On the dynamics of a family of planar discontinuous piecewise linear maps

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Talk based on:

A. Cima, A. G., V. Mañosa, F. Mañosas. Pointwise periodic maps with quantized first integrals. *Communications Nonlinear Science and Numerical Simulations* 108, 106150:1–26, 2022.

A. Cima, A. G., V. Mañosa, F. Mañosas. On some rational piecewise linear rotations. *J. Difference Equ. Appl.* 30, 1577–1589, 2024.

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We consider the family of piecewise linear maps

$$F(x,y) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x - \operatorname{sign}(y) \\ y \end{pmatrix},$$

that has a line of discontinuity $LC_0 = \{y = 0\}$.

When $\alpha \in \mathcal{A} := \{\pi/3, \pi/2, 2\pi/3, 4\pi/3, 3\pi/2, 5\pi/3\}$, it is known that the maps are pointwise periodic but not globally periodic.

In fact, their sets of periods are unbounded.

In complex notation they write as

$$F_\lambda(z) = \lambda(z - H(z)),$$

where $z \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $\lambda = \mathrm{e}^{i\alpha} \in \mathbb{C}$ (thus $|\lambda| = 1$), and

$$H(z) = \begin{cases} 1, & \text{if} \quad \mathrm{Im}(z) \geq 0, \\ -1, & \text{if} \quad \mathrm{Im}(z) < 0. \end{cases}$$

Chang, Cheng and Wang studied the periodic behavior of the solutions of $x_{n+2} + \rho\, x_{n+1} + x_n = \text{sign}(x_{n+1})$ with $\rho = -1, 0$ and 1. These difference equations are conjugate with the maps F with $\alpha = \pi/3, \pi/2$ and $2\pi/3$, respectively.

Y. C. Chang, G. Q. Wang, S. S. Cheng. Complete set of periodic solutions of a discontinuous recurrence equation. *J. Difference Equations and Appl.* 18, 1133–1162, 2012.

Y. C. Chang, S. S. Cheng. Complete periodicity analysis for a discontinuous recurrence equation. *Int. J. Bifurcations and Chaos* 23, 1330012 (34 pages), 2013.

Y. C. Chang, S. S. Cheng. Complete periodic behaviours of real and complex bang bang dynamical systems. *J. Difference Equations and Appl.* 20, 765–810, 2014.

Y. C. Chang, S. S. Cheng, Y. C. Yeh. Abundant periodic and aperiodic solutions of a discontinuous three-term recurrence relation. *J. Difference Equations and Appl.* 25, 1082–1106, 2019.

We found the periodic maps in our family, which are also bijective and area preserving, specially interesting under the light of a classical result of Montgomery (1937) which states that:

"every pointwise periodic homeomorphism F in an euclidian space is globally periodic",

i.e. there exists p such that $F^p = \mathrm{Id}$.

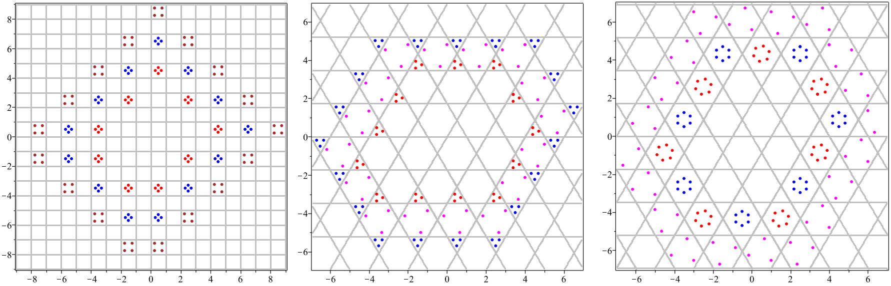
D. Montgomery. Pointwise periodic homeomorphisms. *Amer J Math.* 59, 118–20, 1937.

A similar family of maps was studied also by Goetz and Quas.

A. Goetz, A. Quas. Global properties of a family of piecewise isometries. *Ergod. Th. & Dynam. Sys.* 29, 545–568, 2009.

Numerics: $\alpha = \frac{\pi}{2}$, $\alpha = \frac{2\pi}{3}$ and $\alpha = \frac{\pi}{3}$

The *critical lines LC_{-i} (in grey)*, which are the preimages of the *discontinuity line* $LC_0 = \{y = 0\}$, that is $LC_{-i} = F^{-i}(LC_0)$, form regular tilings.



