Attracting fixed points for the Buchner-Żebrowski equation: the role of negative Schwarzian derivative

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Notation and basic notions

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Our general aim: to study the dynamics (the asymptotic behaviour) of the orbits of (DE).

Local and global attraction

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Our more specific aim: to study whether local attraction may imply global attraction for (DE).

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(S3) $Sf(x) < 0$ for any $x \in I$ (except possibly at c). Here, *the Schwarzian derivative of f at x*, $Sf(x)$, is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

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Theorem (Allwright, Singer 1978)

Consider the order 1 equation

$$x_{n+1} = f(x_n), \quad n \geq 0, \quad x_0 \in I; \tag{FO}$$

If f belongs to the class S and u is a local attractor for (FO), that is, $|f'(u)| \leq 1$, then u is a global attractor of (FO).

A map of the class S: the Ricker map

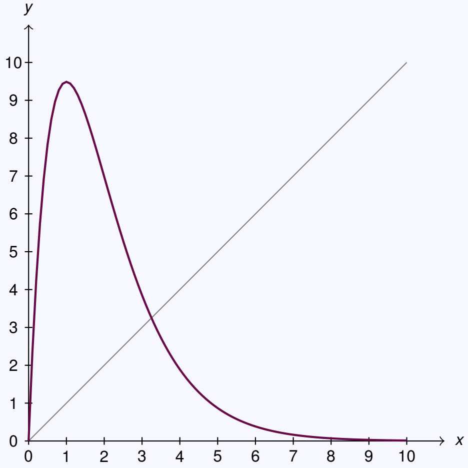

Figure 1: The Ricker map $f(x) = pxe^{-qx}$ with $p = e^{3.25}$, $q = 1$, $u = 3.25$.

The Buchner-Żebrowski equation

The Buchner-Żebrowski equation (Buchner & Żebrowski 2000):

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Motivation:

- Control of chaos: sometimes (FO) behaves “chaotically”, while (BZ_k) does not.
- Applications in digital filter design.

The "physics" of (BZ_k)

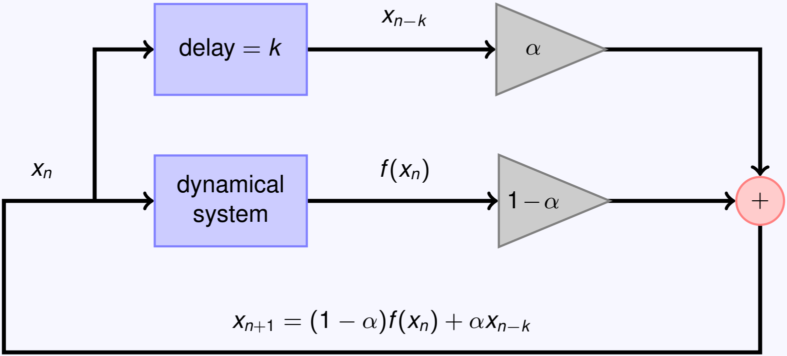

Figure 2: Block diagram for the Buchner-Żebrowski control law.

The domination property

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Theorem (El-Morshedy & Jiménez López 2008)

If u is L.A.S (resp. G.A.S) for (FO), then it is also L.A.S (resp. G.A.S) for (BZ$_k$). In particular, if f belongs to the class S and $|f'(u)| \leq 1$, then u is G.A.S for (BZ$_k$).

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Theorem 1 (E.B. & Jiménez López 2025)

When k is odd, even more is true: the fixed point u is L.A.S (resp. G.A.S) for (FO) $\Leftrightarrow$ u is a L.A.S (resp. G.A.S) for (BZ$_k$).

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When k is odd, even more is true: the fixed point u is L.A.S (resp. G.A.S) for (FO) $\Leftrightarrow$ u is a L.A.S (resp. G.A.S) for (BZ_k).

On the other hand, it is quite possible that, when k is even, u is locally attracting for (BZ_k), while it is unstable (in particular, non-attracting) for (FO).

The specific aim of this work

Our precise aim: to study whether L.A.S. implies G.A.S. for (BZ_k) when k is even, f belongs to the class S and $|f'(u)| > 1$.

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It is also easy to prove that if $f'(u) > -1$, (BZ_k) is unstable. So:

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The Clark equation (Clark 1976):

$$x_{n+1} = \alpha x_n + (1-\alpha) f(x_{n-k}), \quad 0 < \alpha < 1 \qquad (\mathrm{CE}_k)$$

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V. Jiménez Lopez & E. Parreño, *L.A.S. and negative Schwarzian derivative do not imply G.A.S. in Clark's equation*, J. Dyn. Diff. Equat., 2016.

