Abundance of weird quasiperiodic attractors in piecewise linear discontinuous maps

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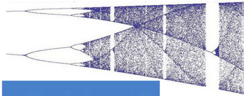

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Overview

- The existence of a new type of attractors, which appear chaotic but are not, has been observed in 2D discontinuous piecewise linear (PWL) homogeneous systems that model economic dynamics (see Gardini *et al.* (2024, 2025c));
- In a recent paper, Gardini *et al.* (2025a), we identified specific regions in the parameter space associated with this type of attractor, called weird quasiperiodic attractor (WQA), focusing on a 2D discontinuous PWL homogeneous map derived from the well-known 2D border collision normal form Simpson (2014); Simpson and Tuffley (2017)).
- **The goal of this work:** show that WQAs can be observed in a broad class of 2D PWL discontinuous maps, regardless of the type of borders of the partitions where different functions are defined.
- We also examine particular nongeneric cases where the attracting sets can be analyzed via a one-dimensional (1D) restriction of the map or a first return map, ultimately leading to a PWL circle map. These nongeneric cases have been recently investigated in Gardini *et al.* (2025b), where we describe the dynamics of the related class of 1D maps.
- Furthermore, we show that WQAs may also occur in three-dimensional maps and, more generally, may exist in n-dimensional (nD) maps.

A 2D discontinuous PWL map with a unique fixed point

Consider a 2D discontinuous PWL map (often referred to as piecewise affine), in which the functions are defined in two partitions and have the same real fixed point $(x, y) = (-\xi, -\eta)$:

$$M : \begin{cases} M_L : \begin{cases} x' = \tau_L x + y + (\tau_L \xi + \eta - \xi) \\ y' = -\delta_L x - (\delta_L \xi + \eta) \end{cases} & \text{for } x < h - \xi \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ M_R : \begin{cases} x' = \tau_R x + y + (\tau_R \xi + \eta - \xi) \\ y' = -\delta_R x \ - (\delta_R \xi + \eta) \end{cases} & \text{for } x > h - \xi \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases}$$

where the prime symbol $'$ denotes the unit time advancement operator, and $h \neq 0$.

The system (1) is topologically conjugate to the 2D discontinuous PWL homogeneous map:

$$T_1 = \begin{cases} T_L : X' = J_L X \ \text{ for } x < h, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \ \text{ for } x > h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{2}$$

Maps M and T_1 have the same dynamics, as they are topologically conjugate.

2D PWL maps with discontinuity set $x = -1$

Consider

$$T_1 = \begin{cases} T_L : X' = J_L X \text{ for } x < h, & J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \text{ for } x > h, & J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{3}$$

Here, the discontinuity set is the vertical line $x = h \neq 0$ and T_R and T_L have the same fixed point $(x, y) = (0, 0)$, denoted by O.

Remark (see Gardini *et al.* **(2025a)):** The peculiarity of this map is the emergence of a WQA: a new type of attractor that does not include any periodic point, thus, it is neither an attracting cycle nor a chaotic attractor.

Definition: A map is chaotic in a closed invariant set A if periodic points are dense in A and there exists an aperiodic trajectory dense in A (so that there is transitivity).

Remark: A WQA appears as the closure of quasiperiodic trajectories, where the term "weird" refers to the rather complex and often intricate geometric structure of these attractors.

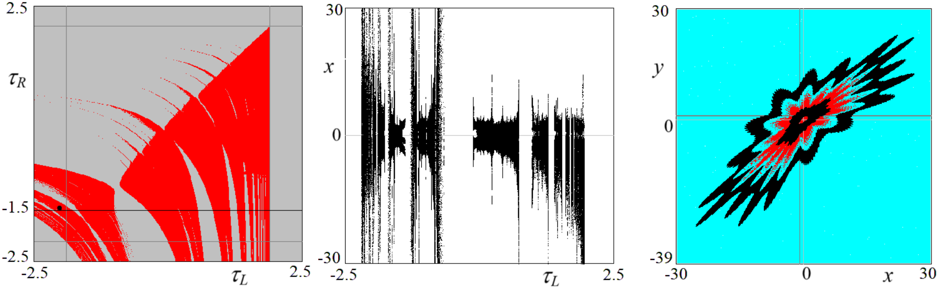
Remark: Note that if a 2D map has other invariant sets, where it is reducible to a 1D map (e.g., a closed invariant curve, or a set consisting of a finite number of segments), then the related attractors are not classified as weird (although these sets may coexist with a WQA).

2D PWL maps with discontinuity set $x = -1$

$$T_1 = \begin{cases} T_L : X' = J_L X \quad \text{for } x < h, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \quad \text{for } x > h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases}$$

In Gardini *et al.* (2025a), $h = -1$ and we observed a WQA when O is:

- attracting for the two functions;
- attracting only for one function;
- repelling for the functions (in Figure below, saddle for T_L and unstable focus for T_R).



(a) (b) (c)

The class of 2D PWL maps

The characteristic property of the class of maps considered in this work is the same in any dimension:

Definition (class of maps)

We consider, for $n \geq 1$, the class of **nD discontinuous PWL maps** defined in a finite number of partitions by linear functions with the same real fixed point.

Remark: According to the definition, we consider PWL maps with a unique fixed point (hyperbolic or nonhyperbolic). Without loss of generality, we assume that the real fixed point is O (PWL map becomes homogeneous).

Application in:

- Economics, see Gardini *et al.* (2024) (WQA coexists with O always nonhyperbolic, resulting in a segment of fixed points)
- Engineering, see Kollar *et al.* (2004).

Remark: The discontinuity set can be a vertical line, a straight line $y = mx + q$ with $q \neq 0$, multiple straight lines, or any curve(s) (as e.g., a circle which we use in some examples).

WQA vs other attractors

Definition (1, WQA)

A WQA $\mathcal{A}$ of a map F:

- Is a topological attractor: a closed invariant ($F(\mathcal{A}) = \mathcal{A}$) set with a dense trajectory and an attracting neighborhood;
- Does not include periodic points;
- F on $\mathcal{A}$ is not homeomorphic to a 1D circle map.

Remark: WQAs are not SNA strange nonchaotic attractors (SNAs, see, e.g., Grebogi *et al.* (1984); Feudel *et al.* (2006); Duan *et al.* (2024))

Remark: We start considering 2D maps in the class defined above where bounded dynamics, not associated with the fixed point O, are either reducible to those of a PWL circle map (related to 1D maps) or give rise to 2D WQA.