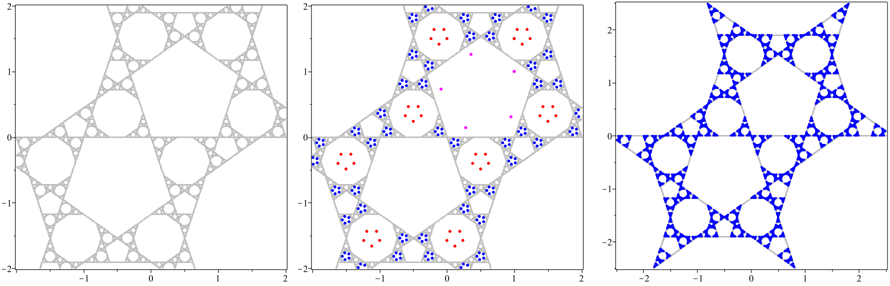
$\alpha = \frac{\pi}{2}$ $\alpha = \frac{2\pi}{3}$ $\alpha = \frac{\pi}{3}$

The orbits in the *critical set* $\mathcal{F} = \bigcup_{i \in \mathbb{N}} LC_{-i}$ are also periodic.

More numerics: $\alpha = 2\pi \frac{q}{p}$

For other cases of α (i. e. $\alpha \notin \mathcal{A}$), we prove that the *critical lines* LC_{-i} do NOT form regular tilings.

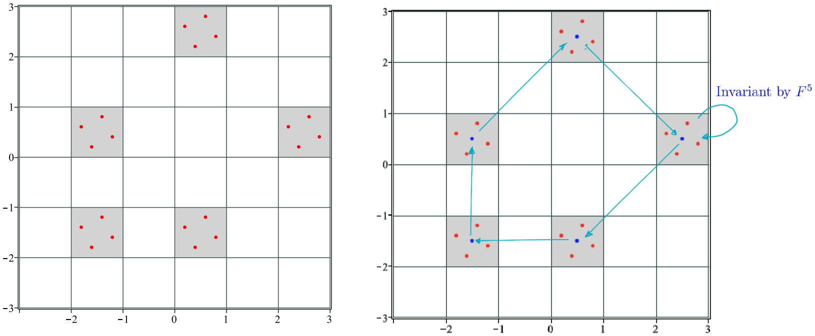
There seems to appear *fractal structures* and that there could exist *non-periodic orbits* in the *critical set*.



The *critical set* $\mathcal{F} = \bigcup_{i\in\mathbb{N}} LC_{-i}$, and some orbits for $\alpha = \frac{8\pi}{5}$.

What is happening in the pointwise periodic cases?

An example with $\alpha = \frac{\pi}{2}$:



There is a *periodic inter-tiles dynamics*.

All the points (except the center) in each *tile* have the same behavior.

After the inter-tiles dynamics is established, there is a *periodic intra-tiles dynamics*.

The tiles are the level sets a *first integral* with discrete energy levels.

Main results in the *integrable* cases corresponding to $\alpha = \frac{\pi}{2}, \frac{2\pi}{3}$ and $\frac{\pi}{3}$.

(i) We characterize the geometry of the *critical sets* $\mathcal{F} = \bigcup_{i\in\mathbb{N}} LC_{-i}$ obtaining some regular tilings.

(ii) We find some first integrals for all the maps.

The dynamics on the regular set $\mathcal{U} = \mathbb{R}^2 \setminus \mathcal{F}$ is explained in terms of the energy levels of of the first integrals.

(iii) We also characterize the dynamics in the critical set.

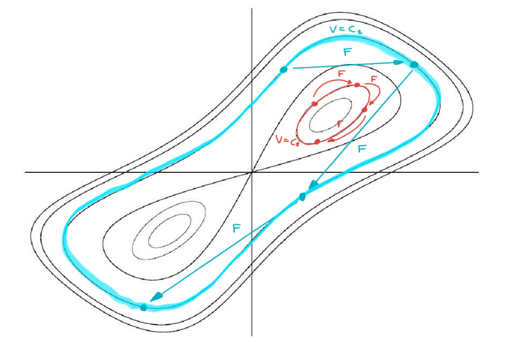
(iv) We *re-obtain* the set of periods of the maps.

First integrals in DDS

A function $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a first integral of a map F if

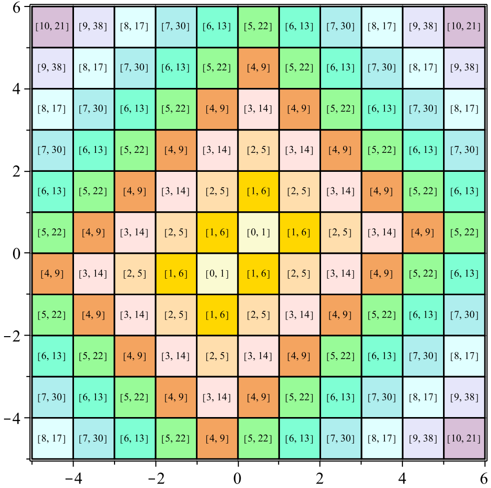
$$V(F(x, y)) = V(x, y)$$

That is, the orbits lie on the *level sets* $\{V(x, y) = c\}$ called *energy levels*.



