On the dynamics of a family of max-type difference equations

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Generalized Lyness max-type difference equations

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}^m},$$

where $k, l, m \in \mathbb{R}$ and $A, x_{-1}, x_0 \in (0, \infty)$.

G. Ladas, *On the recursive sequence* $x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}}$, J. Difference Equ. Appl. **1**(1995), 95-97.

Main goal: To advance in the knowledge of the dynamics of the family of max-type difference equations

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}},$$

where $A, x_{-1}, x_0 \in (0, \infty)$.

Existing literature

J. Feuer, E.J. Janowski, G. Ladas, and C. Teixera, *Global behavior of solutions of* $x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}$, J. Comput. Anal. Appl. **2**(2000), 237-252.

J. Feuer, *Periodic solutions of the Lyness max equation*, J. Math. Anal. Appl. **288**(2003), 147-160.

A. Gelisken, C. Cinar, I. Yalcinkaya, *On the periodicity of a difference equation with maximum*, Discrete Dyn. Nat. Soc. (2008), Article ID 820629, 11 pages.

Outline

Outline

1. Topological conjugacy with piecewise linear equations
2. Case $A = 1$
3. Case $A > 1$
4. Case $0 < A < 1$
5. Numerical simulations and further lines of research

- A **discrete dynamical system** is a pair (X, F), where X is a topological space and $F : X \to X$ is continuous.

- We call **associated dynamical system** to the difference equation

$$x_{n+k} = f\left(x_{n+k-1}, \dots, x_{n+1}, x_n\right)$$

to the pair (X^k, F), where the map $F : X^k \to X^k$ is given by

$$F(x_1, \dots, x_k) = \left(x_2, \dots, x_k, f\left(x_k, \dots, x_2, x_1\right)\right).$$

• For (X, φ) and (Y, ψ), we say that the dynamical systems generated by $\varphi : X \to X$ and $\psi : Y \to Y$ are **topologically conjugate** if there is an homeomorphism $\phi : X \to Y$, such that $\phi(\varphi(x)) = \psi(\phi(x))$ for all $x \in X$.

• We say that two difference equations are topologically conjugate when the associated dynamical systems so are.

• In this case, the difference equations exhibit the same type of dynamics; for instance, they have the same number of equilibrium points or periodic orbits, or have chaotic attractors which are homeomorphic.

A. Linero Bas, D. Nieves Roldán, *On the relationship between Lozi maps and max-type difference equations*, J. Difference Equ. Appl. **29**(2022), no. 9-12, 1015-1044.

Generalized Lozi map:

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha \neq 0$.

Generalized Lyness max-type equations:

$$z_{n+1} = \frac{\max\left\{z_n^{2\alpha}, A\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha \neq 0$ and $A > 0$.

Consider the max-type family of difference equations

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}, \quad \text{with } A > 0.$$

- If $A > 1$, it is topologically conjugate to

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} - 1$$

- If $A = 1$, it is topologically conjugate to

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1}$$

- If $0 < A < 1$, it is topologically conjugate to

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} + 1$$

Outline

The equation

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n x_{n-1}}$$

is globally periodic of period 7.

Therefore, the piecewise linear equation

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1}$$

is globally periodic of period 7 too.

Furthermore, due to the topological conjugacy established between Lyness max-type difference equations and generalized Lozi maps, we get a whole uniparametric family

$$x_{n+1} = \frac{\max\{z_n, B\}}{z_n z_{n-1}} \cdot B^2,$$

for all $B > 0$, which is globally periodic of period 7.

Outline

We limit to the study of the topologically conjugate piecewise linear equation

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} - 1$$

We consider its associate DDS, $(\mathbb{R}^2, F_1)$, where $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F_1(x, y) = \left(y, \frac{1}{2}|y| - \frac{1}{2}y - x - 1 \right).$$

Proposition

F_1 has a unique equilibrium point, namely, $(\bar{x}, \bar{y}) = \left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Proposition

The triangle T of vertices $(0,0), (-1,0)$ *and* $(0,-1)$ *is invariant under F_1. Moreover, every point in T, except the equilibrium point, is periodic of prime period* 3.