Remark: WQAs have properties similar to the quasiperiodic trajectories occurring in 1D PWL circle maps. For this reason, they may be considered a generalization of such dynamics to the 2D phase plane.

A recap on 1D discontinuous PWL homogeneous maps

- The case in Definition 1 for $n = 1$ corresponds to the class of maps considered in Gardini *et al.* (2025b).
- A 2D map within our definition can lead to a 1D first return map in some segment that is a function corresponding to Definition 1 with $n = 1$.

In the case of with only one discontinuity point:

$$F = \begin{cases} F_L : x' = s_L x & \text{for } x < h \\ F_R : x' = s_R x & \text{for } x > h \end{cases} \quad h \neq 0 \tag{4}$$

where $h \neq 0$ is a scaling factor. Set $h > 0$ ($h < 0$ is topologically conjugate).

- When the slopes are positive, bounded asymptotic dynamics distinct from the fixed point O, can occur only if $F_L(h) > h > F_R(h)$, leading to $s_L > 1 > s_R$.
- In this case, the map is a circle homeomorphism ($F_R \circ F_L(h) = s_R s_L h = F_L \circ F_R(h)$) in the invariant absorbing interval $I = [F_R(h), F_L(h)] = [s_R h, s_L h]$.
- Its dynamics depend on the rotation number ρ, which is the same for any point of interval I (see de Melo and van Strien (1991)).
  - If ρ is rational, then I is filled with periodic points (of the same period).
  - If ρ is irrational, then I is filled with quasiperiodic orbits dense in the interval.

A recap on 1D discontinuous PWL homogeneous maps

- The generic case is an irrational rotation.
- Rational rotation is associated with a set of zero Lebesgue measure in the (s_R, s_L) parameter plane.
- In the class of 1D PWL Lorenz maps (with one discontinuity point), the case of a circle map denotes the transition from regular dynamics (in a gap map, where chaos cannot occur) to chaotic dynamics (in an overlapping map), see Rand (1978); Berry and Mestel (1991); Avrutin *et al.* (2019).

Theorem (1, from Gardini *et al.* (2025b))

Let G be a 1D discontinuous PWL homogeneous map as in Definition 1. Then:

(1) *A hyperbolic cycle different from the fixed point O cannot exist (and thus, a chaotic set cannot exist).*

(2) *The only possible bounded invariant sets of map G, different from those related to the fixed point O (whether hyperbolic or nonhyperbolic), are those occurring in a* **PWL circle map: Intervals densely filled with nonhyperbolic cycles or quasiperiodic orbits**. *Coexistence is possible.*

(3) *Quasiperiodic orbits lead to (weak) sensitivity to initial conditions.*

(4) *The Lyapunov exponent is zero.*

Let us move to 2D discontinuous PWL homogeneous maps

Lemma (1, General Properties)

Let T be an 2D map as in Definition 1. Then:

(i) *A hyperbolic cycle different from the fixed point O cannot exist (and thus, no chaotic set).*

(ii) *Segments of straight lines through the fixed point O are mapped into segments of straight lines through O.*

(iii) *Any composition of the linear functions defining map T preserves properties (i) and (ii).*

The same holds for a nD map as in Definition 1.

2D discontinuous PWL homogeneous maps: Main results

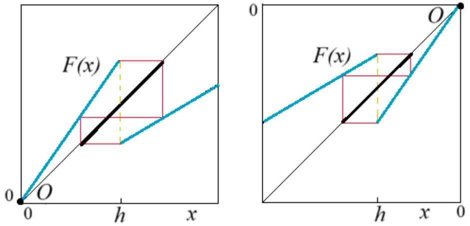
Theorem (2, Dynamics on Invariant Segments)

Let T be a 2D map as in Definition 1. Let r be a straight line through the fixed point O ($y = mx$), such that the first return of map T on a segment of r leads to a 1D map with a finite number of discontinuity points. Then the related dynamics of map T in the phase plane are either nonhyperbolic cycles (with eigenvalue 1) or quasiperiodic trajectories densely filling some segments.

Remark: Thus, when an invariant segment exists, the first return map can be reduced to a 1D map satisfying Definition 1 for $n = 1$.

Remark: A suitable segment exists where the first return is the simplest one, with only one discontinuity point, corresponding to the form of map F in (4) (see Gardini *et al.* (2025b)).

Remark: In case of only one discontinuity point and bounded dynamics, we should have as in the Figure below:



2D discontinuous PWL homogeneous maps: Main results

Theorem (3, Possible Bounded ω-limit Sets)

Let T be a 2D map as in Definition 1. Then:

- (j) *A bounded ω-limit set $\mathcal{A}$ different from the fixed point O and from local invariant sets associated with O when it is nonhyperbolic, can only be one of the following kind:*
  - (ja) *it is a nonhyperbolic k-cycle, $k \geq 2$, belonging to k segments not intersecting any border, filled with k-periodic points (all cycles have the same symbolic sequence);*
  - (jb) *it belongs to an invariant set on which the dynamics are reducible to a discontinuous 1D map;*
  - (jc) *it is a weird quasiperiodic attractor (WQA).*
- (jj) *When $\mathcal{A}$ does not consist in segments filled with cycles, then map T exhibits (weak) sensitivity to initial conditions in $\mathcal{A}$.*

Remark: The attracting sets related to (ja) and (jb) are not structurally stable (not persistent to parameter perturbation);

Remark: Escluding the fixed point, in the parameter space of the considered 2D maps, the generic attractor is a WQA.

2D PWL homogeneous maps with the same discontinuity set $x = -1$.

