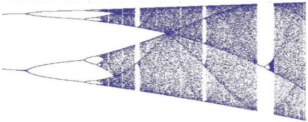

Progress on Difference Equations
International Conference
PODE 2025
Cartagena, 28th-30th May 2025

On the emergence and properties of weird quasiperiodic attractors

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• A new type of dynamic behavior, termed a **weird quasiperiodic attractor, WQA**.

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• L. Gardini, D. Radi, N. Schmitt, I. Sushko, F. Westerhoff. How risk aversion may shape the dynamics of stock markets: A chartist-fundamentalist approach, 2025d (in progress).

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$$F = \begin{cases} F_L : (x, y) \to (\tau_L x + y, -\delta_L x) \text{ if } x < h, \\ F_R : (x, y) \to (\tau_R x + y, -\delta_R x) \text{ if } x > h. \end{cases}$$

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• Among several known definitions of chaos, we use the following one:

Def. A 2D map $F : I \to I$, $I \subseteq \mathbb{R}^2$, is said to be **chaotic** on a closed invariant set $X \subseteq I$ if (a) periodic points of F are dense in X, and (b) F is topologically transitive, i.e., there is a dense aperiodic (neither periodic, nor quasiperiodic) trajectory on X.

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So, we consider a 2D discontinuous PWL map $F : \mathbb{R}^2 \to \mathbb{R}^2$, defined as follows:

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where

$$J_L = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix}, \; J_R = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}.$$

Here, $\delta_i = \det J_i$, $\tau_i = tr J_i$, $i = L, R$, are real parameters.

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$$D_L = \{(x, y) : x < -1\}, \quad D_R = \{(x, y) : x > -1\},$$

separated by the **discontinuity line** $C_{-1} = \{(x, y) : x = -1\}$.

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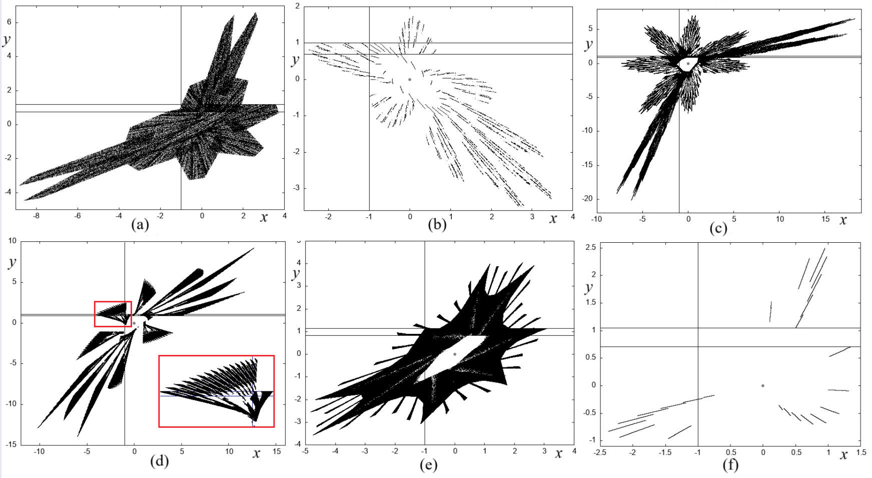
$$C^L = F_L(C_{-1}) = \{(x, y) : y = \delta_L\}, \; C^R = F_R(C_{-1}) = \{(x, y) : y = \delta_R\}.$$