A. Cima, A. Gasull, V. Mañosa, *Global periodicity and complete integrability of discrete dynamical systems*, J. Difference Equ. Appl. **12**(2006), 697-726.

The invariant triangle T is a region of complete integrability for the discrete dynamical system and we get that (T, F_1) is completely integrable with the maps $V_1, V_2 : T \to \mathbb{R}$ given by

$$V_1(x,y) = -x^2 - y^2 - xy - x - y, \quad V_2(x,y) = -x^2y - xy^2 - xy,$$

being first integrals, which are functionally independent, for the DDS.

We focus on the dynamics outside the invariant region T.

- We can assume without loss of generality that the initial conditions, (x_0, y_0), are in the first quadrant.
- Let $\alpha := \max\{x_0, y_0\}$ and define the following segments:

$$
\begin{aligned}
\mathcal{R}_1 &:= \left\{(x,y) \in \mathbb{R}^2 : \; y = \alpha, \; 0 \le x \le \alpha\right\};\\
\mathcal{R}_2 &:= \left\{(x,y) \in \mathbb{R}^2 : \; x = \alpha, \; 0 \le y \le \alpha\right\};\\
\mathcal{R}_3 &:= \left\{(x,y) \in \mathbb{R}^2 : \; x = \alpha, \; -\alpha - 1 \le y \le 0\right\};\\
\mathcal{R}_4 &:= \left\{(x,y) \in \mathbb{R}^2 : \; y = -\alpha - 1, \; 0 \le x \le \alpha\right\};\\
\mathcal{R}_5 &:= \left\{(x,y) \in \mathbb{R}^2 : \; x + y = -\alpha - 1, \; -\alpha - 1 \le x \le 0\right\};\\
\mathcal{R}_6 &:= \left\{(x,y) \in \mathbb{R}^2 : \; x = -\alpha - 1, \; 0 \le y \le \alpha\right\};\\
\mathcal{R}_7 &:= \left\{(x,y) \in \mathbb{R}^2 : \; y = \alpha, \; -\alpha - 1 \le x \le 0\right\}.
\end{aligned}
$$

Proposition

Consider the map F_1. The compact graph Γ_1 determined by the union of the segments $\bigcup_{j=1}^{7} \mathcal{R}_j$ is invariant under F_1.

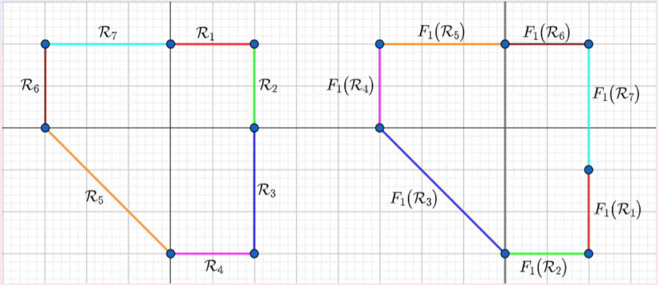

Figure: The evolution of the segments $\mathcal{R}_i$ under F_1.

Corollary

Consider the map F_1. For any pair of arbitrary initial conditions (x_0, y_0), with $x_0, y_0 \geq 0$, its orbit under F_1 is entirely contained in its correspondent graph Γ_1:

$$\left(F_1^n(x_0, y_0)\right)_{n\geq 0} \subseteq \Gamma_1.$$

Moreover, the sequence $\left(F_1^n(x_0, y_0)\right)_{n\geq 0}$ moves in a clockwise direction around the compact graph Γ_1.

Some remarks

- Since each compact graph depends on the initial conditions, we have a foliation of closed curves that cover the plane $\mathbb{R}^2$.
- Due to the fact that they are invariant curves, it must exist a first integral for which these graphs are the corresponding level curves.

$$\tilde{V}(x,y) = -\frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}|x-y| + |x+y+1| + \frac{1}{2}\big|x+y+|x-y|\big|.$$

For every $\alpha > 0$, given a pair of initial conditions $(x_0, y_0) \in Q_1$, it is easy to see that $\tilde{V}(x_0, y_0) = 2\alpha + 1$. Therefore, the compact graph $\Gamma_1(\alpha)$ corresponds with the level curve $\tilde{V}(x,y) = 2\alpha + 1$, for every $\alpha > 0$.