First integral of F for $\alpha = \pi/2$

$$V(x,y) = \max\left(|\mathrm{E}(x)+\mathrm{E}(y)+1|-1,\ |\mathrm{E}(x)-\mathrm{E}(y)|\right),$$



V is a *first integral* of $F_{\pi/2}$. It takes values in $\mathbb{N} \cup 0$ thus it is quantized.

First integral of F for $\alpha = 2\pi/3$

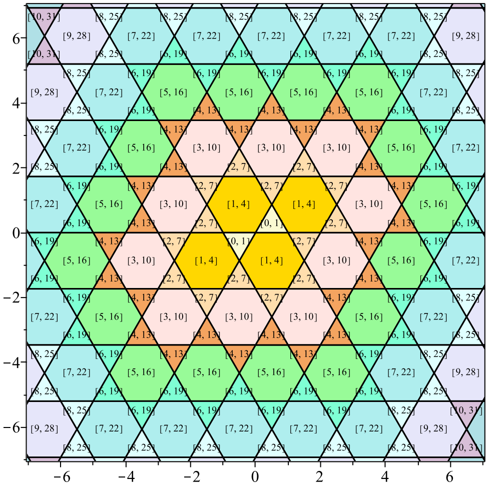
$$V_{2\pi/3}(x,y) = \max\big(|B(x,y) - C(y) + D(x,y)|,\ |B(x,y) + C(y) + D(x,y) + 1| - 1, |-B(x,y) + C(y) + D(x,y)| \big).$$

with

$$B(x,y) = \mathrm{E}\left(\frac{3x - \sqrt{3}y}{6}\right),$$

$$C(y) = \mathrm{E}\left(\frac{\sqrt{3}y}{3}\right) \text{ and}$$

$$D(x,y) = \mathrm{E}\left(\frac{3x + \sqrt{3}y + 3}{6}\right)$$



First integral of F for $\alpha = \pi/3$

$$V_{2\pi/3}(x,y) = \max\big(\,|B(x,y) - C(y) + D(x,y)|\,,$$
$$|B(x,y) + C(y) + D(x,y) + 1| - 1, |-B(x,y) + C(y) + D(x,y)|\,\big)$$

with

$$B(x,y) = \mathrm{E}\left(\frac{3x - \sqrt{3}y + 3}{6}\right),$$

$$C(y) = \mathrm{E}\left(\frac{\sqrt{3}y}{3}\right) \text{ and}$$

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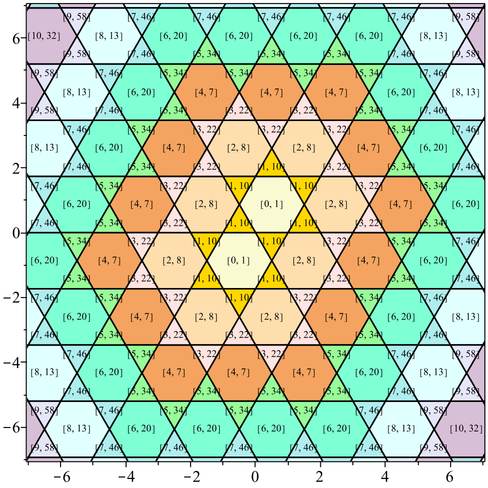
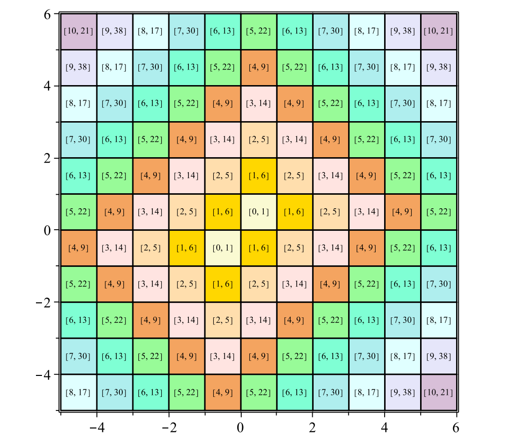


Theorem A (i): case $\alpha = \pi/2$

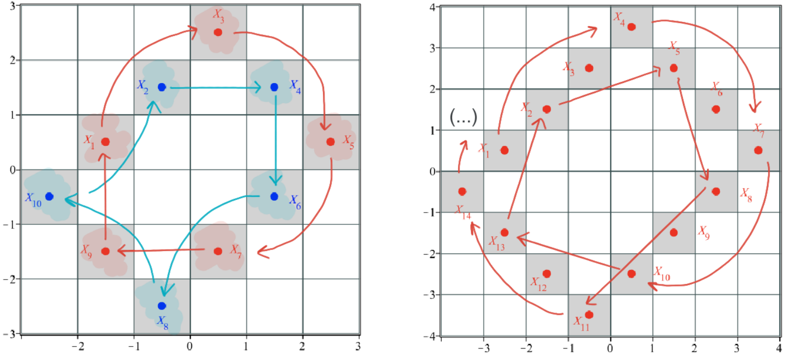
The critical set $\mathcal{F}$ is the square Euclidian uniform tiling in the figure.
$V(x, y) = \max\left(|E(x) + E(y) + 1| - 1, |E(x) - E(y)|\right)$, is a *first integral* on the regular set $\mathcal{U}$.