2D PWL maps with discontinuity set $x = -1$

Consider the map:

$$T_1 = \begin{cases} T_L : X' = J_L X \ \text{ for } x < -1, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \ \text{ for } x > -1, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases}$$

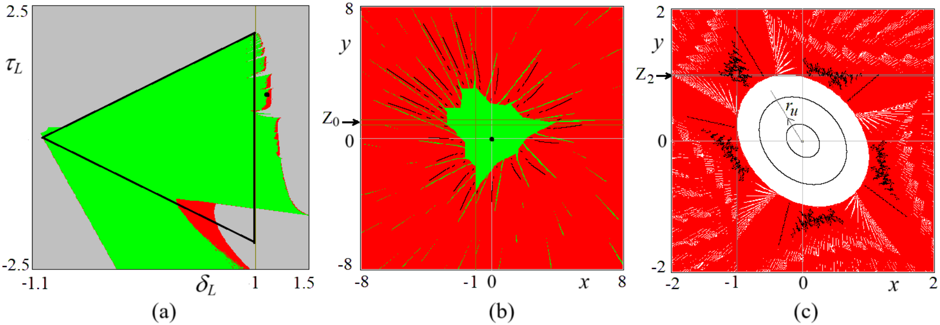
Properties:

- The discontinuity line (critical line) is $x = -1$ and its images by the two linear functions are

$$T_{L/R}(x = -1) : \quad y = \delta_{L/R} \tag{5}$$

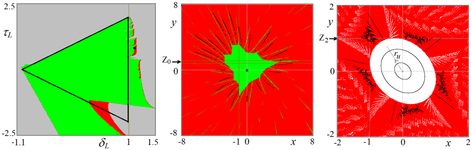
- Assuming that no eigenvalue is equal to zero (i.e. $\delta_{L/R} \neq 0$), each half-plane bounded by $x = -1$ is mapped by the linear function $T_{L/R}$ into a half-plane bounded by $y = \delta_{L/R}$.
- The relative positions of the half-planes determine the classification of the kind of map.
- For $\delta_L \neq \delta_R$, the map may be uniquely invertible or noninvertible.
- The strip bounded by $y = \delta_{L/R}$ may be a so-called region Z_0, whose points have no rank-1 preimage, or a region Z_2, whose points have two different rank-1 preimages.

2D PWL maps with discontinuity set $x = -1$



- Panel (a), stability triangle for map T_L.
- Panel (b), O repelling focus for T_L, O attracting focus (hyperbolic) for the map coexisting with a WQA.
- Panel (c), O is nonhyperbolic, a center (irrational rotation number) with invariant region filled with ellipses (on which the trajectories are quasiperiodic, see Sushko and Gardini (2008)) and bounded by an ellipse tangent to the discontinuity line and its images. A WQA exists.
- Panel (c), disconected basins of attraction as a portion of the invariant region is Z_2.

2D PWL maps with discontinuity set $x = -1$



(a) (b) (c)

Mechanism of appearance of the WQA in Panel (c):

- Consider $T = T_L \circ (T_R)^4$;
- O is a (virtual) saddle of T, with eigenvalues $\lambda_1 \simeq 0.94$ and $\lambda_2 \simeq 1.042$;
- For $0 < \lambda_2 < 1$, the invariant polygon attracts all the points of the phase plane.
- For $\lambda_2 = 1$, there exist five invariant segments (bounded on both sides), filled with 5-cycles, fixed points of map T.
- For $\lambda_2 > 1$, the trajectories on the eigenvector, say r_u, become repelling.
- Let P be the intersection point of eigenvector r_u with the discontinuity line, and P_{-1} its rank-1 preimage (which lies within the R partition).
- T maps segment $P_{-1}P$ into a segment PP_1 (along the eigenvector), which belongs to the L partition, where a different function is applied.
- Then the iterates of this segment converge to a WQA.
- We can say that the WQA is the ω-limit set of $T_1^n(PP_1)$, for $n \to \infty$.

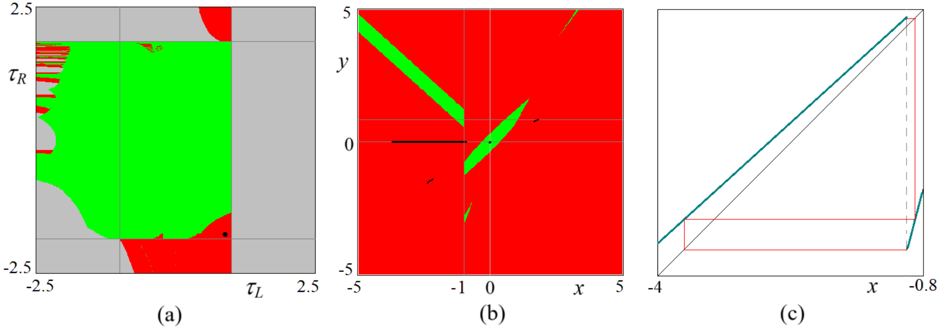
PWL map T_1 with $\delta_L = 0$

$$T_1 = \begin{cases} T_L : X' = J_L X \quad \text{for } x < -1, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ 0 & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \quad \text{for } x > -1, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases}$$

Properties:

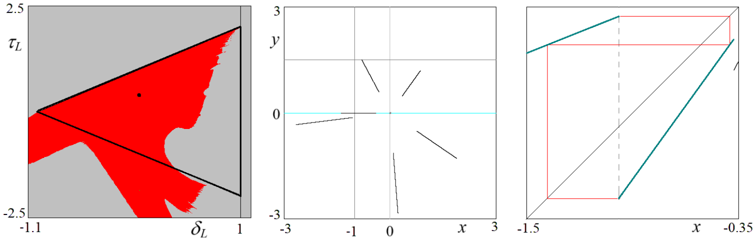
- T_L has an eigenvalue equal to zero, region L is mapped into the corresponding image of the discontinuity line, that is an eigenvector of the Jacobian matrix in that partition;
- Excluding O, bounded invariant set must necessarily have points on that critical line.
- The dynamics of the 2D map can be investigated via a 1D first return map on that line (critical line $y = 0$ in the specific case).

PWL map T_1 with $\delta_L = 0$



- Panel (a), 2D bifurcation diagram.
- Panel (c), first return map in the segment of the attractor that lies along the critical line $y = 0$, confirming that the dynamics of the attractor (distinct from the fixed point O) are those of a PWL circle map;
- Panel (b), attracting O that coexists with an attracting set (no WQA);

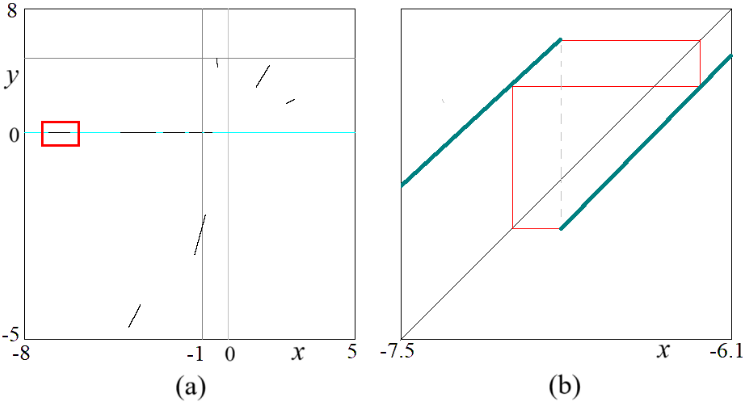
PWL map T_1 with $\delta_L = 0$



(a) (b) (c)