Map F: Preliminaries

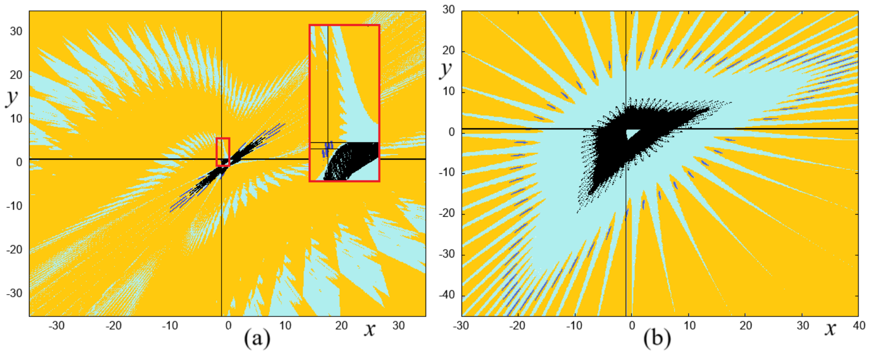
Examples of WQA of map F



(a) $\delta_L = 0.75$, $\delta_R = 1.2$, $\tau_L = -0.7$, $\tau_R = -2.5$; **(b)** $\delta_L = 0.7$, $\delta_R = 1.001$, $\tau_L = 0.3$, $\tau_R = 0.71$; $\delta_L = 0.9$, $\delta_R = 1.1$, $\tau_L = -2.5$, **(c)** $\tau_R = -0.7$; **(d)** $\tau_R = -1.2$; **(e)** $\delta_L = 0.84$, $\delta_R = 1.15$, $\tau_L = -1$, $\tau_R = -1.9$; **(f)** $\delta_L = 1.05$, $\delta_R = 0.7$, $\tau_L = -0.75$, $\tau_R = -1.6$.

Map F: Preliminaries

Examples of coexisting WQAs with basins



(a) $\delta_L = 0.9$, $\delta_R = 1.1$, $\tau_L = 0.3$, $\tau_R = 0.71$; **(b)** $\delta_L = 0.9$, $\delta_R = 1.11$, $\tau_L = -2$, $\tau_R = -1.91$.

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• Other kinds of attractors with quasiperiodic dynamics: a **Cantor set attractor** (as in a gap map),

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Map F: Properties

Property 1

The *fixed point* $O = (0,0) \in D_R$ of map F_R is the unique fixed point of map F, provided no eigenvalue of F_R equals 1, i.e., $1 - \tau_R + \delta_R \neq 0$.

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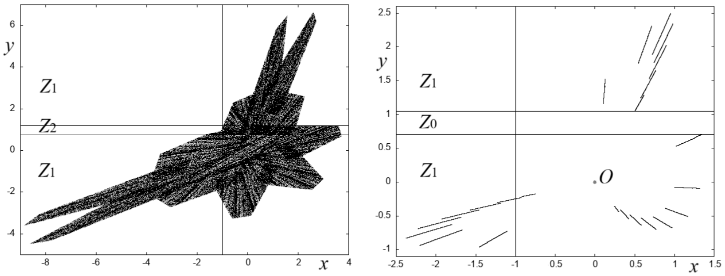
Property 2

Invertibility of map F is of (a) $Z_1 - Z_0 - Z_1$ type for $0 < \delta_R < \delta_L$, or $\delta_L < \delta_R < 0$; (b) $Z_1 - Z_2 - Z_1$ type for $\delta_R < \delta_L < 0$, or $0 < \delta_L < \delta_R$; (c) $Z_0 - Z_1 - Z_2$ type for $\delta_L \delta_R < 0$; (d) $Z_1 - Z_\infty - Z_1 - Z_0$ type for $\delta_L = 0$, $\delta_R \neq 0$; (e) $Z_0 - Z_\infty - Z_0 - Z_1$ type for $\delta_R = 0$, $\delta_L \neq 0$; (f) $Z_1 - Z_2 - Z_1$ type for $\delta_R = \delta_L$.



