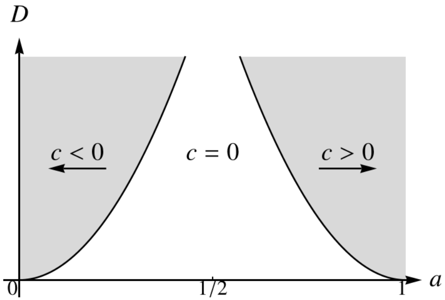
For every $a \in (0,1)$ there exists a value $\bar{D} > 0$ such that for all $D < \bar{D}$ and any double infinite sequence $\{s_i\}_{i\in\mathbb{Z}}$, where $s_i \in \{0,1\}$, there exists a stationary solution $\{u_i\}_{i\in\mathbb{Z}}$ which satisfies $u_i < a$ whenever $s_i = 0$, and $u_i > a$ whenever $s_i = 1$.

Continuous x Discrete - bistable case

PDE



LDE



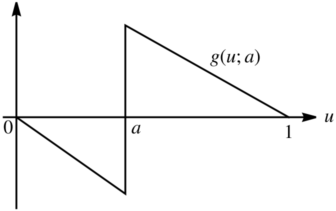
Piecewise linear reactions (caricatures)

McKean(1969)

The interest of Nagumo's problem lies in the hope that it is a reasonable caricature of the Hodgkin-Huxley model, but naturally it is permissible to caricature it still further, provided the essential features are not lost.

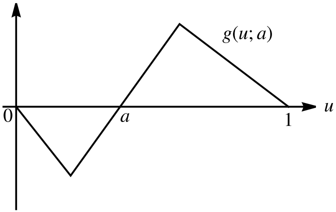
Fáth(1998), Moore(2014):

$$g(u;a) = \begin{cases} -u, & u < a, \\ 1-u, & u \geq a. \end{cases}$$



Elmer(2006), Humphires(2011), Moore(2016)

$$g(u;a) = \begin{cases} -u, & u \leq a/2, \\ u-a, & u \in (a/2, (a+1)/2), \\ 1-u, & u \geq (a+1)/2. \end{cases}$$



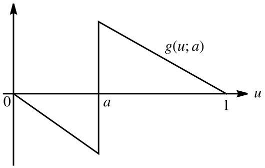
Monotone stationary solutions for piecewise linear reactions

Stationary solutions of LDE: $\{u_i\}_{i\in\mathbf{Z}}$

$$0 = D\left(u_{i-1} - 2u_i + u_{i+1}\right) + g(u_i; a), \quad i \in \mathbb{Z}.$$

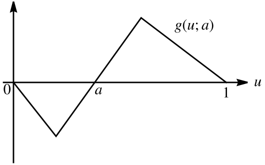
Monotone stationary wave solution for McKean's caricature

$$\begin{array}{rcll} 0 & = & D(u_{i-1} - 2u_i + u_{i+1}) - u_i, & i < i^*, \\ 0 & = & D(u_{i-1} - 2u_i + u_{i+1}) - u_i + 1, & i \geq i^*. \end{array}$$



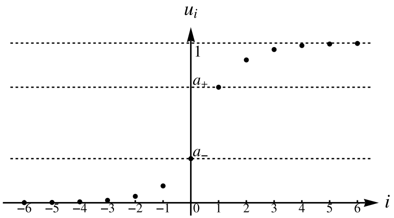
Analogously, for the sawtooth nonlinearity

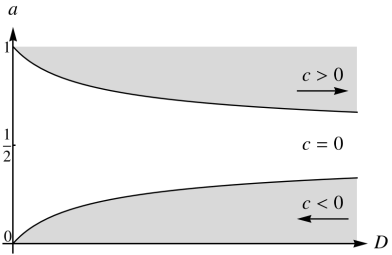
$$\begin{array}{rcll} 0 & = & D(u_{i-1} - 2u_i + u_{i+1}) - u_i, & i < i^*, \\ 0 & = & D(u_{i-1} - 2u_i + u_{i+1}) + u_i - a, & i \in \left[i^*, i^* + n\right], \\ 0 & = & D(u_{i-1} - 2u_i + u_{i+1}) - u_i + 1, & i > i^* + n. \end{array}$$



