# Fractional variable-order discrete-time equation with distributed delays

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# Outline

# Oblivion function

Given $k, l \in \mathbb{Z}$ and an order function $\nu(\cdot)$, we define *oblivion function* as a discrete function of two variables, specified by its values for any $l \in \mathbb{Z}$. In particular, for $k < 0$ we set $a^{\nu(i)}(k) = 0$, and $a^{\nu(i)}(0) = 1$, while for $k > 0$ we define

$$a^{\nu(i)}(k) = (-1)^k \frac{\nu(i)\,(\nu(i)-1)\cdots(\nu(i)-k+1)}{k!}\,. \tag{1}$$

# Oblivion function

Formula (1) can also be expressed using the following recurrence relation for $k \in \mathbb{N}$:

$$\begin{aligned} a^{\nu(i)}(0) &= 1\,, \\ a^{\nu(i)}(k) &= a^{\nu(i)}(k-1)\left[1 - \frac{\nu(i)+1}{k}\right] \quad \text{for } k \geqslant 1\,. \end{aligned} \tag{2}$$

# Fractional variable-order sum of convolution type (FVOS)

## Definition 1

*Let $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$. For a function $x : \mathbb{Z} \to \mathbb{R}$, the fractional variable-order sum of convolution type (FVOS) is defined as:*

$$\Delta^{-\nu(\cdot)} x(k) := \sum_{i=0}^{k} a^{\nu(i)}(i) x(k-i) \, .$$

Note that the FVOS can be interpreted as a discrete convolution:

$$\Delta^{-\nu(\cdot)} x(k) = (a * x)(k) \, .$$

## $\mathcal{Z}$-transform of FVOS

Given the convolution structure, the Z-transform of the fractional sum can be written as:

$$\mathcal{Z}\left[\Delta_h^{-\nu(\cdot)}x\right](z) = X(z)\mathcal{A}_1(z)\,, \tag{3}$$

where $X(z) := \mathcal{Z}[x](z)$ and

$$\mathcal{A}_1(z) := \sum_{i=0}^{\infty}(-1)^i\binom{-\nu(i)}{i}z^{-i}\,.$$

- If the order function is constant, i.e. $\nu(k) \equiv \alpha$, then equation (3) simplifies to:

$$\mathcal{Z}[y](z) = \left(\frac{z}{z-1}\right)^{\alpha} X(z)\,.$$

# The Caputo-type fractional variable-order difference operator of convolution type

## Definition 2

*Let $\nu : \mathbb{Z} \to (q-1, q]$, where $q \in \mathbb{N}_1$. The Caputo fractional variable-order h-difference operator of convolution type (CFVOD) with order function $\nu(\cdot)$ for a function $x : \mathbb{Z} \to \mathbb{R}$ is defined as*

$$\Delta^{\nu(\cdot)} x(k) = \Delta^{-(q-\nu(\cdot))} \Delta^{q} x(k) . \tag{4}$$

## Remarks

- For $q = 1$:

$$\Delta_h^{\nu(\cdot)} x(kh) = \Delta_h^{-(1-\nu(\cdot))} \Delta_h x(k) \, .$$

- If $\nu(k) \equiv q \in \mathbb{N}_1$, then:

$$\Delta^{\nu(\cdot)} x(k) = \Delta^q x(k) \, .$$

- For $q = 1$, the Z-transform representation is:

$$\mathcal{Z}\left[\Delta^{\nu(\cdot)} x\right](z) = \left((z-1)X(z) - zx(0)\right) \mathcal{A}(z) \, ,$$

where $X(z) = \mathcal{Z}[x](z)$,

$$\mathcal{A}(z) := \sum_{i=0}^{\infty} (-1)^i \binom{\nu(i) - 1}{i} z^{-i} \, .$$

# Discrete-time systems with the Caputo fractional variable-order operator

Let us consider a discrete-time system with variable-order dynamics of the form:

$$\Delta^{\nu(\cdot)}x(kh) = f(x(k)), \quad k \geq 1, \tag{5}$$

with the initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $\nu : \mathbb{Z} \to \mathbb{R}_+ \cup \{0\}$ is the order function, and $x : \mathbb{N}_0 \to \mathbb{R}^n$ is the state variable. Then

$$x(1) = x(0) + f(x(0)),$$

$$x(k) = x(k-1) + f(x(k-1)) - \sum_{i=1}^{k-1} a^{\nu(k-i)-1}(k-i)\,(x(i) - x(i-1))$$

$k \geq 2$

with the initial value $x(0) = x_0 \in \mathbb{R}^n$ given.