Map F: Properties

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In the (δ_i, τ_i)-parameter plane, $i = L, R$, the eigenvalues $\lambda^i_{1,2} = \frac{1}{2}(\tau_i \pm \sqrt{\tau_i^2 - 4\delta_i})$ of matrix J_i satisfy $\left|\lambda^i_{1,2}\right| < 1$ in the **stability triangle** T^i, defined as
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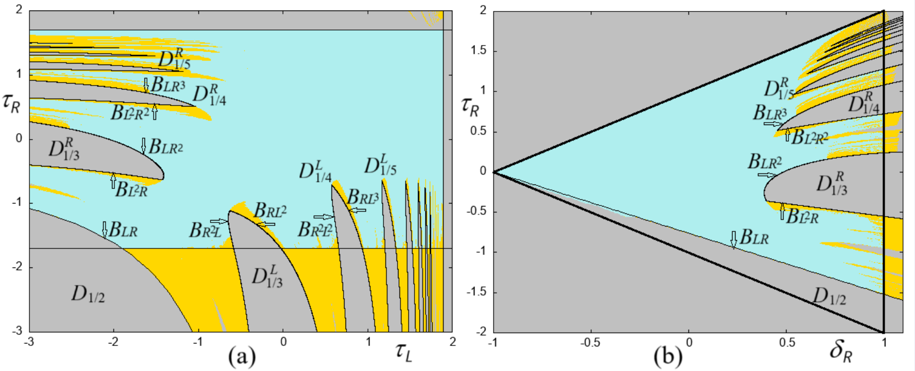
Rotation number of a cycle

For a (nonhyperbolic) cycle with a symbolic sequence σ, by its **rotation number** $\rho = m/n$, we mean that along the cycle the trajectory makes m turns around the origin in n iterations.

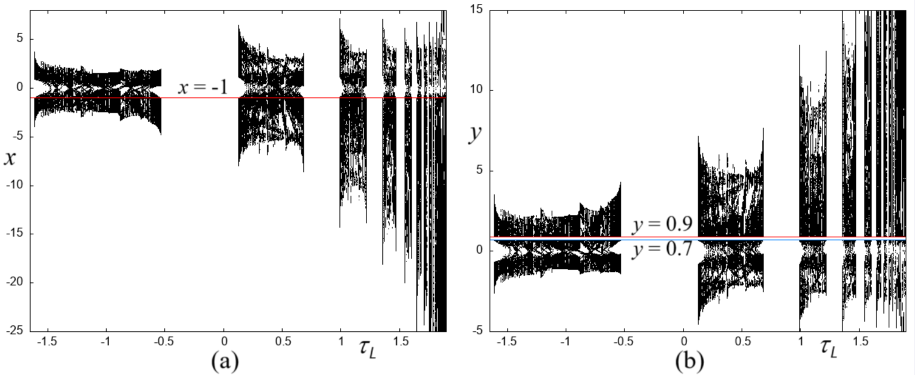
Bifurcation structure of map F in the (τ_L, τ_R)- and (δ_R, τ_R)-parameter plane

Blue and yellow parameter regions indicate convergence to the fixed point O and to a WQA, respectively. Gray regions indicate divergence.



(a) the (τ_L, τ_R)-parameter plane for $\delta_L = 0.9$, $\delta_R = 0.7$, and (b) the (δ_R, τ_R)-parameter plane for $\delta_L = 0.9$, $\tau_L = -2.5$. The boundaries $B_{LR^{n-1}}$ and $B_{L^2R^{n-2}}$ of the divergence region $D^R_{1/n}$, and the boundaries $B_{RL^{n-1}}$ and $B_{R^2L^{n-2}}$ of the divergence region $D^L_{1/n}$ are shown for $n = 2, \ldots, 9$.