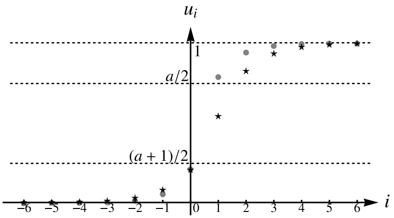
Results overview (visualization)

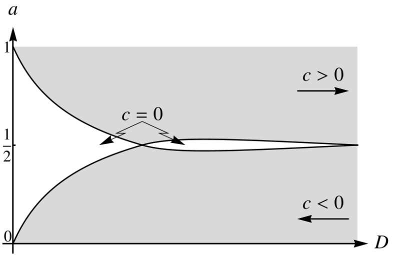
McKean's caricature





Sawtooth nonlinearity





Mirroring technique

Transformation of the second order difference equation

$$d(u_{i-1} - 2u_i + u_{i+1}) + g(u_i; a) = 0, \quad i \in \mathbb{Z}.$$

by

$$v_i = u_{2i}, \quad w_i = u_{2i+1}, \quad i \in \mathbb{Z},$$

to system

$$\begin{aligned} v_{i+1} =& 2w_i - v_i - \tfrac{1}{d} g(w_i; a), \\ w_{i+1} =& 2v_{i+1} - w_i - \tfrac{1}{d} g(v_{i+1}; a). \end{aligned}$$

Which can be rewrite

$$\begin{aligned} v_{i+1} - \varphi(w_i) =& \varphi(w_i) - v_i, \\ w_{i+1} - \varphi(v_{i+1}) =& \varphi(v_{i+1}) - w_i, \end{aligned}$$

by setting

$$\varphi(u) = \varphi(u; a, d) = u - \tfrac{1}{2d} g(u; a).$$

Mirroring technique (Unbounded solutions)

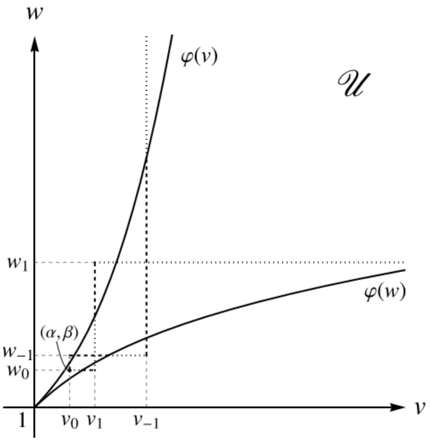
Forward mirroring scheme

$$\begin{aligned} v_{i+1} - \varphi(w_i) &= \varphi(w_i) - v_i, \\ w_{i+1} - \varphi(v_{i+1}) &= \varphi(v_{i+1}) - w_i, \end{aligned}$$

where

$$\varphi(u) = u - \frac{1}{2d} g(u; a).$$

$$\mathscr{U} = \Big\{ (v, w) \in \mathbb{R}^2 : \\ v > 1 \ \wedge \ \varphi^{-1}(v) \leq w \leq \varphi(v) \Big\}$$



Two sided unbounded solutions

Theorem

Every point $(\alpha, \beta) \in \mathscr{U}$ *determines an equivalence class* $\left[u^{\alpha,\beta}\right]$ *of stationary solutions* u *of Nagumo LDE with* g *given by cubic nonlinearity such that*

$$u_i > 1 \quad \text{for every} \quad i \in \mathbb{Z} \qquad \text{and} \qquad \lim_{i \to \pm\infty} u_i = +\infty$$

represented by a solution $u^{\alpha,\beta}$ *such that* $u_0^{\alpha,\beta} = \alpha$, $u_1^{\alpha,\beta} = \beta$, *and:*