# Interpretation of final term

The final term in the second equation can be interpreted as a type of input or actuator signal:

$$u(k) = -\sum_{i=1}^{k-1} a^{\nu(k-i)-1}(k-i)\,(x(i) - x(i-1))$$

We also assume $\nu(0) = 1$ in numerical simulations.

# Type of an order function

The order function can be given a piecewise-constant function over specific time intervals:

$$\nu(k) = \begin{cases} \nu_1, & \text{for } k < k_1, \\ \nu_2, & \text{for } k_1 \leq k < k_2, \\ \vdots & \\ \nu_n, & \text{for } k_{n-1} \leq k < k_n, \\ 1, & \text{for } k \geq k_n, \end{cases} \tag{6}$$

where $k_i \in \mathbb{N}$ and $0 \leq k_1 < k_2 < \ldots < k_n$.
Then, the sum in $\mathcal{A}(z)$ is finite, and the convergency domain of the series is the whole plane.

# Conditions for stability of linear discrete-time

Let us consider the system with a variable-order:

$$\Delta^{\nu(\cdot)}x(k) = Ax(k)\,,\ \ k \geq 1 \tag{7}$$

## Proposition 3 ( [11])

*Let* $\mathrm{spec}(A) = \{\lambda_i :\ \ i = 1, \ldots, k\}$, $k \leq n$.

(a) *If for all* $i = 1, \ldots, k$ *we have that*

$$\lambda_i \in \{(z-1)\mathcal{A}(z) :\ \ |z| < 1\}\ , \tag{8}$$

*then system (7) is asymptotically stable.*

(b) *If there exists* $i \in \{1, \ldots, k\}$ *such that*

$$\lambda_i \in \{(z-1)\mathcal{A}(z) :\ \ |z| > 1\}\ , \tag{9}$$

*then system (7) is unstable.*

## Scalar equations

Now we consider the situation for scalar equations. Let us consider such equation with a variable-order of the following form:

$$\Delta^{\nu(\cdot)}x(k) = \lambda x(k)\,,\ \ k \geq 1, \tag{10}$$

with initial condition $x(0) = x_0 \in \mathbb{R}$, where $\nu : \mathbb{Z} \to [0,1]$ is an order function, $x : \mathbb{N}_0 \to \mathbb{R}$ is a state function and $\lambda \in \mathbb{R}$. Then, the condition for the asymptotic stability of equation (10).

### Proposition 4 ( [11])

*If*

$$-2\sum_{i=0}^{\infty}\binom{\nu(i)-1}{i} < \lambda < 0\,, \tag{11}$$

*then equation (10) is asymptotically stable.*

## Equations with one distributed delay

We consider the following delay differential equation with one distributed delay:

$$\Delta^{\nu(\cdot)}x(k) = f\left(x(k), \sum_{s=0}^{\infty} h(s)x(k-s)\right), \quad k \in \mathbb{N}, \tag{12}$$

where the delay kernel $h : [0, \infty) \to [0, \infty)$ is a discrete probability function.

The delay kernel is assumed to be bounded, satisfying

$$\sum_{s=0}^{\infty} h(s) = 1,$$

with the average time delay $\tau = \sum_{s=0}^{\infty} sh(s) < \infty$.
The initial condition associated with the difference equation with infinite delay (12) is of the following form:

$$x(k) = \psi(k), \quad t \in (-\infty, 0], \tag{13}$$

where $\psi : (-\infty, 0] \to \mathbb{R}$ is a discrete function with the property that there exists $\mu > 0$ such that $\lim_{t\to-\infty} e^{\mu t}\psi(t) = 0$. In other words, the initial function $\psi$ belongs to the Banach space $C_{0,\mu}(\mathbb{R}_-, \mathbb{R})$ endowed with the norm:

$$\|\psi\|_{\infty,\mu} = \sup_{t\in(-\infty,0]} e^{\mu t}|\psi(t)|.$$

## Linearization

Assume that equation (12) has an equilibrium point $x^\star$, that is, $f(x^\star, x^\star) = 0$. Linearizing the system around this equilibrium point, leads to a linear fractional difference equation with distributed delay:

$$\Delta^{\nu(\cdot)}x(k) = \alpha x(k) + \beta \sum_{s=0}^{\infty} h(s)x(k-s) \qquad (14)$$

where $\alpha = \frac{\partial f}{\partial x_1}(x^\star, x^\star)$ and $\beta = \frac{\partial f}{\partial x_2}(x^\star, x^\star)$.

## $\mathcal{Z}$-transform of the linearized equation

Taking $\mathcal{Z}$-transform we receive

$$((z-1)\mathcal{A}(z)-\alpha-\beta H(z))\,X(z)=zx(0)\mathcal{A}(z)\,, \qquad (15)$$

where $E(z)=\mathcal{Z}[\epsilon(\cdot)](z)$ and $H(z)=\mathcal{Z}[H(\cdot)](z)$.
Hence, we obtain the associated characteristic equation:

$$\Delta(z):=(z-1)\mathcal{A}(z)-\alpha-\beta H(z)=0. \qquad (16)$$

We investigate the stability region of the equilibrium point $x_*$ in terms of the characteristic parameters $\alpha$ and $\beta$. To have more flexibility of solutions we also consider the kernel on finite support. Then, the following result is possible.

## Proposition 5

*Let* $\beta \geq 0$, $A := \sum_{i=0}^{\infty} \binom{\nu(i)-1}{i}$, *and* $\beta < 2A + \beta H(-1)$, *then for* $p := -\alpha$, *and*

$$p \in (\beta, 2A + \beta H(-1)) \,, \tag{17}$$

*then equation (14) is asymptotically stable.*

## Example

Considering the expression: $2\sum_{i=0}^{\infty}\binom{\nu(i)-1}{i}+\beta H(-1)$ we have:

1. The order function $\nu(i)$ is defined piecewise as:

$$\nu(i)=\begin{cases}0.8 & \text{for } i\in[0,10],\\ 0.85 & \text{for } i\in[11,15],\\ 0.9 & \text{for } i\in[16,20],\\ 0.95 & \text{for } i\in[21,30],\\ 1.0 & \text{for } i>30\end{cases}$$

2. We compute an approximate sum up to $N=50$:

$$A:=\sum_{i=0}^{50}\binom{\nu(i)-1}{i}\approx 0.87585$$

3. We evaluate $H(-1)$ for the memory kernel:

$$h(0)=0,\quad h(k)=\frac{2^{k-1}}{63}\quad \text{for } k=1,\dots,6$$

We compute the final expression for selected values of $\beta$:

$$\beta = 0.8: \qquad 2A + \beta H(-1) \approx 2.0184$$
$$\beta = 0.1: \qquad 2A + \beta H(-1) \approx 1.7850$$
$$\beta = 0.01: \qquad 2A + \beta H(-1) \approx 1.7550$$

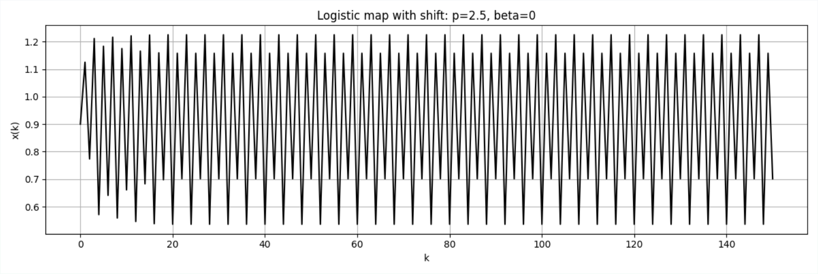


Figure 1: Solution $x(k)$ to logistic equation with shift $x(k) = x(k-1) + \alpha x(k)(1 - x(k))$ with $\alpha = 2.5, \beta = 0$.

