Exploring the κ-logistic Growth Model: Dynamics and Insights

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Economic growth

- Economic growth is a fundamental branch of Macroeconomics
- Two pioneering works:
  - The optimal-growth model by Ramsey
  - The Solow-Swan model
- Among later works, the model proposed by Böhm and Kaas is a fundamental milestone

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The Böhm and Kaas model: a recap

- $t \in \mathbb{N}$
- x_t: the capital per-capita level at time $t \in \mathbb{N}$
- $n \geq 0$: the labor force growth rate
- $\delta \in [0, 1]$: the capital depreciation rate
- $s_w \in [0, 1]$: the saving rate for workers
- $s_r \in [0, 1]$: the saving rate for stakeholders
- $f : \mathbb{R}^+ \to \mathbb{R}^+$: the production function
- $w : \mathbb{R}^+ \to \mathbb{R}$: the wage rate
- $\pi : \mathbb{R}^+ \to \mathbb{R}$: the profit share

$$x_{t+1} = F(x_t) = \frac{1}{1+n}\left[(1-\delta)x_t + s_w f(x_t) + x_t f'(x_t)(s_r - s_w)\right]$$

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The novelty

- The novelty of the present work is the use of the κ-logistic function, denoted by σ_κ, to describe the production function in the Böhm and Kaas growth model.
- Such a function is a generalization of the well-known sigmoidal logistic function, frequently used in machine learning and it is obtained from the κ-exponential function (or, more concisely, $\exp_\kappa$) introduced by Kaniadakis

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The $\exp_\kappa$ function - I

The $\exp_\kappa$ function:

- It is a generalization of the exponential function
- It depends on a parameter κ
  - When the parameter κ approaches zero, it tends to the exp function
  - With κ large enough, it tends to infinity like a power function
- Its popularity is due to the capability to take rare events into account
  - Ordinary events follow an exponential law
  - Rare events are characterized by a Pareto (or power-tail) law

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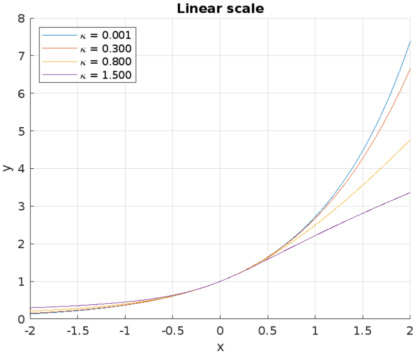
The $\exp_\kappa$ function - II

The $\exp_\kappa$ function is a function mapping $\mathbb{R}$ onto $\mathbb{R}^{++}$ given by

$$\exp_\kappa(x) = \left(\sqrt{1+\kappa^2x^2} + \kappa x\right)^{\frac{1}{\kappa}}$$



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The logistic (or sigmoidal) function

$$\sigma : \mathbb{R} \to (0, 1), \qquad \sigma(x) = \frac{1}{1 + e^{-x}}$$

- It maps $\mathbb{R}$ to the interval $(0, 1)$, thus transforming a number into a probability
- For this reason, this function is widely used in the field of machine learning
- Its first derivative is always positive, meaning that larger numbers correspond to probabilities closer to one and vice versa

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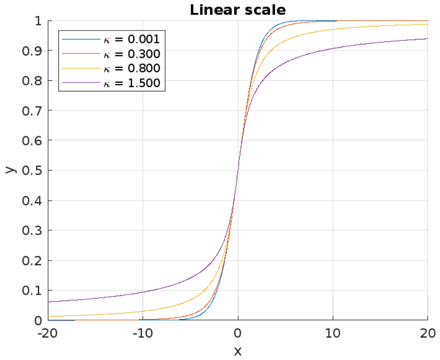
The κ-logistic function

$$\sigma_\kappa : \mathbb{R} \to (0,\, 1), \qquad \sigma_\kappa(x) = \frac{1}{1 + \exp_\kappa(-x)}$$



Some properties of the σ_κ function

a) $\lim_{x \to -\infty} \sigma_\kappa(x) = 0.$

b) $\lim_{x \to +\infty} \sigma_\kappa(x) = 1.$

c) $\sigma_\kappa(x) \in (0,\, 1) \;\; \forall\, x \in \mathbb{R}.$

d) $\sigma_{-\kappa}(x) = \sigma_\kappa(x) \;\; \forall\, \kappa \in \mathbb{R}.$

e) $\frac{d}{dx}\sigma_\kappa(x) = \frac{1}{\sqrt{1+\kappa^2 x^2}}\sigma_\kappa(x)\,(1 - \sigma_\kappa(x)).$

f) σ_κ is a strictly monotonic increasing function.

g) $x_f = 0$ is an inflection point for σ_κ. Specifically, σ_κ is convex for $x < x_f$ and concave for $x > x_f$.

h) A line intersects the σ_κ function in at most three points.

i) σ_κ' is an even function and its maximum is at the origin with a value of $\frac{1}{4}$.