1. *if* $\varphi^{-1}(\alpha) < \beta < \varphi(\alpha)$, *then* $\left[u^{\alpha,\beta}\right] \neq [u^{\tilde{\alpha},\tilde{\beta}}]$ *for every* $(\tilde{\alpha}, \tilde{\beta}) \neq (\alpha, \beta)$, $(\tilde{\alpha}, \tilde{\beta}) \in \mathscr{U}$,
2. *if either* $\varphi^{-1}(\alpha) = \beta$, *or* $\beta = \varphi(\alpha)$, *then* $[u^{\alpha,\beta}] = [u^{\beta,\alpha}]$.

Moreover, every stationary solution u *such that* $u_i > 1$ *for every* $i \in \mathbb{Z}$ *and* $\lim_{i \to \pm\infty} u_i = +\infty$ *is an element of an equivalence class* $\left[u^{\alpha,\beta}\right]$ *determined by a point* $(\alpha, \beta) \in \mathscr{U}$.

Two-sided unbounded stationary solutions (incresing sequences in both directions) can be identified with points in $\mathcal{U}$.

Functional mirroring

From the mirroring equation

$$u_{n+1} - \varphi(u_n) = \varphi(u_n) - u_{n-1}$$

we derive functional mirroring

$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u)$$

with initial conditions

$$\begin{aligned} \bar{f}_0(u) &= 2\varphi(u) - 1 & \rightarrow \qquad & \left\{\bar{f}_n(u)\right\}_{n\in\mathbb{N}}, \\ \underline{f}_0(u) &= 2\varphi(u) - \varphi^{-1}(u) & \rightarrow \qquad & \{\underline{f}_n(u)\}_{n\in\mathbb{N}}. \end{aligned}$$

Lemma (Well-defined sequences)

Let φ be a C^1-function and satisfy $\varphi(1) = 1$ and $\varphi'(u) > 1$. If f_n is a C^1-function and satisfy $f_n(1) = 1$ and $f_n'(u) > 1$, then f_{n+1} defined by functional mirroring is well-defined C^1-function and satisfies $f_{n+1}(1) = 1$ and $f_{n+1}'(u) > 1$ as well.

Functional mirroring

Sequences $\{\bar{f}_n(u)\}_{n\in\mathbb{N}}$ and $\{\underline{f}_n(u)\}_{n\in\mathbb{N}}$ generated via the functional mirroring

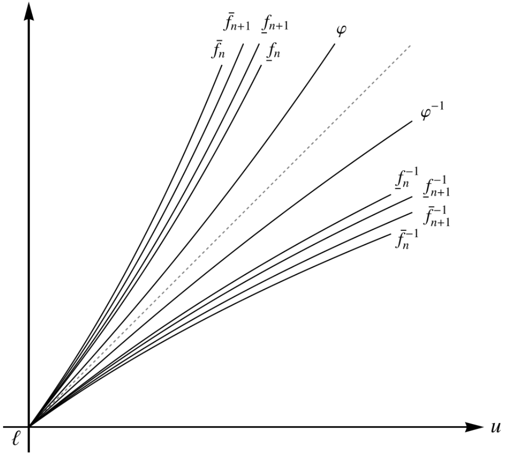
$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u)$$

with initial conditions

$$\bar{f}_0(u) = 2\varphi(u) - 1, \quad \text{and} \quad \underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u).$$

converge to the same continuous limit

$$\lim_{n\to+\infty} \bar{f}_n(u) = \lim_{n\to+\infty} \underline{f}_n(u) = f(u).$$



Result

Theorem

Let g be a C^1-function and satisfy $g(1;a)=0$, $g'(u;a)<0$ for all $u>1$. There exists a unique function $f:[1,\infty)\to[1,\infty)$ which is continuous, strictly increasing with $f(1)=1$, $\lim_{u\to\infty} f(u)=+\infty$, and $f(u)>u$ for all $u>1$ such that:

1. *every pair $(u,f(u))$ determines a strictly increasing stationary solutions $(u_i)_{i\in\mathbb{Z}}$ of the LDE satisfying $u_i>1$ for all $i\in\mathbb{Z}$ and*

$$\lim_{i\to-\infty} u_i = 1, \qquad \lim_{i\to+\infty} u_i = +\infty,$$

2. *every pair $(u,f^{-1}(u))$ determines a strictly decreasing stationary solutions $(u_i)_{i\in\mathbb{Z}}$ of the LDE satisfying $u_i>1$ for all $i\in\mathbb{Z}$ and*

$$\lim_{i\to-\infty} u_i = +\infty \quad \text{and} \quad \lim_{i\to+\infty} u_i = 1$$

.

One-sided unbounded stationary solutions (incresing sequences in both directions) can be identified with points of the graph of the function f.

Functional equation and stationary solutions

The functional mirroring

$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u)$$

in limit leads to the functional equation

$$\frac{f(u) + f^{-1}(u)}{2} = \varphi(u)$$

Since $f(u_i) = u_{i+1}$ and $f^{-1}(u_i) = u_{i-1}$ we have

$$\begin{aligned}
0 =& u_{i-1} - 2u_i + u_{i+1} + \frac{g(u_i)}{d} \\
=& f^{-1}(u_i) + f(u_i) - 2u_i + \frac{g(u_i)}{d} \\
=& f^{-1}(u_i) + f(u_i) - 2\varphi(u_i).
\end{aligned}$$

General existence result

Lemma

Let g be a reaction function of the heterogeneous inferior type and φ be given by

$$\varphi(u) = u - \frac{g(u;a)}{2D},$$

then:

1. *there exists unique $f_1(u) \geq u$ and $\eta \in (0,1)$ such that*

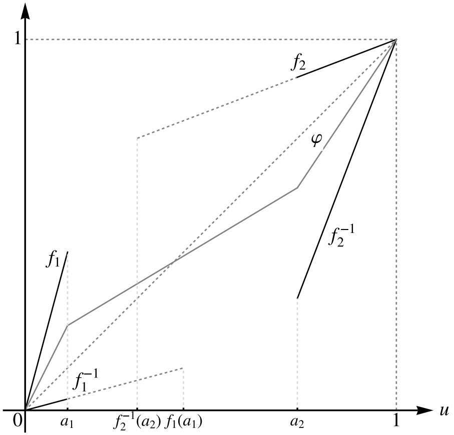
$$\frac{f_1(u) + f_1^{-1}(u)}{2} = \varphi(u), \quad u \in [0,\eta] \tag{1}$$

   and

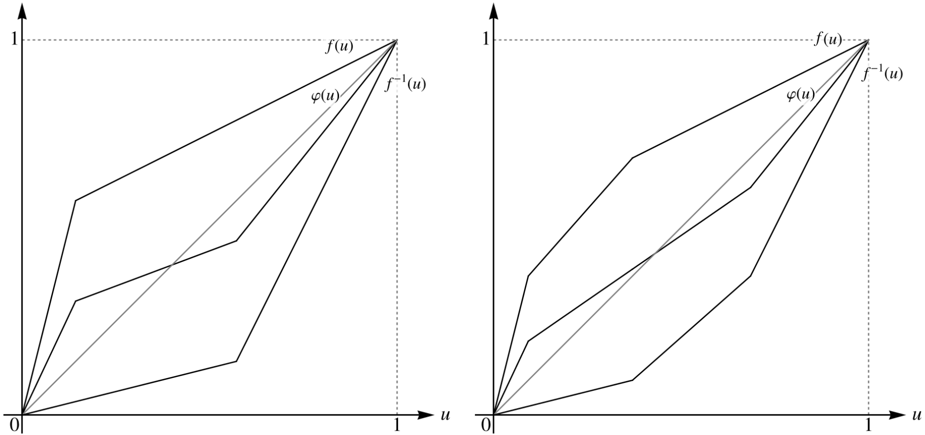
2. *there exists unique $f_2(u) \geq u$ and $\zeta \in (0,1)$ such that*