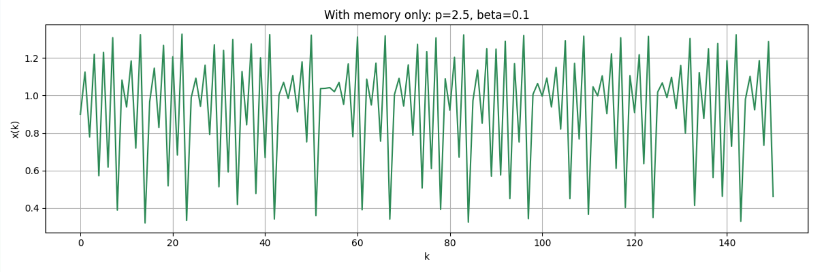


Figure 2: Solution $x(k)$ to logistic equation with memory, $\alpha = 2.5, \beta = 0.1$.

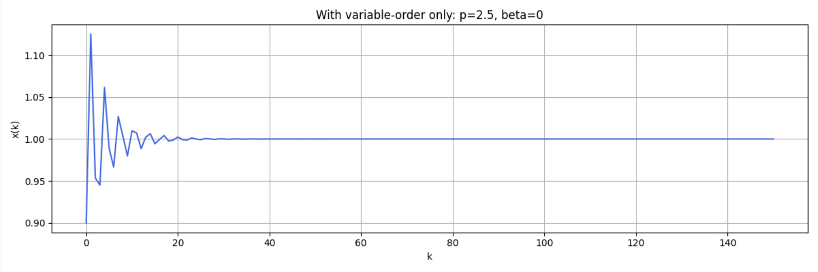


Figure 3: Solution $x(k)$ to logistic equation with variable order only, $\alpha = 2.5, \beta = 0$.

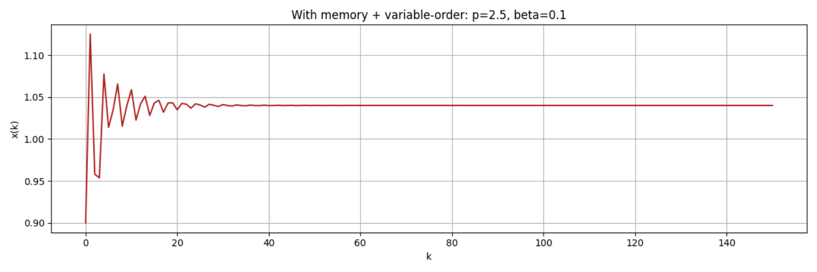


Figure 4: Solution $x(k)$ to logistic equation with memory and variable order, $\alpha = 2.5, \beta = 0.1$.

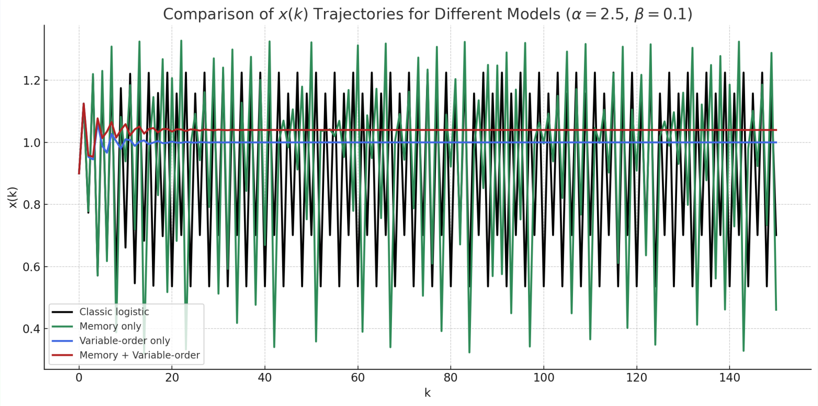


Figure 5: Solution $x(k)$ -comparison, $\alpha = 2.5, \beta = 0.1$.

## Conclusions

- The work addresses the area of fractional discrete-time systems with Caputo-type operators and infinite delay.
- The proposed framework lays a foundation for further investigation into the existence, uniqueness, and stability of solutions in such systems.
- It offers valuable information on the correct initialization of fractional differential equations with delay.
- Future efforts will focus on the development of numerical schemes for simulation and control design, as well as the practical implementation of these models in real-world applications.

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