The level sets $\{V(x, y) = c\}, c \in \mathbb{N}$, are necklaces with $4c + 2$ squares

Theorem A: $\alpha = \pi/2$. Dynamics of the centers

If $V(x,y) = c$ then $F(X_i) = X_j$ with $j \equiv i + c \mod (4c+2)$.



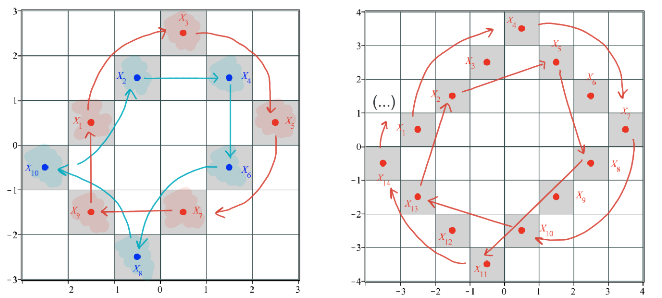
Dynamics of the centers X_j in the levels $c = 2$ and $c = 3$

Theorem A: $\alpha = \pi/2$. Dynamics of the centers

The set of centers of a level is an invariant set, and the map F from these set to itself is conjugated to

$$h : \mathbb{Z}_{4c+2} \to \mathbb{Z}_{4c+2} \text{ given by } h(i) = i + c.$$

- If c is odd, the centers are $(4c+2)$-periodic points.
- If c is even, the centers are $(2c+1)$-periodic points. There are two different periodic orbits.

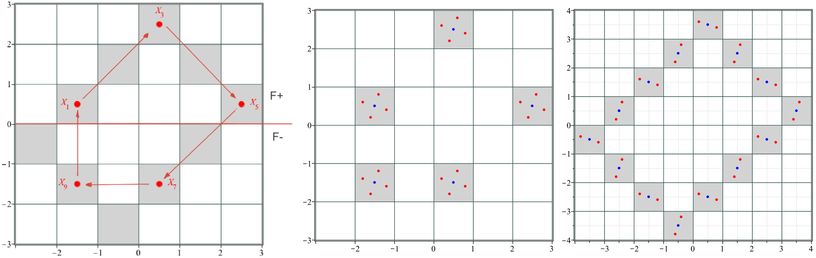


Theorem A (ii): Dynamics on the regular set

• When c is even, each square in this level is invariant by F^{2c+1} which is a rotation of order 4 around the center.

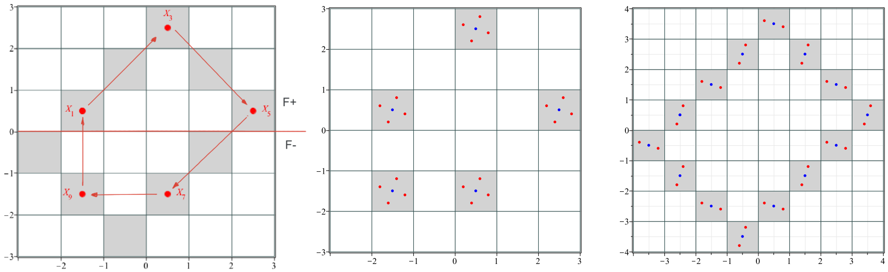
• When c is odd, each tile in this level is invariant by F^{4c+2} which is a rotation of order 2 around the center.

• The orbits in the regular set are periodic of period $2c+1$ or $4c+2$ (the centers) according if c is even or odd; or $8c+4$ otherwise, where $c \in \mathbb{N}$.



$c = 2$ $\qquad$ $c = 3$

Theorem A (iii): Dynamics on the regular set



$c = 2$ $c = 3$

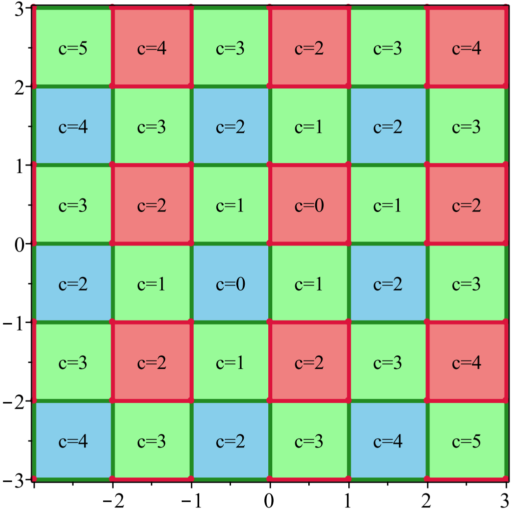
This is because the itinerary maps are rotations.
If $c = 2$, for X_1:

$$X_1 \xrightarrow{F_+} X_3 \xrightarrow{F_+} X_5 \xrightarrow{F_+} X_7 \xrightarrow{F_-} X_9 \xrightarrow{F_-} X_1$$

Its itinerary map is $F_-^2 \circ F_+^3$ which is a *rotation* centered at X_1 of order 4.