Local attraction for the Buchner-Żebrowski equation

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Let $(r_k(\Theta), \alpha_k(\Theta))$ be given by

$$r_k(\Theta) = -\frac{\sin(\frac{\Theta}{2})}{\sin(\frac{(k-1)\Theta}{2(k+1)})}, \qquad \alpha_k(\Theta) = \frac{\sin(\frac{\Theta}{k+1})}{\sin(\frac{k\Theta}{k+1})}, \qquad \Theta \in [0, \pi].$$

Note that $r_k(\Theta)$ maps increasingly $[0, \pi]$ onto $[-\frac{k+1}{k-1}, -\frac{1}{\cos(\frac{\pi}{k+1})}]$. The curve $(r_k(\Theta), \alpha_k(\Theta))$ can also be seen as the graph of an increasing function $\alpha = a_k(r)$, $r \in [-\frac{k+1}{k-1}, -\frac{1}{\cos(\frac{\pi}{k+1})}]$, with

$$a_k(-\tfrac{k+1}{k-1}) = \tfrac{1}{k}, \quad a_k(-\tfrac{1}{\cos(\frac{\pi}{k+1})}) = 1.$$

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Theorem (Kuruklis 1994)

Let $r = f'(u)$. Then u is locally attracting (respectively, non-attracting) for (BZ_k) if $\frac{r+1}{r-1} < \alpha < a_k(r)$ (respectively, $\alpha > a_k(r)$ or $\alpha < \frac{r+1}{r-1}$).

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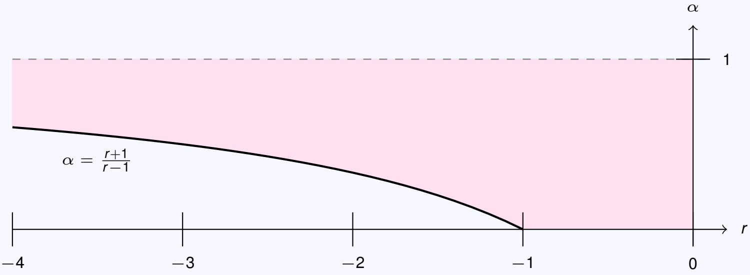

Figure 5: Local attraction (in green) for the Buchner-Żebrowski equation; “pink” means that this attraction is known to be global (when f belongs to the class S).

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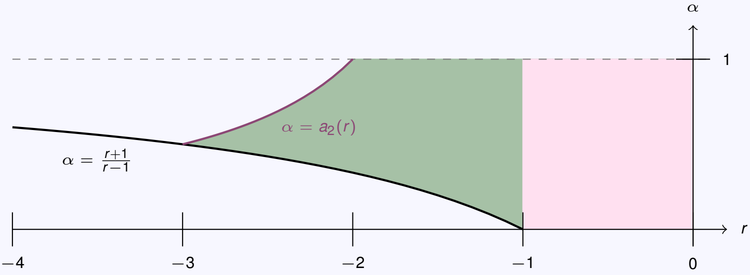

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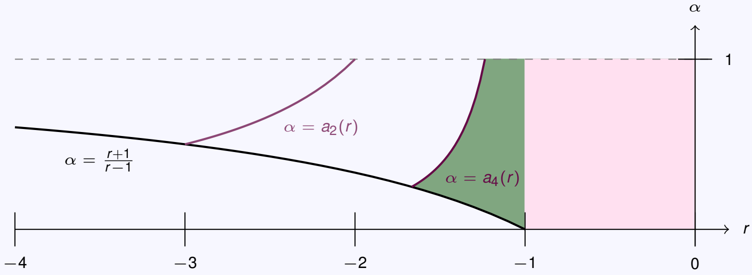

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On the Neimark-Sacker bifurcation

A natural way to investigate the conjecture is to study the bifurcation arising at $\alpha = a_k(r)$. It turns out that, under generic conditions, a *Neimark-Sacker bifurcation* arises involving the appearance of an invariant curve near the fixed point u.