$$\frac{f_2(u) + f_2^{-1}(u)}{2} = \varphi(u), \quad u \in [\zeta,1]. \tag{2}$$

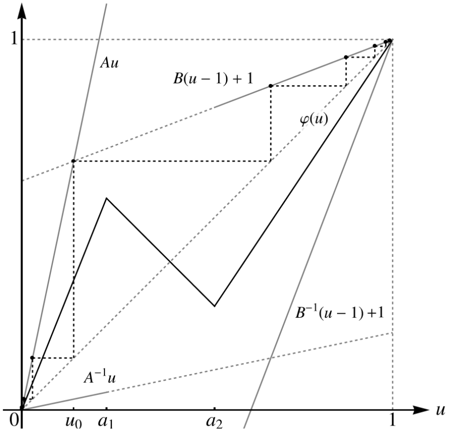
Illustration

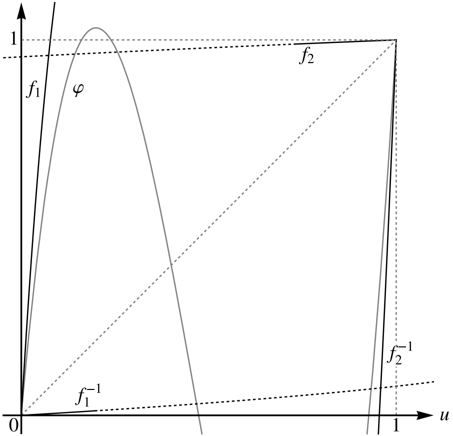


Complete solutions for piecewise linear reaction

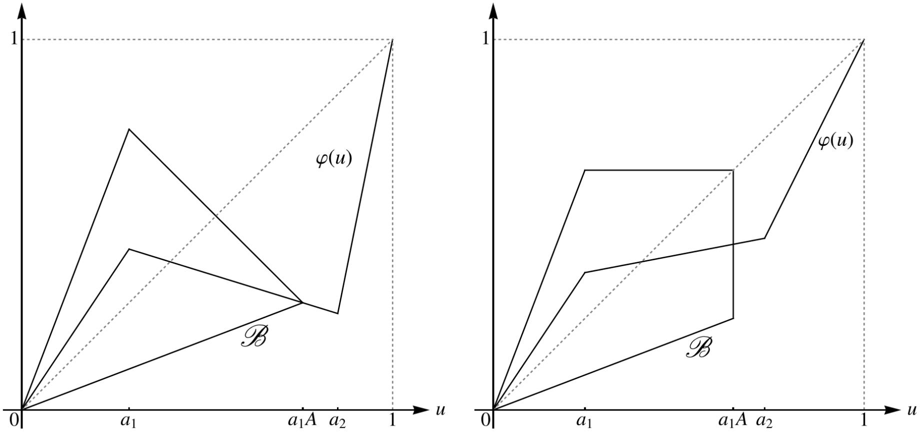


Single solution

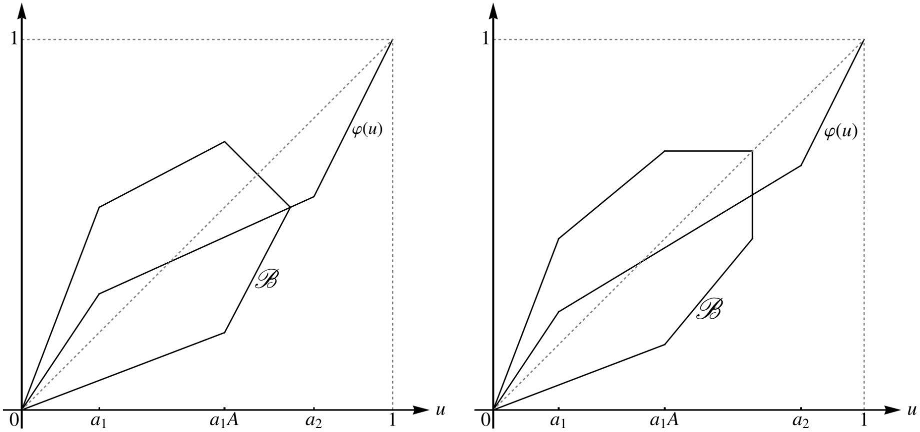