Theorem A (iii): Dynamics on the *critical set*

All orbits with initial condition on $\mathcal{F}$ are $(8n+4)$-periodic for all $n \in \mathbb{N}$.



Theorem A (iv): Set of periods

The map F is pointwise periodic.

Its set of periods is

$$\mathrm{Per}(F) = \{4n+1;\ 8n+4;\ \text{and } 8n+6 \text{ for all } n \in \mathbb{N}\}.$$

Theorem A (iv) was previously obtained in:

Y. C. Chang, S. S. Cheng. Complete periodic behaviours of real and complex bang bang dynamical systems. *J. Difference Equations and Appl.* 20, 765–810, 2014.

Theorem B: case $\alpha = 2\pi/3$:

(ii a) For c even, $\{V(x,y) = c\}$ is a formed by $6c+2$ triangles, whose dynamics is conjugated with $h : \mathbb{Z}_{6c+2} \to \mathbb{Z}_{6c+2}$, $h(i) = i + 2c$.

Each triangle is invariant by F^{3c+1} which is a rotation of order 3 around the center $\Rightarrow$ $9c+3-$periodic points.

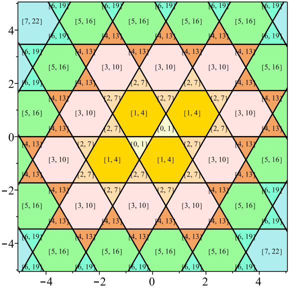
(ii b) For c odd, $\{V(x,y) = c\}$ is formed by $3c+1$ hexagons whose dynamics is conjugated with $h : \mathbb{Z}_{3c+1} \to \mathbb{Z}_{3c+1}$, $h(i) = i + c$.

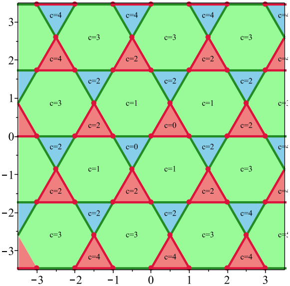
Each hexagon is invariant by F^{3c+1}: a rotation of order 3 around the center of the tile $\Rightarrow$ $9c+3-$periodic points.

(iii) All orbits on $\mathcal{F}$ are $9n+3$-periodic.

(iv) The map F is pointwise periodic.

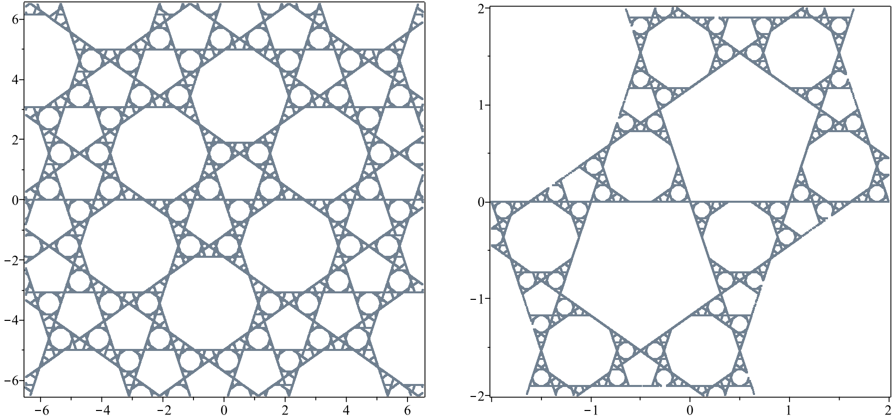
$$\mathrm{Per}(F) = \{3n+1 \text{ and } 9n+3 \text{ for all } n \in \mathbb{N}_0\}.$$





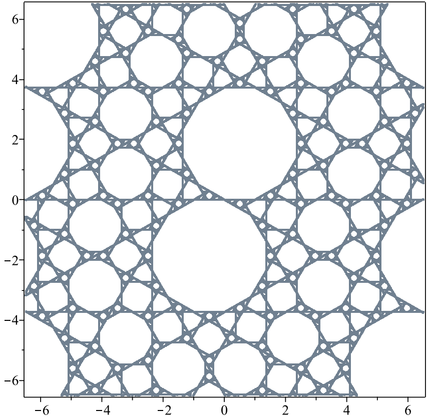
Theorem B (iv) was previously obtained by Chang, Wang & Cheng. J. Difference Eq. and Appl. 18 (2012).