Proposition

Consider the DDS $(\mathbb{R}^2, F_1)$*. If* $\alpha := \max\{x_0, y_0\} = \frac{p}{q}$*, with* $p, q \in \mathbb{N}$ *and* $gcd(p, q) = 1$*, then the solution generated from the initial conditions* (x_0, y_0) *under* F_1 *is periodic with period* $7p + 3q$*.*

-PROOF-

- The orbit generated by the initial conditions (x_0, y_0) under F_1 is contained in the corresponding invariant compact graph Γ_1.
- We make a partition of Γ_1: we divide the segments $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_4$ and $\mathcal{R}_6$ in p segments of equal length, while we divide $\mathcal{R}_3, \mathcal{R}_5$ and $\mathcal{R}_7$ in $p + q$ segments. Then we study the images of the elements of the partition under F_1.

On the one hand,

$$\begin{aligned}
\mathcal{R}_{1,j} &= \left\{(x,y) \in \mathcal{R}_1 : \frac{j-1}{q} \leq x \leq \frac{j}{q}\right\};\\
\mathcal{R}_{2,j} &= \left\{(x,y) \in \mathcal{R}_2 : \frac{p-j}{q} \leq y \leq \frac{p-j+1}{q}\right\};\\
\mathcal{R}_{4,j} &= \left\{(x,y) \in \mathcal{R}_4 : \frac{p-j}{q} \leq x \leq \frac{p-j+1}{q}\right\};\\
\mathcal{R}_{6,j} &= \left\{(x,y) \in \mathcal{R}_6 : \frac{j-1}{q} \leq y \leq \frac{j}{q}\right\};
\end{aligned}$$

for every $j = 1, \ldots, p$.

$$
\begin{aligned}
\mathcal{R}^0_{3,i} &= \left\{(x,y) \in \mathcal{R}_3 : \frac{-i}{q} \leq y \leq \frac{-i+1}{q}\right\};\\
\mathcal{R}_{3,j} &= \left\{(x,y) \in \mathcal{R}_3 : -1 - \frac{j}{q} \leq y \leq -1 - \frac{j-1}{q}\right\};\\
\mathcal{R}^0_{5,i} &= \left\{(x,y) \in \mathcal{R}_5 : \frac{-i}{q} \leq x \leq \frac{-i+1}{q}\right\};\\
\mathcal{R}_{5,j} &= \left\{(x,y) \in \mathcal{R}_5 : -1 - \frac{j}{q} \leq x \leq -1 - \frac{j-1}{q}\right\};\\
\mathcal{R}^0_{7,i} &= \left\{(x,y) \in \mathcal{R}_7 : -1 - \frac{p-i+1}{q} \leq x \leq -1 - \frac{p-i}{q}\right\};\\
\mathcal{R}_{7,j} &= \left\{(x,y) \in \mathcal{R}_7 : \frac{-p+j-1}{q} \leq x \leq \frac{-p+j}{q}\right\};
\end{aligned}
$$

with $j = 1, \ldots, p$ and $i = 1, \ldots, q$.

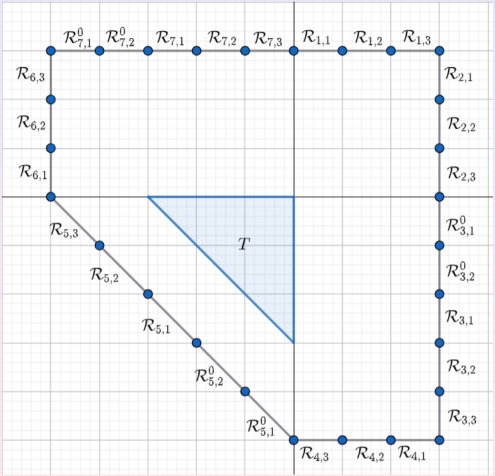

Figure: The compact invariant graph Γ_α with $\alpha = \frac{p}{q} = \frac{3}{2}$.