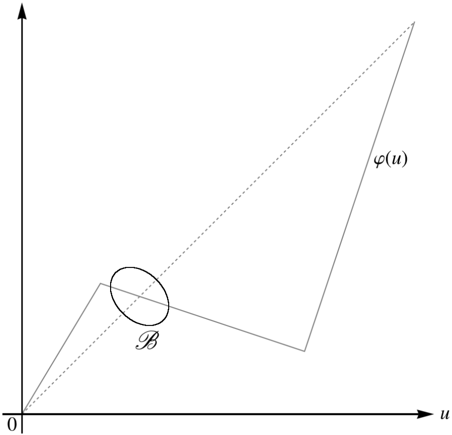
Pulses

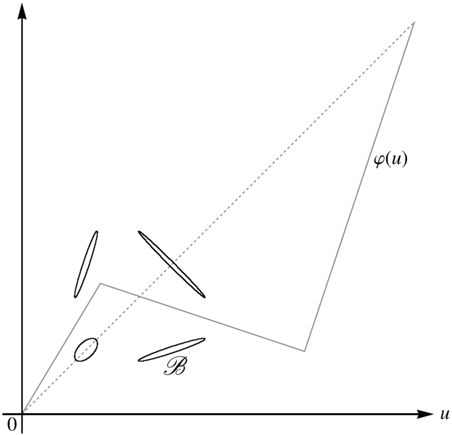

1
0
a_1
a_1A
a_2
1
u
$\varphi(u)$
$\mathscr{B}$
1
0
a_1
a_1A
a_2
1
u
$\varphi(u)$
$\mathscr{B}$

Pulses

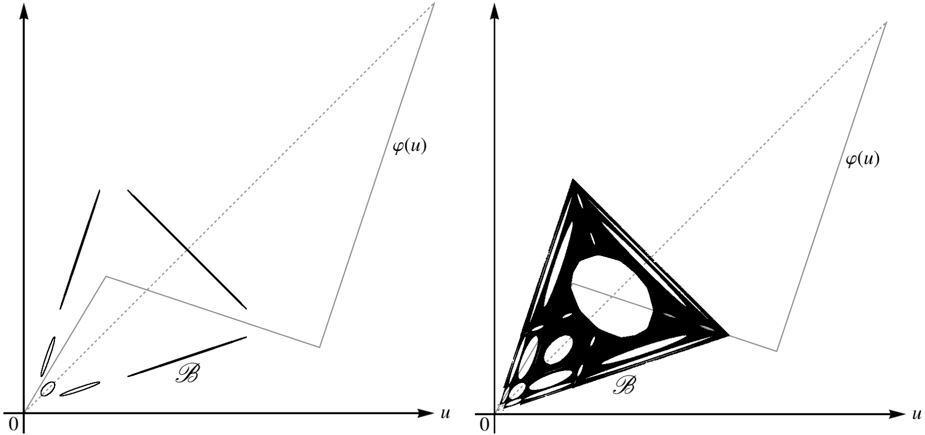


Bi-symmetric sets





Bi-symmetric sets



Thank you for your attention