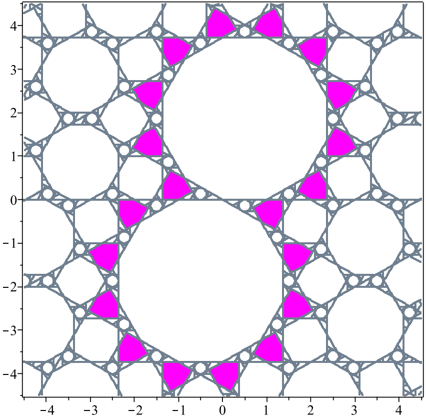
The singular set $\mathcal{F}$ when $\alpha = \frac{8\pi}{5}$



Critical set $\mathcal{F} = \bigcup_{i\in\mathbb{N}} LC_{-i}$.

The singular set $\mathcal{F}$ when $\alpha = \frac{11\pi}{6}$





Critical set $\mathcal{F} = \bigcup_{i \in \mathbb{N}} LC_{-i}$.

We prove the existence of the magenta non-regular tiles.

A fast overview of our results and simulations

Some results

- Existence of non-regular polygonal tiles in $\mathcal{U}$ and shape of the tiles.
- Periodic dynamics (with inter-tile and intra-tile dynamics) in $\widetilde{\mathcal{U}} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$.
- The orbits must be non-periodic in $\overline{\mathcal{F}} \setminus \mathcal{F}$, but we do not know whether this set is empty or not.

Some numerical evidences

- Fractalization of the critical set $\mathcal{F}$.
- Periodic dynamics (with inter-tile and intra-tile dynamics) in the regular set $\mathcal{U} = \mathbb{R}^2 \setminus \mathcal{F}$
- Unboundedness of the periods in compact sets. Both in $\mathcal{U}$ and $\mathcal{F}$.
- Existence of non-periodic points in $\mathcal{F}$.

We will give ideas of the proofs of the points in red

Some results

- Existence of non-regular polygonal tiles in $\mathcal{U}$ and shape of the tiles.
- Periodic dynamics (with inter-tile and intra-tile dynamics) in $\widetilde{\mathcal{U}} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$.
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- Existence of non-periodic points in $\mathcal{F}$.

A known result

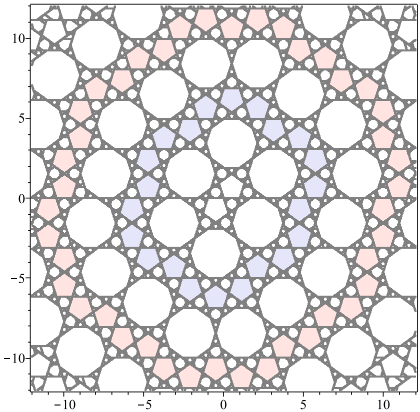
In A. Goetz, A. Quas. Ergod. Th. & Dynam. Sys. **29** (2009) it is proved (constructively) the existence of what they call *periodic islands*.

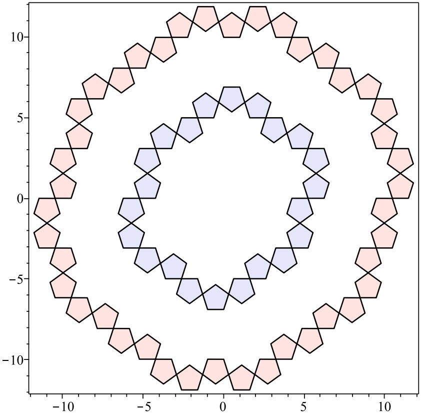
Theorem (Goetz-Quas)

When $\alpha = 2\pi\frac{p}{q}$ there exists a sequence of open invariant nested necklaces that tend to infinity, whose beads are polygons, and where the dynamics of F is given by a composition of two rotations.

Although the adherence of the union of all these invariant necklaces does not fill the full plane, it allows to prove that all orbits of F are bounded.

Some necklaces of Goetz and Quas' paper





Some invariant nested necklaces

Main results for $\alpha = 2\pi\frac{p}{q}$ with $(p, q) = 1$

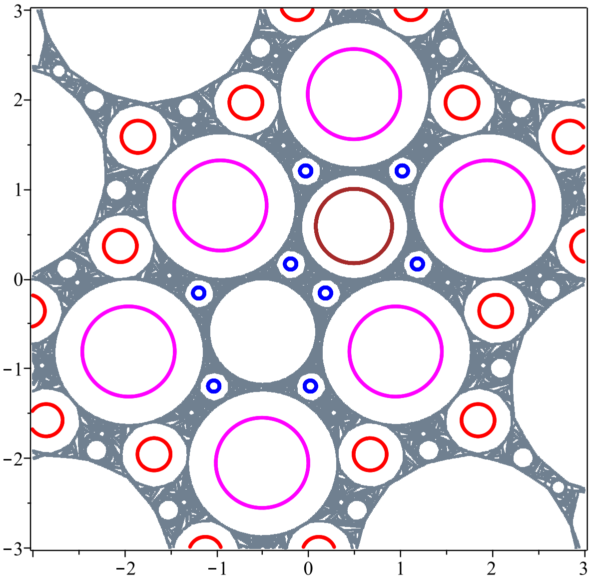
Theorem C: geometry of the tiles

(i) Let V be a connected component of $\widetilde{\mathcal{U}} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$. Then ∂V is a *convex polygon* with at most q sides if q is even and at most $2q$ sides when q is odd.