Proposition

Consider the DDS $\left(\mathbb{R}^2, F_1\right)$. *If* $\alpha := \max\{x_0, y_0\} \in \mathbb{R} \setminus \mathbb{Q}$, *then the solution generated from the initial conditions* (x_0, y_0) *under* F_1 *is non-periodic. Moreover, its orbit is dense in* $\Gamma_1(\alpha)$.

-**PROOF**- The evolution of the terms in the segment $\mathcal{R}_1 \cup \mathcal{R}_7$ of the compact graph Γ_1 is an irrational rotation map whose rotation number is $\frac{\alpha+2}{2\alpha+1} \in \mathbb{R} \setminus \mathbb{Q}$, implying that the orbit is dense in the compact graph.

Theorem: Consider the piecewise linear difference equation

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} - 1.$$

Its dynamics is given by:

- A unique equilibrium point, $\bar{y} = -\frac{1}{3}$.
- An invariant triangle $T = \{(x, y) \in \mathbb{R}^2 : x + y \geq -1, x \leq 0, y \leq 0\}$, where every solution, except the equilibrium, is a 3-cycle of the form $(\alpha, \beta, -\alpha - \beta - 1)$.
- An infinite number of periodic orbits whenever the initial conditions are outside T and $\alpha = \max\{y_{-1}, y_0\} \in \mathbb{Q}$. Furthermore, if $\alpha = \frac{p}{q}$ with $\gcd(p, q) = 1$, the orbit is periodic with period $7p + 3q$.
- An infinite number of non-periodic orbits whenever the initial conditions are outside T and $\alpha = \max\{y_{-1}, y_0\} \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, the non-periodic orbits are dense in a compact invariant graph surrounding T.

Theorem: Consider the max-type family of difference equations

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}, \quad \text{with } 1 < A \in \mathbb{R}$$

and initial conditions $x_{-1}, x_0 \in (0, \infty)$. Its dynamics is given by:

- A unique equilibrium point, $\bar{x} = \sqrt[3]{A}$.
- An invariant region $R_1 = \left\{(x, y) \in \mathbb{R}^2 : x, y \in \left[\frac{1}{A}, A\right], xy \geq 1\right\}$, where every solution, except the equilibrium, is a 3-cycle of the form $\left(\alpha, \beta, \frac{A}{\alpha\beta}\right)$.
- An infinite number of periodic orbits whenever the initial conditions are outside R_1 and, if $\alpha = \max\{x_{-1}, x_0\}$, $A^{1+2\alpha} \in \mathbb{Q}$. Furthermore, if $A^{1+2\alpha} = \frac{p}{q}$, the orbit is periodic with period $7p + 3q$.
- An infinite number of non-periodic orbits whenever the initial conditions are outside R_1 and $A^{1+2\alpha} \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, the non-periodic orbits are dense in a compact invariant graph surrounding R_1.

Outline

1. Topological conjugacy with piecewise linear equations
2. Case $A = 1$
3. Case $A > 1$
4. Case $0 < A < 1$
5. Numerical simulations and further lines of research

We analyze the topologically conjugate piecewise linear equation

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} + 1.$$

We consider its associate DDS, $(\mathbb{R}^2, F_2)$, where $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F_2(x, y) = \left(y, \frac{1}{2}|y| - \frac{1}{2}y - x + 1\right).$$

Proposition

F_2 has a unique equilibrium point, namely, $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Proposition

The square $I^2 = [0,1] \times [0,1]$ *is invariant under* F_2. *Moreover, every point in* I^2, *except the equilibrium point, is periodic of prime period* 4.