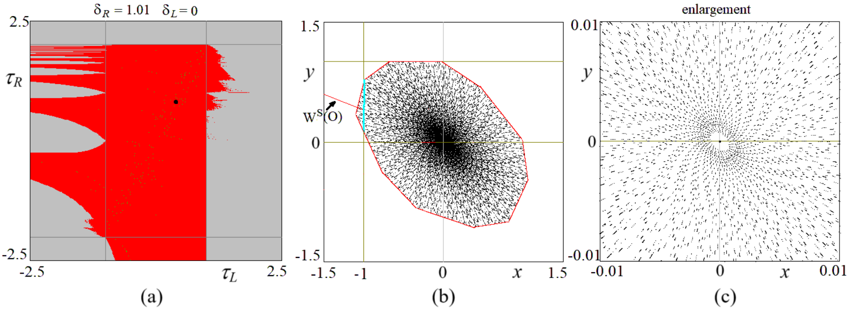
- Panel (a), O is a repelling focus.
- Panel (a), stability triangle of function T_L and that another attractor may exist for a wide range of parameter values.
- Panel (a), the black dot s.t. O attracting for T_L with $\delta_L = 0$.
- Panel (b), the corresponding attracting set;
- Panel (c), the first return map on a segment on the critical line $y = 0$ confirms that the dynamics of the attractor are those of a PWL circle map (no WQA).

PWL map T_1 with $\delta_L = 0$



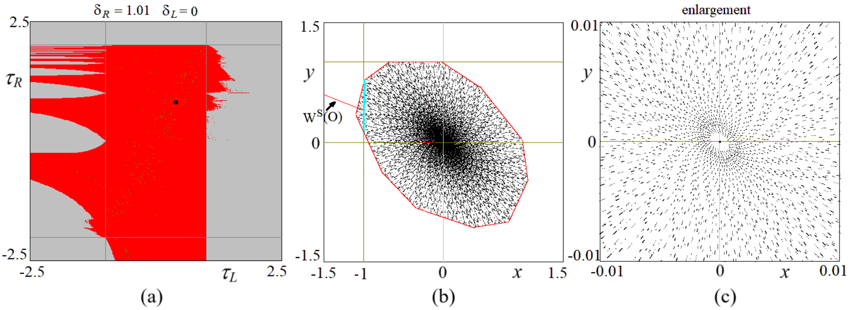
- Panel (a), the attracting set consists of several intervals along the critical line $y = 0$;
- Panel (b), the first return map can be defined on a single interval, resulting in a PWL circle map (no WQA).

The condition $\delta_L = 0$ (which causes the left partition to be mapped onto the critical line $y = 0$) does not necessarily imply that an attractor consists of a finite number of segments.



- Panel (a), $\delta_L = 0$ and O is repelling.
- In this case, we are not able to define a suitable first return map on the critical line $y = 0$.
- Panel (b), attracting set (appears to be a WQA) and O repelling focus.

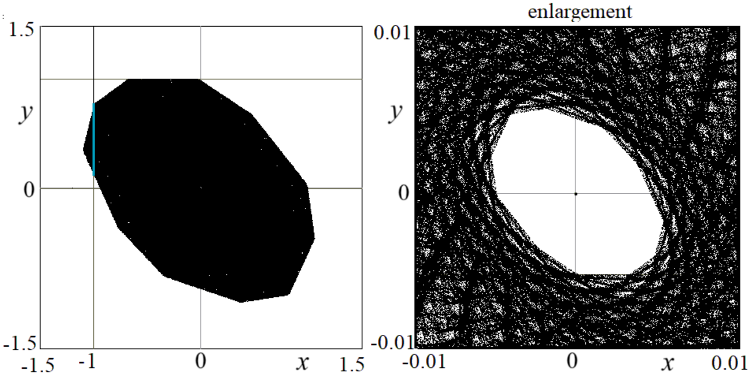
PWL map T_1 with $\delta_L = 0$



- Panel (b), an invariant polygon (in red) including the WQA via a finite number of images of the segment (in cyan) of the discontinuity set (see Mira *et al.* (1996)).
- Panel (b), in red the stable set of O, $W^S(O)$. That half-line in the L partition is mapped (in one iteration) into the fixed point O.
- Panel (b), it appears that no point of the WQA belongs to the half-line $W^S(O)$.
- Panel (c), a neighborhood of the origin without points of the WQA. Hence, there are no points of the WQA close to the stable set of O (half-line in the L partition, shown in red in panel (b)).

PWL map T_1 with $\delta_L = -0.01$

It persists for $\delta_L = -0.01$.



(a) (b)

- Panel (a), WQA, the points now fill the existing invariant polygon (although not densely).
- Panel (b), no points of the WQA in a suitable neighbohood of the fixed point O.

PWL map T_1 with $\delta_L = -0.01$

Explanation of the existence of WQA:

- The half-line stable set of O is now an eigenvector of T_L with slope -0.4236 (corresponding to the eigenvalue $\lambda = 0.0236$) whose points tend toward the virtual fixed point O.
- This eigenvector of T_L intersects the discontinuity line at a point P, inside the invariant region. Point P also has a rank-1 preimage via the inverse T_L^{-1}, denoted by P_{-1}, which belongs to the eigenvector within the left partition.
- Consequently, T_L maps segment $P_{-1}P$ into segment PP_1 along the eigenvector.
- However, segment PP_1 belongs to the right partition, where the right function applies, and in a finite number of iterations, the points are mapped again to the left partition.
- From there, due to the shape of the related eigenvectors, they are mapped again to the right partition, and so forth.
- This iterative process leads to a WQA: ω-limit set of the iterations of segment PP_1, i.e., the ω-limit set of $(T_1)^n(PP_1)$, for $n \to \infty$.