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For the Buchner-Żebrowski equation:

If $\epsilon > 0$ is small enough, then two possibilities arise:

- if $a_k(r) < \alpha < a_k(r) + \epsilon$, then there is an invariant (attracting) curve near u; if $a_k(r) - \epsilon < \alpha \leq a_k(r)$, then there is no invariant curve near u (*supercritical N-S bifurcation*).
- if $a_k(r) \leq \alpha < a_k(r) + \epsilon$, then there is no invariant curve near u; if $a_k(r) - \epsilon < \alpha < a_k(r)$, then there is a (non-attracting) invariant curve near u (*subcritical N-S bifurcation*)

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- if $a_k(r) \leq \alpha < a_k(r) + \epsilon$, then there is no invariant curve near u; if $a_k(r) - \epsilon < \alpha < a_k(r)$, then there is a (non-attracting) invariant curve near u (*subcritical N-S bifurcation*)

In the supercritical case the conjecture is reinforced; in the subcritical case the conjecture is disproved!

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$$B_k(\Theta) = \frac{(k+1)\sin(\frac{\Theta}{k+1})\sin(\frac{k\Theta}{k+1})}{(k+1)\sin(\frac{\Theta}{k+1})\cos(\frac{k\Theta}{k+1}) - \sin\Theta} \cdot \frac{4\sin(\frac{\Theta}{2(k+1)})\cos(\frac{k\Theta}{2(k+1)})\cos(\frac{(k-1)\Theta}{2(k+1)})}{1 + 8\sin(\frac{\Theta}{2(k+1)})\cos(\frac{k\Theta}{2(k+1)})\sin(\frac{\Theta}{2})},$$

$$D_k(\Theta) = \frac{\sin(\frac{\Theta}{2})}{\sin(\frac{k\Theta}{2(k+1)})\cos(\frac{\Theta}{2(k+1)})}.$$

On the Neimark-Sacker bifurcation

The role of Schwarzian derivative

$$\Sigma f(u) := \frac{f'''(u) f'(u)}{(f''(u))^2}$$

If $f''(u) = 0$:

- if $f'''(u) < 0$, then $\Sigma f(u) := \infty$
- if $f'''(u) = 0$, then $\Sigma f(u) := \frac{3}{2}$
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Theorem 2 (E.B. & Jiménez López & 2025)

Let Θ be such that $f'(u) = r = r_k(\Theta)$. Then (BZ_k) exhibit a supercritical (respectively, a subcritical) Neimark-Sacker bifurcation at $\alpha = a_k(r) = \alpha_k(\Theta)$ if $\Sigma f(u) < N_k(\Theta)$ (respectively, if $N_k(\Theta) < \Sigma f(u)$).

Does L.A. imply G.A.?: the Buchner-Żebrowski equation case

The important things about the maps $N_k(\Theta)$:

- they are strictly increasing;
- we have

$$N_k(0) = \frac{3(k-3)(k+1)}{2(k-1)k}$$
$$< \frac{3 + 4\sin^2(\frac{\pi}{2(k+1)})(4 + \cos(\frac{\pi}{k+1}) + \sin(\frac{\pi}{k+1})\tan(\frac{k\pi}{k+1}))}{(2 + 16\sin^2(\frac{\pi}{2(k+1)}))\cos^2(\frac{\pi}{2(k+1)})} = N_k(\pi) < \frac{3}{2};$$

- $N_k(\Theta) \to \frac{3}{2}$ as $k \to \infty$ (uniformly).

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For instance,

- $N_2(\Theta) \geq -9/4$,
- $N_4(\Theta) \geq 5/8$,
- $N_6(\Theta) \geq 21/20$...

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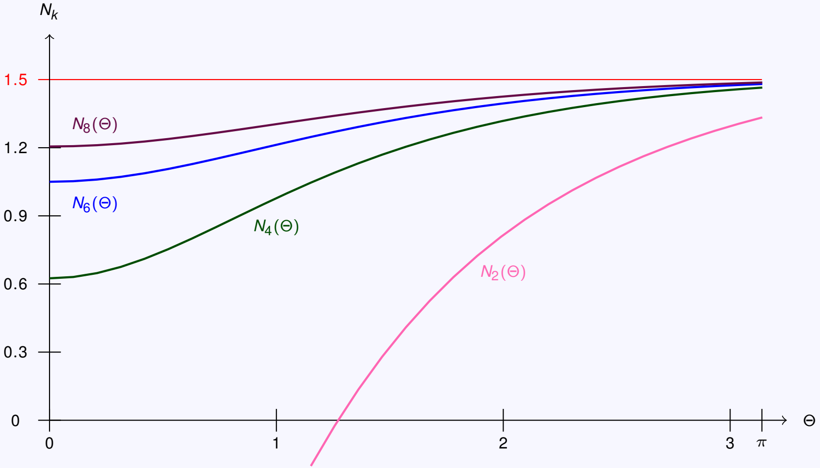

Figure 7: Graphs of maps $N_k(\Theta)$, $k = 2, 4, 6, 8$.