Again, let $\alpha := \max\{x_0, y_0\}$, with $\alpha > 1$ in order to avoid being in I^2, and define the following segments:

$$\begin{aligned}
\mathcal{M}_1 &:= \{(x,y) \in \mathbb{R}^2 : y = \alpha,\ 0 \leq x \leq \alpha\},\\
\mathcal{M}_2 &:= \{(x,y) \in \mathbb{R}^2 : x = \alpha,\ 0 \leq y \leq \alpha\},\\
\mathcal{M}_3 &:= \{(x,y) \in \mathbb{R}^2 : x = \alpha,\ 1-\alpha \leq y \leq 0\},\\
\mathcal{M}_4 &:= \{(x,y) \in \mathbb{R}^2 : y = 1-\alpha,\ 0 \leq x \leq \alpha\},\\
\mathcal{M}_5 &:= \{(x,y) \in \mathbb{R}^2 : x + y = 1-\alpha,\ 1-\alpha \leq x \leq 0\},\\
\mathcal{M}_6 &:= \{(x,y) \in \mathbb{R}^2 : x = 1-\alpha,\ 0 \leq y \leq \alpha\},\\
\mathcal{M}_7 &:= \{(x,y) \in \mathbb{R}^2 : y = \alpha,\ 1-\alpha \leq x \leq 0\}.
\end{aligned}$$

Proposition

Consider the map F_2. The compact graph Γ_2 determined by the segments $\bigcup_{i=1}^{7} \mathcal{M}_i$ is invariant under F_2.

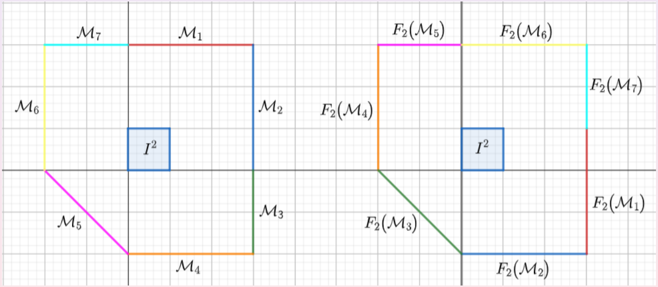

Figure: The evolution of the segments $\mathcal{M}_i$ under F_2.

Proposition

Consider the DDS $(\mathbb{R}^2, F_2)$. *If* $\alpha := \max\{x_0, y_0\} = \frac{p}{q}$, *with* $p, q \in \mathbb{N}$ *and* $\gcd(p, q) = 1$, *then the solution generated from the initial conditions* (x_0, y_0) *under* F_2 *is periodic with period* $7p + 4q$.

Proposition

Consider the DDS $(\mathbb{R}^2, F_2)$. *If* $\alpha := \max\{x_0, y_0\} \in \mathbb{R} \setminus \mathbb{Q}$, *then the solution generated from the initial conditions* (x_0, y_0) *under* F_2 *is non-periodic. Moreover, its orbit is dense in* $\Gamma_2(\alpha)$.

Theorem: Consider the piecewise linear difference equation

$$y_{n+1} = \frac{1}{2}|y_n| - \frac{1}{2}y_n - y_{n-1} + 1.$$

Its dynamics is given by:

- A unique equilibrium point, $\bar{y} = \frac{1}{2}$.
- An invariant square $I^2 = [0,1]^2$, where every solution, except the equilibrium, is a 4-cycle of the form $(\alpha, \beta, 1-\alpha, 1-\beta)$.
- An infinite number of periodic orbits whenever the initial conditions are outside I^2 and $\alpha = \max\{y_{-1}, y_0\} \in \mathbb{Q}$. Furthermore, if $\alpha = \frac{p}{q}$ with $\gcd(p,q) = 1$, the orbit is periodic with period $7p + 4q$.
- An infinite number of non-periodic orbits whenever the initial conditions are outside I^2 and $\alpha = \max\{y_{-1}, y_0\} \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, the non-periodic orbits are dense in a compact invariant graph surrounding I^2.