PWL map T_1 with $\delta_L = -0.01$

Why the structure may appear quite weird:

- Segment PP_1 belongs to an invariant region, all of its iterates remain in that region (divergence is not possible);
- Linear homogeneous functions map segments belonging to straight lines through the origin into segments belonging to straight lines through the origin (Lemma 1).
- Thus, applying T_1 to segment PP_1, in a finite number of iterations segment $(T_1)^k(PP_1)$ crosses the discontinuity line, leading to 2 segments.
- Each of these segments is then iterated similarly, each one (after a different number of iterations) is crossing the discontinuity line, and thus a further division occurs, leading to 2^2 segments.
- This process continues indefinitely; the iterates of the original segment generate 2^n segments, for any n.
- No point can be mapped into itself in a finite number of iterations since cycles do not exist, and all the segments belonging to $(T_1)^n(PP_1)$ are on straight lines through the origin.
- The ω-limit set of $(T_1)^n(PP_1)$ for $n \to \infty$ gives the attractor, a WQA.
- This attractor belong to an invariant area, a polygon bounded by a finite number of critical segments. These critical segments are the images of the segment on the discontinuity line included in the area (shown in azure in panel (a)).

Further 2D PWL maps

Same discontinuity set $x = -1$ but **triangular map** with two different linear functions having a **common eigenvector** $y = 0$ **associated with eigenvalues** a_l **and** a_r:

$$T_2 = \begin{cases} T_L : X' = J_L X \quad \text{for } x < h, \quad J_L = \begin{bmatrix} a_l & b_l \\ 0 & d_l \end{bmatrix} \\ \\ T_R : X' = J_R X \quad \text{for } x > h, \quad J_R = \begin{bmatrix} a_r & b_r \\ 0 & d_r \end{bmatrix} \end{cases} \quad h = -1 \tag{6}$$

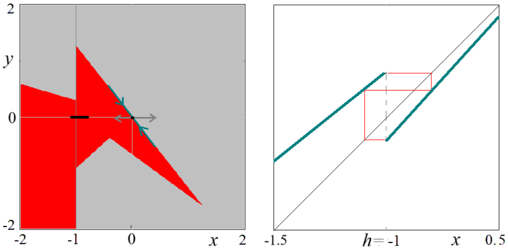
Properties:

- The restriction of map T_2 on $y = 0$ leads to the 1D PWL circle map:

$$x' = a_l x \ \ for\ x < -1 \ \ \text{and} \ \ x' = a_r x \ \ for\ x > -1 \tag{7}$$

- Hence, on eigenvector $y = 0$, the dynamics are those occurring in map F in (4) with discontinuity point $x = -1$.
- Clearly, the global behavior in the phase plane depends on the other eigenvalues of the two linear functions, and their associated eigenvectors (below two examples)

Further 2D PWL maps



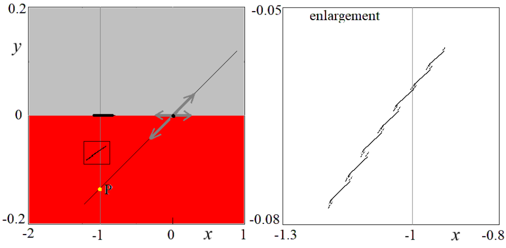
(a) (b)

- O is a virtual attracting node, with both eigenvalues positive;
- O is a real saddle with eigenvalue $\lambda_1 = a_r = 1.1$, related to the eigenvector $y = 0$, and eigenvalue $\lambda_2 = d_r = -0.8$, related to the eigenvector with slope $s_2 \simeq -1.267$.
- Panel (a) shows that the only attractor is a segment on eigenvector $y = 0$.
- Panel (b) first return map on that segment is a PWL circle map $x' = a_l x = 0.8x$ *for* $x < -1$ and $x' = a_r x = 1.1x$ *for* $x > -1$.
- The basin of attraction depends on the global dynamics:

  The boundary between the two basins (the basin of divergent trajectories and the basin of the unique attractor) consists of segments of the stable set of the origin (the eigenvector associated with $\lambda_2 = -0.8$), and segments of the discontinuity line and their preimages.

Further 2D PWL maps

We modify only parameter d_r (the second eigenvalue in the right partition)



(a) (b)

- O is a virtual attracting node, with both eigenvalues positive;
- O is a real repelling node with positive eigenvalues;
- The restriction of the map to the eigenvector $y = 0$ remains unchanged (PWL circle map).
- The global dynamics now differs. Since the fixed point O is a repelling node, eigenvector $y = 0$ is repelling.
- The unstable eigenvector associated with eigenvalue $d_r = 1.3$, with slope $s_2 \simeq 0.1333$, has the upper branch going to infinity, while the opposite branch intersects the discontinuity line at a point P, so that a segment $P_{-1}P$ on that eigenvector in the right partition is mapped into PP_1 in the left partition.

Further 2D PWL maps

Same discontinuity set $x = -1$ and two triangular maps but with different eigenvectors!!!

Consider

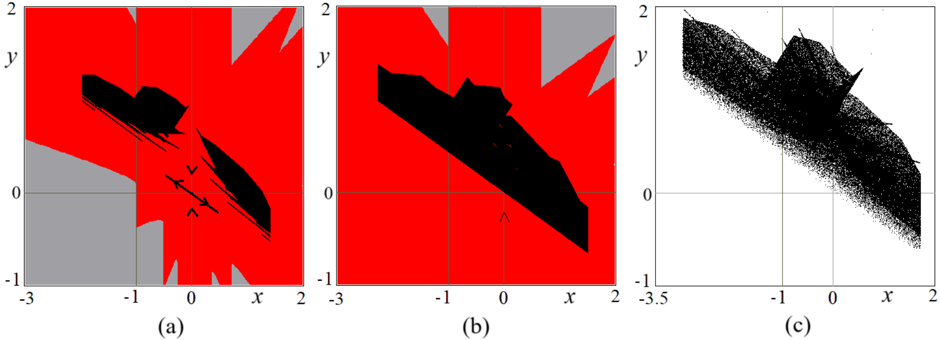
$$T_3 = \begin{cases} T_L : X' = J_L X \qquad \text{for } x < h, \quad J_L = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \\ \\ T_R : X' = J_R X \qquad \text{for } x > h, \quad J_R = \begin{bmatrix} a_2 & 0 \\ b_2 & c_2 \end{bmatrix} \end{cases} \qquad h = -1 \tag{8}$$

Peculiarity: The functions in the two partitions do not have a common eigenvector

- J_L has the eigenvector $y = 0$ associated with eigenvalue a_1;
- J_R has the eigenvector $x = 0$ associated with eigenvalue c_2.