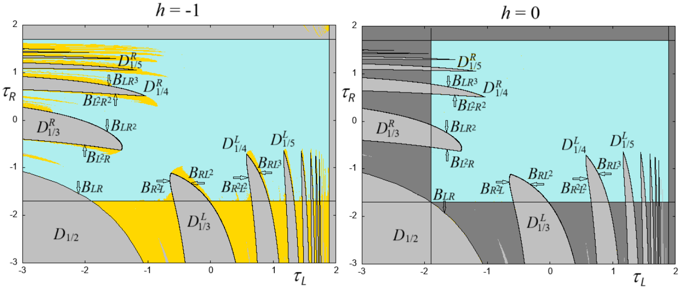
Bifurcation structure of map F: WQA and divergence



A 1D bifurcation diagram of map F showing x versus τ_L in (a), and y versus τ_L in (b) for $\delta_L = 0.9$, $\delta_R = 0.7$, and $\tau_R = -2$.

Divergence regions and dangerous BCB of O at $h = 0$

An analogue of a dangerous BCB in the 2D BCBNF (see e.g., Ganguli & Banerjee 2005)



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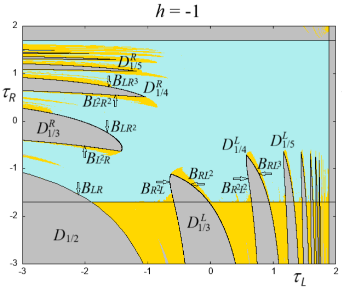
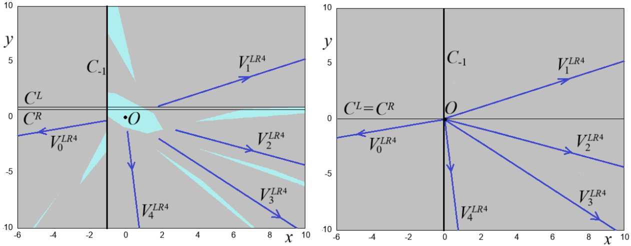
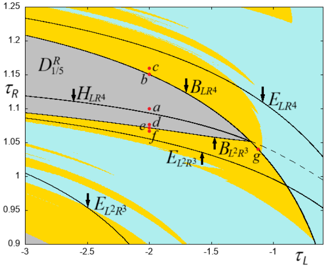
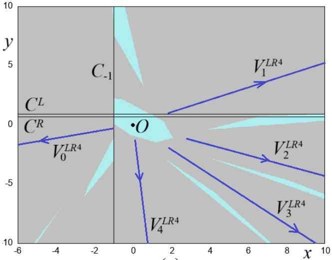


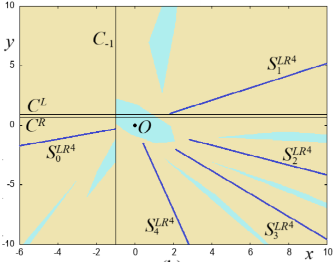


Dynamics of map F near the divergence region $D^R_{1/5}$

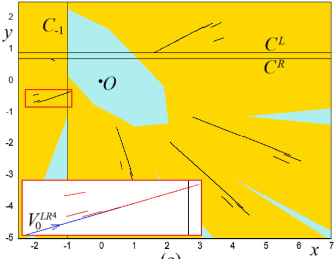




(a)



(b)



(c)

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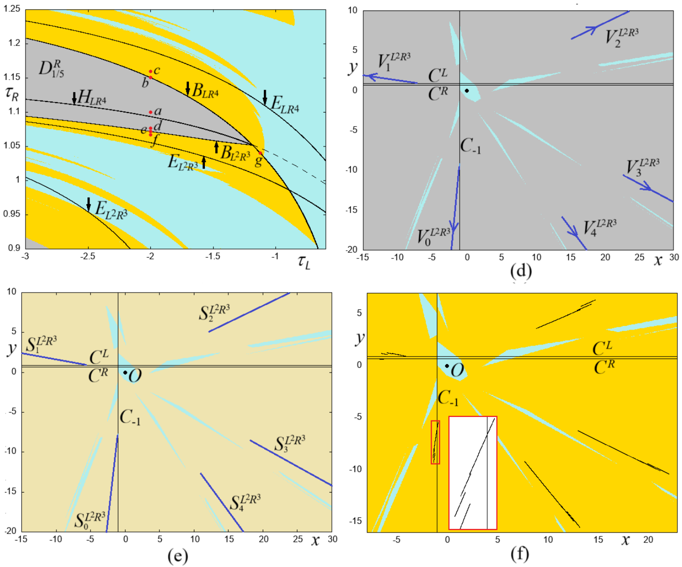