Theorem: Consider the max-type family of difference equations

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n x_{n-1}}, \quad \text{with } 0 < A < 1$$

and initial conditions $x_{-1}, x_0 \in (0, \infty)$. Its dynamics is given by:

- A unique equilibrium point, $\bar{x} = 1$.
- An invariant region $R_2 = \left\{(x, y) \in \mathbb{R}^2 : x, y \in \left[A, \frac{1}{A}\right]\right\}$, where every solution, except the equilibrium, is a 4-cycle of the form $\left(\alpha, \beta, \frac{1}{\alpha}, \frac{1}{\beta}\right)$.
- An infinite number of periodic orbits whenever the initial conditions are outside R_2 and, if $\alpha = \max\{x_{-1}, x_0\}$, $A^{1-2\alpha} \in \mathbb{Q}$. Furthermore, if $A^{1-2\alpha} = \frac{p}{q}$, the orbit is periodic with period $7p + 4q$.
- An infinite number of non-periodic orbits whenever the initial conditions are outside R_2 and $A^{1-2\alpha} \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, the non-periodic orbits are dense in a compact invariant graph surrounding R_2.

Outline

• To analyze other particular cases of the family of Lyness max-type difference equations.

$$x_{n+1} = \frac{\max\{x_n, A\}}{x_n^2 x_{n-1}}$$

- If $A = 1$, it is globally periodic of period 8.
- If $A \neq 1$, the dynamics are analogous as the case studied before, but with the invariant compact graphs being trapeziums surrounding the invariant regions. Furthermore, for $A > 1$ the periodic solutions have periods of the form $8p + 2q$; and for $0 < A < 1$, the periods of the periodic solutions are of the form $8p + 4q$. The non-periodic solutions are dense in the compact graphs.

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}}$$

- If $A = 1$, it is globally periodic of prime period 9.

M. Crampin, *Piecewise linear recurrence relations*, The Math. Gazette **76**(1992), 355-359.

- If $0 < A < 1$, it is the well-known **Gingerbreadman equation**.

R.L. Devaney, *A piecewise linear model for the zones of instability of an area-preserving map*, Physica **10D**(1984), 387-393.

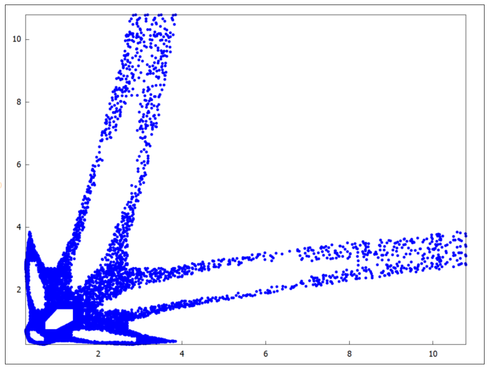

Figure: Numerical simulation with $A = 0.5$, $x_{-1} = 1.35$, $x_0 = 1.74$ and 10000 iterations.

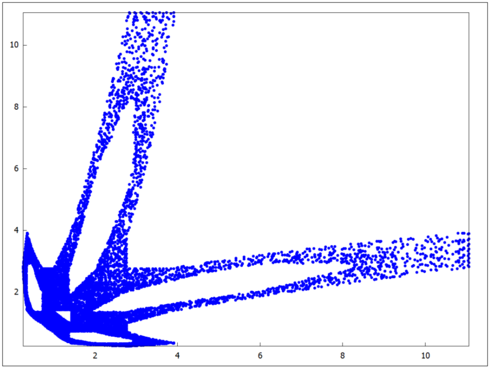

Figure: Numerical simulation with $A = 2$, $x_{-1} = 1.35$, $x_0 = 1.74$ and 10000 iterations.

- To study the family of piecewise linear maps

$$F_\alpha(x, y) = \big(y, \alpha|y| - \alpha y - x + \delta\big),$$

with $\alpha \in \mathbb{R}$ and $\delta \in \{-1, 0, 1\}$.

In the sequel we develop some numerical simulations for the particular case $\delta = -1$.

For $\alpha \geq 1$, it seems that the orbits diverge by three different branches.