Does L.A. imply G.A.?: the Buchner-Żebrowski equation case

Recall:

- If $\Sigma f(u) < N_k(\Theta)$ (resp., if $N_k(\Theta) < \Sigma f(u)$), then the bifurcation is supercritical (resp., subcritical);
- $\Sigma f(u) < \frac{3}{2}$ means that $Sf(u) < 0$;
- $N_k(\Theta)$ is smaller than $\frac{3}{2}$ and goes to $\frac{3}{2}$ as $k \to \infty$.

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Theorem 3 (E.B. & Jiménez López 2025)

We have:

(a) the larger the delay k, the higher the chances that the bifurcation is supercritical;

(b) If $Sf(u) > 0$, then the bifurcation, regardless k, is always subcritical;

(c) If $Sf(u) < 0$ and k is large enough, then the bifurcation is always supercritical.

Does L.A. imply G.A.?: the Buchner-Żebrowski equation case

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- $k = 2$: the bifurcation is supercritical if $20.0855 = e^3 < p < 38.7047$ and subcritical if $38.7047 < p < e^4 = 58.5982$;

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- $k = 2$: the bifurcation is supercritical if $20.0855 = e^3 < p < 38.7047$ and subcritical if $38.7047 < p < e^4 = 58.5982$;
- $k \geq 4$: the bifurcation is supercritical.

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THANK YOU VERY MUCH
FOR YOUR KIND ATTENTION!

The Clark equation

The Clark equation (Clark 1976):

$$x_{n+1} = \alpha x_n + (1-\alpha) f(x_{n-k}), \quad 0 < \alpha < 1 \qquad (\mathrm{CE}_k)$$

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Motivation:

- x_n represents the number of adult members of the population in the year n, α is the annual survival rate, and $h = (1-\alpha)f$ is the recruitment function, which depends on the number of adults k years before.
- It is the discretization of some famous delay differential equations (Gurney, Blythe & Nisbet 1980, — Nicholson's blowflies—, Mackey-Glass 1977 —hematopoiesis—...).

Does local attraction imply global attraction?: the Clark equation case

Theorem

Let Θ be such that $f'(u) = r = r_k(\Theta)$. Then (BZ_k) exhibit a supercritical (respectively, a subcritical) Neimark-Sacker bifurcation at $\alpha = a_k(r) = \alpha_k(\Theta)$ if $\Sigma f(u) < N_k(\Theta)$ (respectively, if $N_k(\Theta) < \Sigma f(u)$).

The important things about the maps $N_k(\Theta)$:

$$N_k(\pi) = 3/2$$

$$N_k'(\pi) = \frac{1}{4\sin(\frac{\pi}{k+1})}(1 - \cos(\tfrac{\pi}{k+1}))(2\cos(\tfrac{\pi}{k+1}) - 1)$$

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In particular, $N_k'(\pi) > 0$ for any $k \geq 3$.

Does local attraction imply global attraction?: the Clark equation case

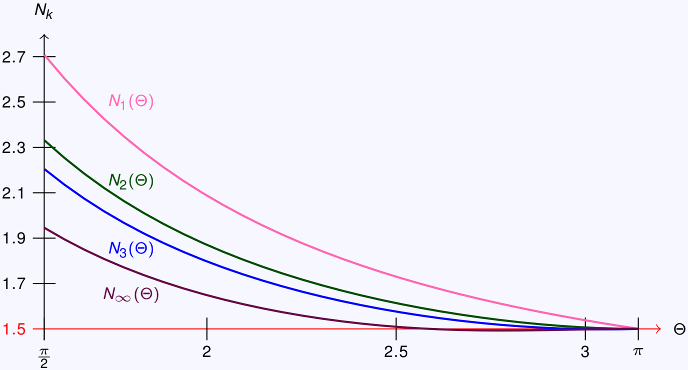

Figure 6: Graphs of maps $N_k(\Theta)$, $k = 1, 2, 3$, and $N_\infty(\Theta) := \lim_{k\to\infty} N_k(\Theta)$.