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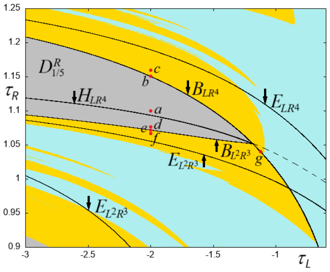
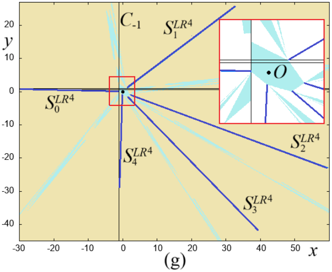
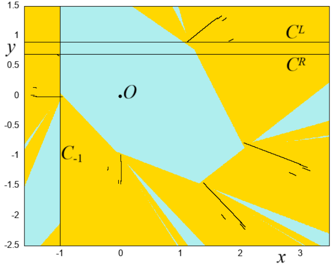
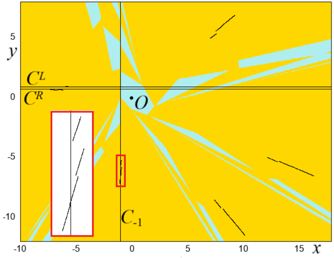
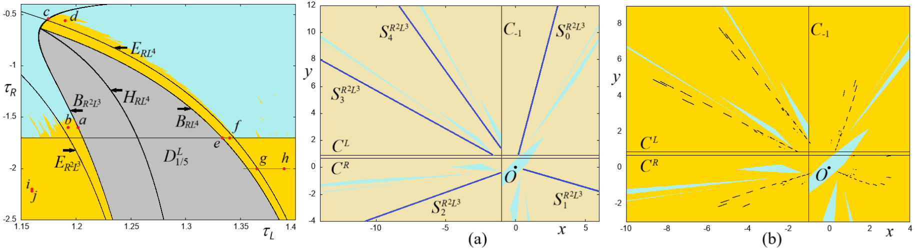








Dynamics of map F near the divergence region $D^L_{1/5}$



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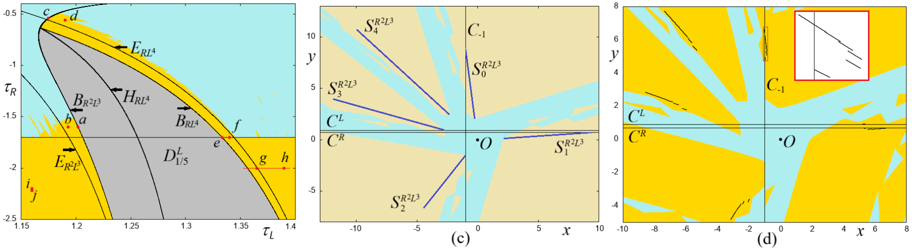

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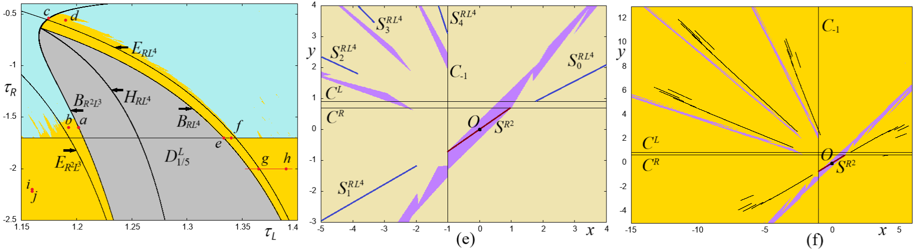