Further 2D PWL maps

- Panel (a)-(c), O is a saddle for both partition;
- Panels (a)-(b), WQAs (as well as divergen trajectories).
- Panel (c), after the contact between the WQA and its basin boundary, we observe only the "ghost" of the former WQA.



Further 2D PWL maps

Another map: Same discontinuity set $x = -1$, but no triangular map, proportional eigenvalues but not the same eigenvectors!!!!

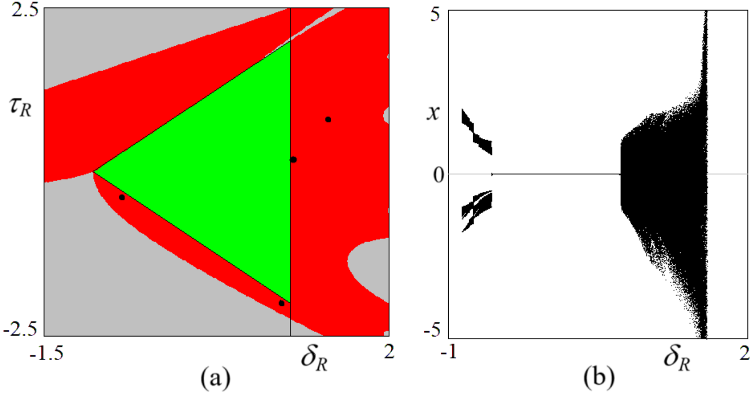
Consider

$$T_4 = \begin{cases} T_L : X' = J_L X \quad \text{for } x < h, \quad J_L = \begin{bmatrix} \alpha\tau_R & 1 \\ -\alpha^2\delta_R & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \quad \text{for } x > h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \quad h = -1 \tag{9}$$

Properties: The linear maps in the two partitions have *proportional eigenvalues*, *but not the same eigenvectors*.

Remark: This property allows for the existence of WQAs, as shown in the next slide. These attractors may occur when O is **unstable** and the eigenvalues are either real or complex conjugate.

Further 2D PWL maps

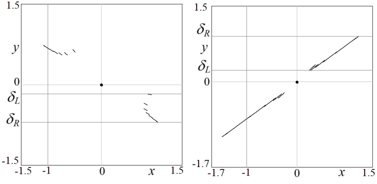


- Panel (a), 2D bifurcation diagram in the parameter plane (δ_R, τ_R), at $\alpha = 0.5$;
- Panel (a), the stability triangle of the real fixed point O is well evidenced, while the red region denotes the existence of WQAs.
- Panel (b), the 1D bifurcation diagram as a function of δ_R at fixed $\tau_R = -0.5$, clearly evidences the existence of WQAs, occurring both for $\delta_R < 1$ and for $\delta_R > 1$.

Remark: The dynamics in the phase plane at the four black dots marked in Figure in the next slide.

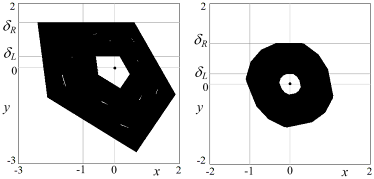
Further 2D PWL maps

Remark: O saddle and virtual attracting node.



(a) (b)

Remark: O unstable focus and virtual attracting focus. WQA belongs to an invariant polygon (determined by a finite number of images of a segment of the discontinuity line).



(a) (b)

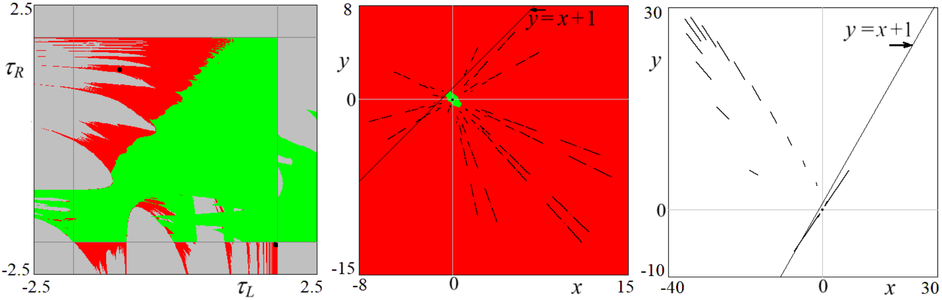
2D PWL homogeneous maps with different discontinuity sets

Discontinuity sets I: straight lines

Consider (discontinuity line $y = x + h$, where $h = 1$, index R to the region below the line, and L to the region above it): assign :

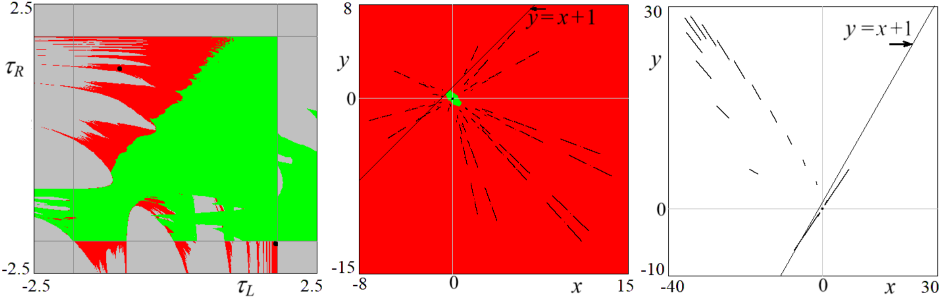
$$T_5 = \begin{cases} T_L : X' = J_L X & \text{for } y > x + h, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X & \text{for } y < x + h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{10}$$

- Panel (a), an example of a 2D bifurcation diagram in the parameter plane (τ_L, τ_R) at fixed $\delta_R = 0.9$ and $\delta_L = 0.8$.