(ii) If k is the period of V and $(k, q) = 1$, then ∂V has either q sides and ∂V is a *regular polygon* or q is odd and ∂V has $2q$ sides.

(iii) When $\alpha/2\pi$ is irrational then ∂V is a circle.

The case $\alpha = \sqrt{2}$: a simulation



Main results for $\alpha = 2\pi\frac{p}{q}$ with $(p, q) = 1$

Theorem D: dynamics on the regular set

Any connected component of $\widetilde{\mathcal{U}} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$ is open, bounded and periodic. Moreover any point in $\widetilde{\mathcal{U}}$ is periodic.

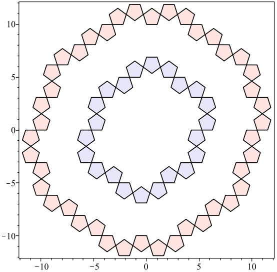
Theorem E: existence of non-periodic orbits

Any point in $\overline{\mathcal{F}} \setminus \mathcal{F}$ is not periodic.

We SUSPECT that when $\alpha = 2\pi\frac{p}{q}$ then $\overline{\mathcal{F}} = \mathcal{F}$, that is, $\mathcal{F}$ is closed and $\overline{\mathcal{F}} \setminus \mathcal{F} = \emptyset$.

Sketch of the proof of Theorem D

- Lemma: any connected component V of $\widetilde{\mathcal{U}}$ is bounded [GQ].
- Due to the existence of the invariant necklaces we obtain that $\cup_{k=0}^{\infty} F^k(V)$ is contained in a bounded set.



- $F^k\big|_V$ is an affine isometry $\Longrightarrow$ the area of $F^k(V)$ is constant.
- Hence, *there must be an overlapping*, i.e. there exist n, m such that

$$F^n(V) \cap F^{n+m}(V) \neq \emptyset.$$

Sketch of the proof of Theorem D

• Lemma: all the points in a connected set that does not intersect $\mathcal{F}$ share itinerary.

If $\underline{I}$ is the itinerary of $F^n(V)$ then $\underline{I} = S^m(\underline{I})$, where S is the shift $\Rightarrow$ $\underline{I}$ is periodic.

• Lemma: The set of points in $\widetilde{\mathcal{U}}$ sharing itinerary is convex $\Rightarrow$ V is convex.

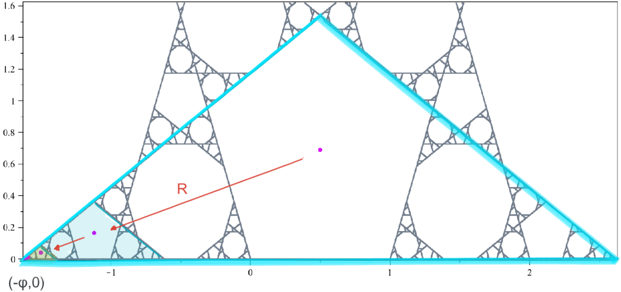
• Lemma: if $\underline{I(z)} = (s_1 \ldots, s_\ell)^\infty$ is periodic, then z is periodic.

It holds that for some natural ℓ, $F^\ell(z) = \lambda z + b$, with $\lambda = \mathrm{e}^{i\ell 2\pi p/q}$ and $b \in \mathbb{C}$. If $\lambda \neq 1$, the map is a rotation with center in the set (because of its *convexity*) and, in consequence, periodic. If $\lambda = 1$ then $b = 0$ due to the *boundeness* and so, again, periodic.

Finally, all points in $F^n(V)$ are periodic $\Rightarrow$ by bijectivity, all points in V are periodic.

Numerical evidences: fractalization and unboundedness of periods in compact sets.

For $\alpha = 8\pi/5$, we found an scale factor of $1/\varphi^3$ between the blue triangle and a (seemingly) infinite sequence of nested triangles, where φ is the *golden ratio*.



In fact, $R(x,y) = ((2\varphi - 3)x + 2 - 2\varphi, (2\varphi - 3)y)$.

This will allow to obtain a (seemingly) infinite sequence of periodic points with unbounded periods in a compact set of $\widetilde{\mathcal{U}} = \mathbb{R}^2 \setminus \overline{\mathcal{F}}$.