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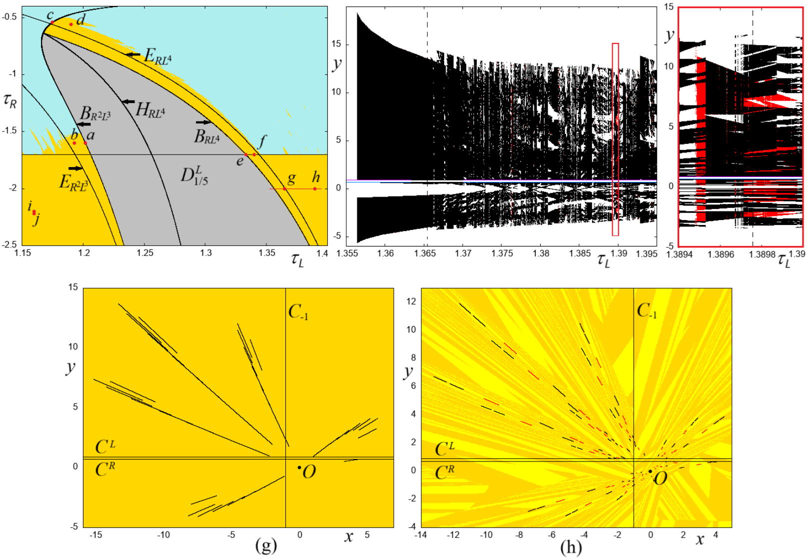

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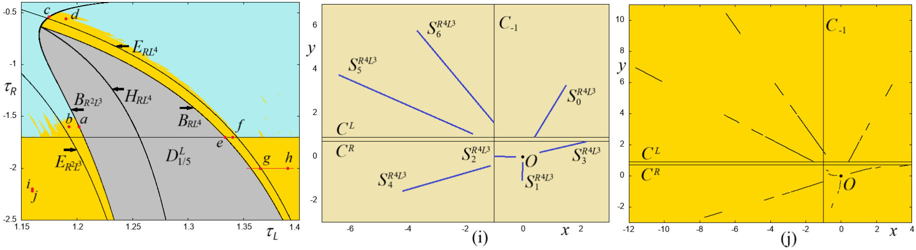
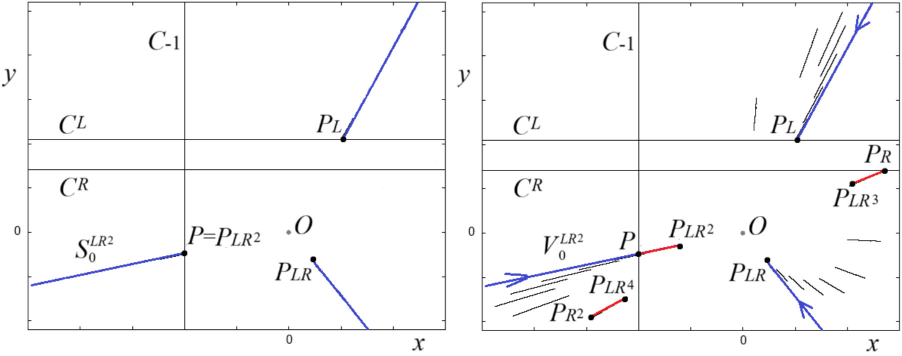


In the yellow regions, there is a dense set of curves associated with rational rotation numbers and the existence of corresponding bounded segments filled with nonhyperbolic cycles, but the general case is the existence of WQAs associated with irrational rotation numbers.

A mechanism of the appearance of a WQA (schematically)



WQA can be seen as a limit set of the iterations of the segment (P, P_σ), where P is an intersection point of the discontinuity line and the eigenvector V_0^σ (associated before with a segment S_0^σ of nonhyperbolic σ-cycles), and $P_\sigma = F^\sigma(P)$.
In the above example, a dense quasiperiodic trajectory on the WQA includes critical point $P_R = F_R(P)$, so that WQA is a closure of the trajectory with the initial point P_R.