Does local attraction imply global attraction?: the Clark equation case

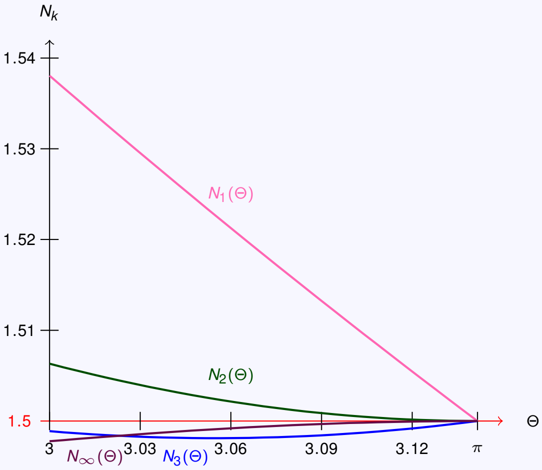

Figure 6: Graphs of maps $N_k(\Theta)$, $k = 1, 2, 3, \infty$ (detail).

L.A. and negative Schwarzian derivative *should* imply G.A.!

Recall:

- If $\Sigma f(u) < N_k(\Theta)$ (resp., if $N_k(\Theta) < \Sigma f(u)$), then the bifurcation is supercritical (resp., subcritical);
- $\Sigma f(u) < \frac{3}{2}$ means that $Sf(u) < 0$;
- $N_k(\Theta)$ is "almost always" greater than $\frac{3}{2}$.

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Theorem (Jiménez López & Parreño 2016)

Assume that one of the following conditions holds:

(a) $k \leq 2$ and $Sf(u) < 0$;

(b) $f'(u) < -1.18$ and $Sf(u) < 0$;

(c) $\Sigma f(u) < 1.49$.

Then (CE_k) exhibits a supercritical Neimark-Sacker bifurcation at $\alpha = a_k(r)$, $r = f'(u)$.

L.A. and negative Schwarzian derivative *should* imply G.A.!

An example: the Ricker map

The Ricker map $f(x) = pxe^{-qx}$, $x \in I = (0, \infty)$, belongs to the class S for any $p > 1, q > 0$. We have:

- $u = \frac{\log p}{q}$,
- $f'(u) = 1 - \log p$,
- $\Sigma f(u) = 1 - \frac{1}{(2-\log(p))^2}$;

hence $f'(u) < -1$ and $\Sigma f(u) < 1$ whenever $p > e^2$, $q > 0$.

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hence $f'(u) < -1$ and $\Sigma f(u) < 1$ whenever $p > e^2$, $q > 0$.

In particular, the bifurcation is always supercritical.

L.A. and negative Schwarzian derivative need not imply G.A.!

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Theorem 3 (Jiménez López & Parreño 2014)

Let f_ϵ, $0 < \epsilon < \epsilon_0$, be C^4 maps. Assume that for any ϵ there is $u_\epsilon \in I$ such that the following conditions are satisfied for $D(\epsilon) := f'_\epsilon(u_\epsilon)$, $T(\epsilon) := \Sigma f_\epsilon(u_\epsilon)$:

(i) $f_\epsilon(u_\epsilon) = u_\epsilon$;

(ii) $\lim_{\epsilon\to 0} D(\epsilon) = -1$, $\lim_{\epsilon\to 0} D'(\epsilon) = d < 0$;

(iii) $\lim_{\epsilon\to 0} T(\epsilon) = 3/2$, $\lim_{\epsilon\to 0} T'(\epsilon) = 0$.

Then, if $k \geq 3$, $\epsilon > 0$ is small enough and we put $h = h_\epsilon$, $u = u_\epsilon$, (CE$_k$) exhibits a subcritical Neimark-Sacker bifurcation at $\alpha = a_k(r)$, $r = f'(u)$. In particular, if $\alpha > a_k(r)$ is close enough to $a_k(r)$, then u is a local, but not global, attractor of (CE$_k$).

L.A. and negative Schwarzian derivative need not imply G.A.!

A simple example belonging to the class S

Let

$$f_\epsilon(x) = \frac{1}{(1-2\epsilon)(\epsilon+(1-\epsilon)x)+2\epsilon(\epsilon+(1-\epsilon)x)^2},$$

with:

- $k = 3$,
- $\epsilon = 0.00167086$,
- $u_\epsilon = 1$,
- $\alpha = 0.00573994$.

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In this case, the bifurcation is subcritical.