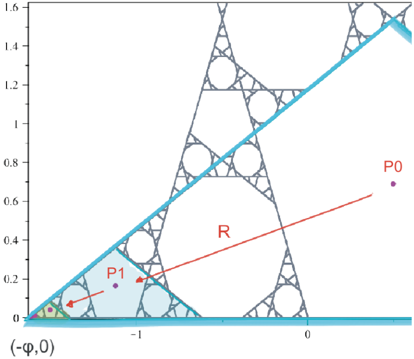
This sequence seems to converge to the point $(-\varphi, 0)$.

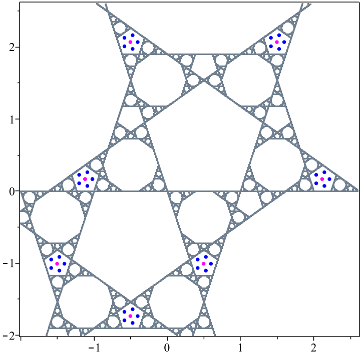
We consider the fixed point which is the center of the big pentagon $P_0 = \left(\frac{1}{2}, \frac{1}{10}(2+\varphi)^{\frac{3}{2}}\right)$.

Then $P_1 = R(P_0)$ is the center of a second pentagon and it is 7-periodic.

Hence we have a 7-periodic inter-tile dynamics. Its itinerary map is a 5-order rotation

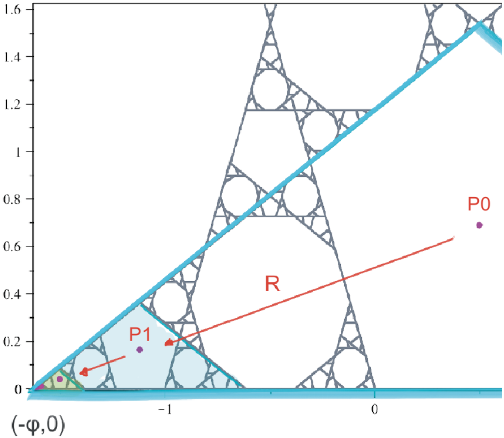
In consequence, there exists 7 pentagons filled by 35-periodic orbits.



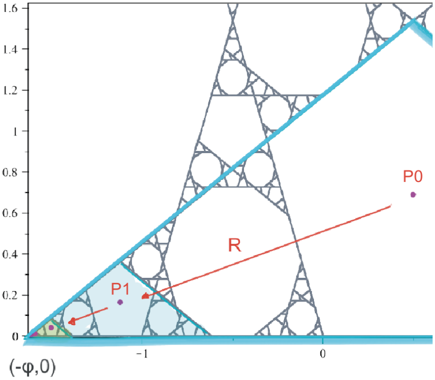


$P_2 = R^2(P_0)$ is 38-periodic, $P_3 = R^3(P_0)$ is 232-periodic,
$P_4 = R^4(P_0)$ is 1338-periodic, $P_5 = R^5(P_0)$ is 8332-periodic,
$P_6 = R^6(P_0)$ is 49988-periodic, $P_7 = R^7(P_0)$ is 299932-periodic,
$P_8 = R^8(P_0)$ is 1799588-periodic, $P_9 = R^9(P_0)$ is 10797532-periodic...

It seems that $\frac{\text{period}(P_{n+1})}{\text{period}(P_n)} \to 6$.

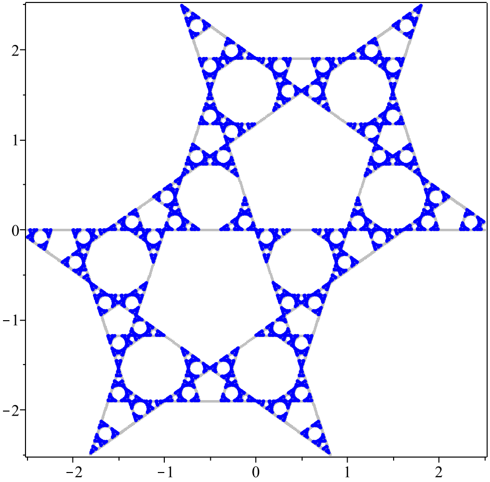


In a similar way, we can obtain points with (apparently) arbitrarily large periods in the critical set $\mathcal{F}$.



The sequence $P_{n+1} = R(P_n)$ converges to the point $(-\varphi, 0) \in \mathcal{F}$, which is a candidate to be a non-periodic point.

In fact, in Chang, Cheng, Yeh, J. Difference Eq. Appl. 25 (2019), it is claimed that this point is not-periodic, but from our point of view their proof is not complete.



Some points of the orbit of the point $(-\varphi, 0)$ in $\mathcal{F}$.

We can keep track the orbit using symbolic computation because

$$F^n(-\varphi, 0) = \left(a_n + b_n\varphi,\ (c_n + d_n\varphi)\ \sqrt{2+\varphi}\right) \text{ with } a_n, b_n, c_n, d_n \in \mathbb{Q}.$$

Thank you very much for you attention!