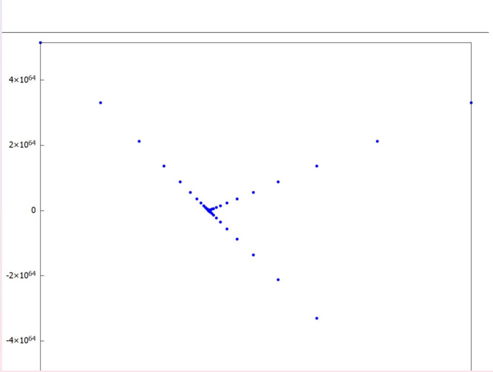

Figure: Simulation for $\alpha = 1.1$ and $(x_0, y_0) = (0, 0)$. 3000 iterations.

For $\alpha \leq -1$, it seem that the orbits diverge by the third quadrant.

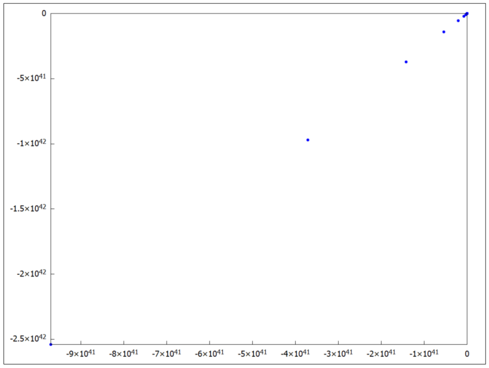

Figure: Simulation for $\alpha = -1.5$ and $(x_0, y_0) = (3.25, -1.6)$.

The interesting dynamics occur when $\alpha \in (-1, 1)$.

- If $\alpha = 0$, the map is globally periodic of period 4.
- The scenario $\alpha = \frac{1}{2}$ was analyzed in detail in this talk.
- If $\alpha = -\frac{1}{2}$ the dynamics is similar to the one presented in this talk, with a region of global periodicity of period 6.

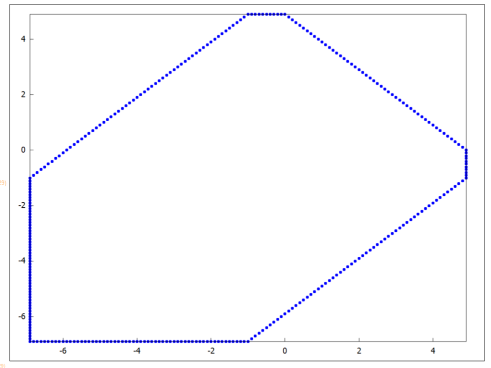

Figure: Simulation for $\alpha = -0.5$ and $(x_0, y_0) = (2.4, -3.5)$. 3000 iterations.

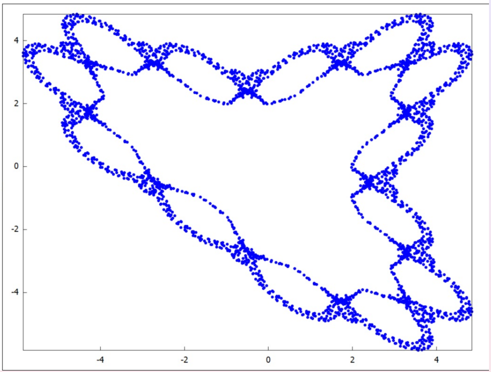

Figure: Simulation for $\alpha = 0.7$ and $(x_0, y_0) = (3, -1)$. 3000 iterations.

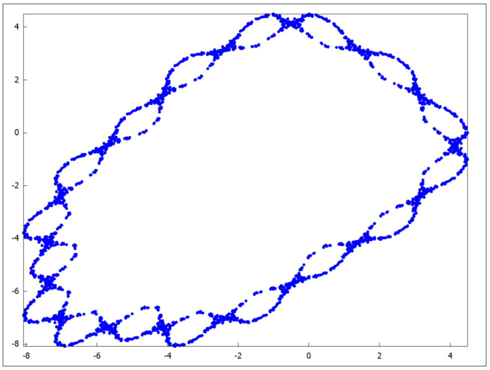

Figure: Simulation for $\alpha = -0.6$ and $(x_0, y_0) = (1.85, -3.45)$. 3000 iterations.

THANK YOU FOR YOUR KIND ATTENTION

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