(a) (b) (c)

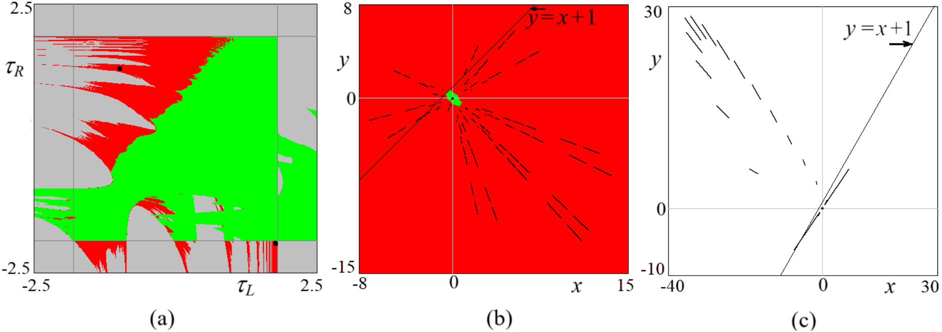
Discontinuity sets I: straight lines



(a) (b) (c)

- Panel (b), O is an attracting focus for both linear functions, and a WQA exists.
- Panel (b), the boundary of the basin of attraction of the fixed point O consists of a segment of the discontinuity line and a finite number of its preimages;
- Panel (b), divergence does not occur; a bounded attracting set must exist.
- Panel (b), basin of attraction of O has no point in the L partition, where the map has a virtual attracting focus at the origin.
- Panel (b), consequently, the points from the L partition are mapped to the R partition, and rotate back to the L partition again, and so forth.

Discontinuity sets I: straight lines

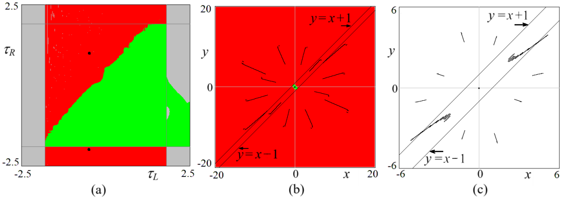


- Panel (c), the origin is a saddle for both linear functions.
- Panel (c), WQA results from the unstable eigenvector of the origin entering the left partition.

Discontinuity sets I: straight lines

Consider (two discontinuity lines, $y = x + h$ and $y = x - h$, with $h = 1$, index R refers to the partition between the two straight lines, and L outside of that):

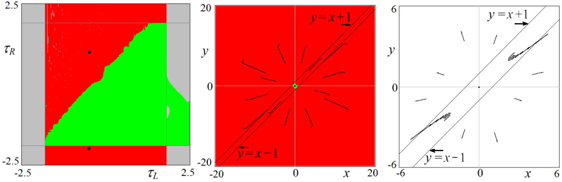
$$T_6 = \begin{cases} T_L : X' = J_L X & \text{for } |y - x| > h, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X & \text{for } |y - x| < h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{11}$$



- Map is now symmetric with respect to the fixed point O. It follows that an invariant set must be either symmetric with respect to O, or the symmetric one also exists.
- Panel (a), 2D bifurcation diagram in the parameter plane (τ_L, τ_R) at fixed parameter values $\delta_R = 0.9$ and $\delta_L = 0.8$.
- Panel (b)-(c), WQAs.

Consider (two discontinuity lines, $y = x + h$ and $y = x - h$, with $h = 1$, index R refers to the partition between the two straight lines, and L outside of that):

$$T_6 = \begin{cases} T_L : X' = J_L X & \text{for } |y-x| > h, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X & \text{for } |y-x| < h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{12}$$



(a) (b) (c)

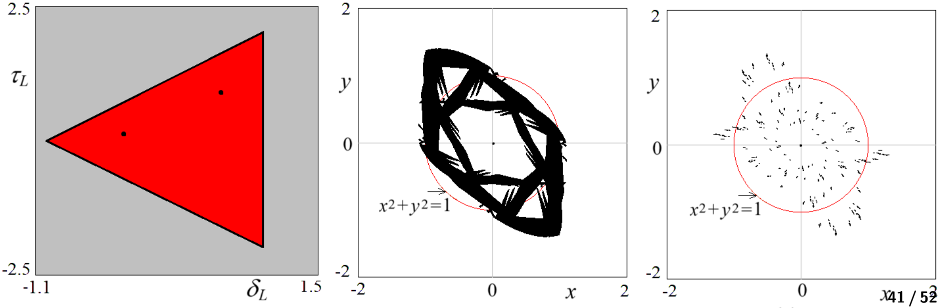
- Panel (b), O attracting focus.
- Panel (c), O is a saddle for T_R and an attracting focus for T_L.
- Similar results are observed when the two straight lines representing the discontinuity sets are vertical. Examples of such cases are shown in Gardini *et al.* (2024), Gardini *et al.* (2025c).

Discontinuity sets II: circles

Consider (discontinuity set: $x^2 + y^2 = 1$, index R the partition inside the circle, while L corresponds to the region outside):

$$T_7 = \begin{cases} T_L : X' = J_L X & \text{for } x^2+y^2>1, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ T_R : X' = J_R X & \text{for } x^2+y^2<1, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \quad (13)$$

- O is a repelling focus.
- Panel (b), O virtual stable focus.
- Panel (c), O virtual stable node.

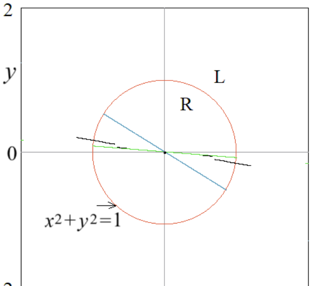


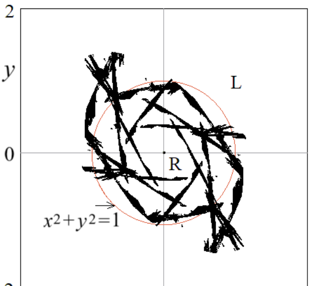
Discontinuity sets II: circles (Cont's)

Consider (discontinuity set: $x^2 + y^2 = 1$, index R the partition inside the circle, while L corresponds to the region outside):

$$T_8 = \begin{cases} T_L : X' = J_L X & \text{for } x^2 + y^2 > 1, \quad J_L = \begin{bmatrix} \alpha\tau_R & 1 \\ -\alpha^2\delta_R & 0 \end{bmatrix} \\ T_R : X' = J_R X & \text{for } x^2 + y^2 < 1, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{14}$$

- Panel (a), WQAs, O is a saddle (its eigenvectors are shown inside the circle), and is a virtual attracting node.
- Panel (b), WQAs, O is a repelling focus and a virtual attracting focus.
- Existence of a WQA may be connected to the absence of divergent trajectories.