Bifurcation boundaries in the parameter space of map F

Recall: the existence of a set $S^\sigma = \{S_i^\sigma\}_{i=0}^{i=n-1}$ of n segments filled with nonhyperbolic σ-cycles is associated with the conditions $P_\sigma(1) = 0$, moreover, the related admissibility conditions must be satisfied: all the segments must be located in the proper partitions.

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Existence of S^σ for $\sigma = LR^{n-1}$, $\sigma = L^2R^{n-2}$, $n \geq 3$

$B_{LR^{n-1}}$: $P_{LR^{n-1}}(1) = 1 - \tau_L a_{n-1} + (\delta_L + \delta_R)a_{n-2} + \delta_L \delta_R^{n-1} = 0$ and $(A^{LR^{n-1}})$ holds,

$B_{L^2R^{n-2}}$: $P_{L^2R^{n-2}}(1) = 1 + (\tau_L(\delta_L + \delta_R) - \delta_L\tau_R)a_{n-3} - (\tau_L^2 - 2\delta_L)a_{n-2} + {}$

$+ \delta_L^2\delta_R^{n-2} = 0$ and $(A^{L^2R^{n-2}})$ holds,

where $(A^{LR^{n-1}})$: $S_0^{LR^{n-1}} \subset D_L$, $S_j^{LR^{n-1}} \subset D_R$, $j = \overline{1, n-1}$,

$(A^{L^2R^{n-2}})$: $S_0^{L^2R^{n-2}} \subset D_L$, $S_1^{L^2R^{n-2}} \subset D_L$, $S_j^{L^2R^{n-2}} \subset D_R$, $j = \overline{2, n-1}$,

and a_k is the solution of the second-order homogeneous linear difference equation $a_k = \tau_R a_{k-1} - \delta_R a_{k-2}$, $a_0 = 1$, $a_1 = \tau_R$, $k = 2, 3, \ldots$.

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$B_{LR^{n-1}}$ and $B_{L^2R^{n-2}}$ are **boundaries of the divergence region** $D_{1/n}^R$ if there exists corresponding σ-cycle at infinity (i.e., all the segments are admissible and one-side unbounded).

Bifurcation boundaries in the parameter space of map F

Transition **from real to complex-conjugate eigenvalues** of matrix $J_{LR^{n-1}} = J_R^{n-1} J_L$ and matrix $J_{L^2R^{n-2}} = J_R^{n-2} J_L^2$ occurs crossing a parameter set $E_{LR^{n-1}}$ and $E_{L^2R^{n-2}}$, respectively, defined as follows:

$$E_{LR^{n-1}} : \quad (\tau_L a_{n-1} - (\delta_L + \delta_R) a_{n-2})^2 - 4\delta_L \delta_R^{n-1} = 0,$$

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$$H_{LR^{n-1}} : \quad \delta_L a_{n-3} - \tau_L a_{n-2} = 0, \quad H_{LR^{n-1}} \subset D_{1/n}^R.$$

• On one side of boundary $H_{LR^{n-1}}$, it holds that the unstable eigenvectors $V_j^{LR^{n-1}}$ are admissible, while eigenvectors $V_j^{L^2R^{n-2}}$ are not admissible (so that an attracting LR^{n-1}-cycle at infinity exists, while the L^2R^{n-2}-cycle is virtual);

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$$+ \delta_R^2 \delta_L^{n-2} = 0 \text{ and } (A^{R^2L^{n-2}}) \text{ holds,}$$

where

$$(A^{RL^{n-1}}): \quad S_0^{RL^{n-1}} \subset D_R, \; S_j^{RL^{n-1}} \subset D_L, \quad j = \overline{1, n-1},$$
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