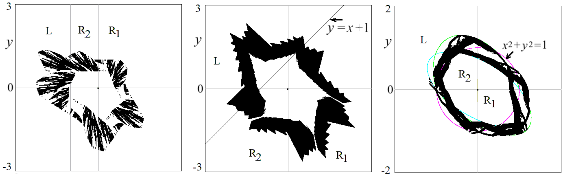


2D PWL homogeneous map with O in two partitions

Consider ($D_R = R_1 \cup R_2$ is split in two regions, Jacobian matrices, in R_1 and R_2, differ in terms of trace and determinant following the standard 2D normal form structure):

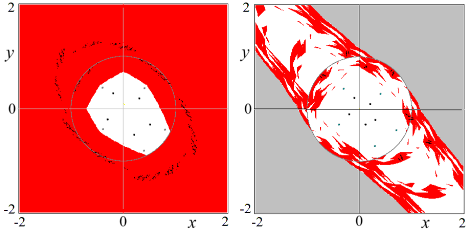
$$T_9 = \begin{cases} T_L : X' = J_L X & \text{for } X \in D_L, & J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ T_{R2} : X' = J_{R2} X & \text{for } X \in R_2 (x \le 0), & J_{R2} = \begin{bmatrix} \tau_{R2} & 1 \\ -\delta_{R2} & 0 \end{bmatrix} \\ T_{R1} : X' = J_{R1} X & \text{for } \in R_1 (x \ge 0), & J_{R1} = \begin{bmatrix} \tau_{R1} & 1 \\ -\delta_{R1} & 0 \end{bmatrix} \end{cases} \tag{15}$$

- O (repelling focus for T_{R1} and T_{R2}, attracting (virtual) focus for T_L) lies on the border of two partitions, where the map is continuous, while an additional discontinuity set is introduced.
- The three cases differ in the definition of D_L: In Panel (a), D_L corresponds to region $x < -1$; in Panel (b) to region $y > x + 1$; in Panel (c) to region $x^2 + y^2 > 1$.



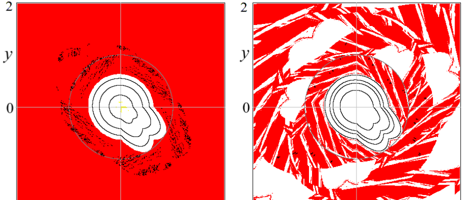
2D PWL homogeneous map with O in two partitions

- The parameters of map correspond to a rational rotation number leading to 5-cycles and a region filled with 5-cycles coexists with a WQA.



(a) (b)

- The parameters of map correspond to an irrational rotation number, leading to a suitable region filled with closed invariant curves on which the trajectories are quasiperiodic, and there is coexistence with a WQA.



nD PWL homogeneous maps.

Generalization to $\mathbb{R}^n$

WQA generalized to the class of $n-$dimensional maps for $n > 2$:

- **nD weird quasiperiodic attractor** is an attractor $\mathcal{A}$ of an nD map T that is a closed invariant set which does not contain any periodic point.
- The dynamics of T on $\mathcal{A}$ cannot be studied by means of a first return map or by the restriction to a set of lower dimension.

Conjecture

Let T be an nD map as given in Definition 1. Then:

(j) *A bounded ω-limit set $\mathcal{A}$ different from the fixed point O and from related local invariant sets of O when it is nonhyperbolic, can only be one of the following:*

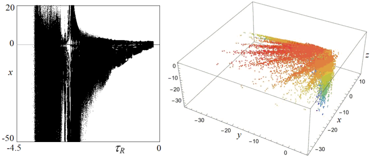
- (ja) *a nonhyperbolic k-cycle, $k \geq 2$ (this occurs in m-dimensional sets, $m < n$, non intersecting any border and filled with cycles of the same symbolic sequence);*
- (jb) *a finite number of m-dimensional sets, $m < n$, filled with quasiperiodic orbits;*
- (jc) *an invariant set, not structurally stable, on which the dynamics are reducible to a discontinuous k-dimensional map, $k < n$;*
- (jd) *an mD weird quasiperiodic attractor, where $2 \leq m \leq n$.*

(jj) *When no cycles exist filling $\mathcal{A}$ densely, then $\mathcal{A}$ exhibits (weak) sensitivity to initial conditions.*

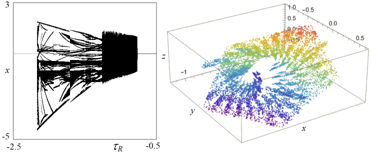
Examples of 3D WQA

Let us consider a 3D example, with $X = (x, y, z)$. The simplest 3D map defined in two partitions reads as follows:

$$T = \begin{cases} T_L : X' = J_L X \qquad \text{for } x < -1, \quad J_L = \begin{bmatrix} \tau_L & 1 & 0 \\ -\sigma_L & 0 & 1 \\ \delta_L & 0 & 0 \end{bmatrix} \\ \\ T_R : X' = J_R X \qquad \text{for } x > -1, \quad J_R = \begin{bmatrix} \tau_R & 1 & 0 \\ -\sigma_R & 0 & 1 \\ \delta_R & 0 & 0 \end{bmatrix} \end{cases} \tag{16}$$



(a) (b)



Further research

(a) Shape of WQAs and sensitivity to parameter perturbation.

(b) Existence of WQAs in a broader class of maps.

(c) Maximum Lyapunov exponent in discontinuous maps.

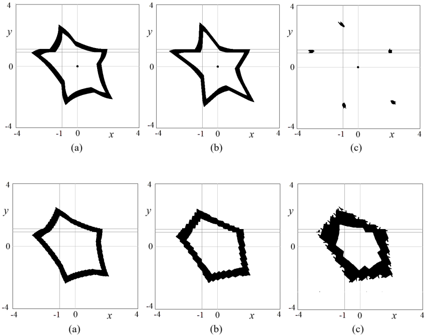
(d) Transition from regular to chaotic dynamics.

Further research

Consider the attractors that are numerically obtained when one of the linear functions is modified into an affine function. Consider the perturbed version of the map as follows:

$$T_{1a} = \begin{cases} T_L : \begin{cases} x' = \tau_L x + y + \mu_L \\ y' = -\delta_L x \end{cases} & \text{for } x < h, \quad J_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} \\ \\ T_R : \begin{cases} x' = \tau_R x + y \\ y' = -\delta_R x \end{cases} & \text{for } x > h, \quad J_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} \end{cases} \tag{17}$$

WQA for $\mu_L = 0$, the qualitative shape of the attracting set remains unchanged for both $\mu_L > 0$ and $\mu_L < 0$, when close to 0. However, (for $\mu_L \neq 0$) the attractors may be chaotic.



Thanks for your attention!!!!

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