Our production function

We propose a modified version of the κ-logistic function for the production function such that:

- is able to describe both strictly concave or convex-concave technologies
- takes into account rare events, as for instance, in economics, natural disasters, such as earthquakes that reduce the supply of capital, or epidemics or other external shocks influencing the supply of intermediate inputs, human or physical capital.
- considers economies at different development levels according to an upper bound to the maximum production level

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The modified κ-logistic production function

$$f(x_t) = M\frac{\sigma_\kappa(x_t - x_c) - \sigma_\kappa(-x_c)}{1 - \sigma_\kappa(-x_c)}$$

- Higher values of M correspond to more developed economies
- Positive values of x_c are associated to non-strictly concave production functions
- Higher κ values are associated to situations in which rare events are more likely to emerge.

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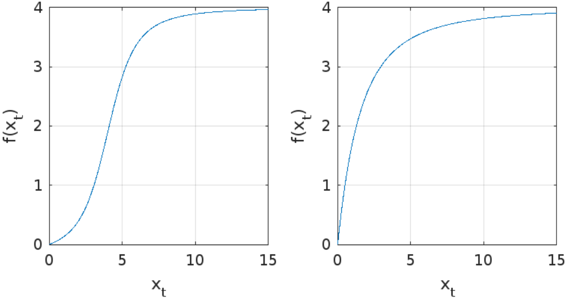
Some properties of f

The modified κ-logistic function has the following characteristics:

- f is convex on $(-\infty, x_c)$ and concave on $(x_c, +\infty)$.
- $\forall\, m,\ q \in \mathbb{R}$, the equation $f(x_t) = mx_t + q$ has at most three roots.



Taking into account the previous formulas, the map of our growth model becomes:

$$x_{t+1} = F(x_t),$$

with

$$\begin{aligned} F(x_t) \quad &= \quad \frac{1}{1+n}\Bigg\{(1-\delta)x_t + s_w M \frac{\sigma_\kappa(x_t - x_c) - \sigma_\kappa(-x_c)}{1-\sigma_\kappa(-x_c)} + \\ &\quad + \frac{(s_r - s_w)M}{1-\sigma_\kappa(-x_c)} \frac{\sigma_\kappa(x_t - x_c)\left(1-\sigma_\kappa(x_t - x_c)\right)x_t}{\sqrt{1+\kappa^2(x_t - x_c)^2}}\Bigg\}. \end{aligned}$$

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Equilibrium points - I

Consider the map $x_{t+1} = F(x_t)$, then:

a) $x^* = 0$ is always an equilibrium point to the map.

b) If $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, there exists at least a $x^* > 0$ such that $F(x^*) = x^*$.

c) If $s_r = s_w$, there are at most two positive equilibrium points. Moreover, under this condition:

  c.1) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists only one positive equilibrium point.

  c.2) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) \leq (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, there are no positive equilibrium points.

d) There exists $\epsilon > 0$ such that if $|s_r - s_w| < \epsilon$, then there are at most two positive equilibrium points. Moreover, under this condition:

  d.1) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists only one positive equilibrium point.

  d.2) If $x_c < 0$ and $s_r M\sigma_\kappa(-x_c) \leq (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there are no positive equilibrium points.

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Equilibrium points - II

e) If $s_w = 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} \le \frac{1}{4}$, then there exists at most two positive equilibrium points. In particular, if $s_w = 0$, $x_c > 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} = \frac{1}{4}$, there exists exactly one positive equilibrium point equal to $x^* = x_c$. Vice versa, if $s_w = 0$, $x_c \le 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} = \frac{1}{4}$, there are no positive equilibrium points. Moreover, if $s_w = 0$ and $\frac{(n+\delta)(1-\sigma_\kappa(-x_c))}{Ms_r} > \frac{1}{4}$, then there are no positive equilibrium points.

f) There exists $\underline{s}_w > 0$ such that for all $s_w \in (0, \underline{s}_w)$, the map has at most two positive equilibrium points.

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Stability of the origin

Consider the map $x_{t+1} = F(x_t)$, with F with the equilibrium point at the origin, then:

a) If $s_r M \sigma_\kappa(-x_c) < (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then the origin is asymptotically stable.

b) If $s_r M \sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then the origin is unstable.

c) If $s_r M \sigma_\kappa(-x_c) = (n+\delta)\sqrt{1+\kappa^2 x_c^2}$ and $x_c \neq 0$ and $s_w \neq 2s_r$, then the origin is unstable.

d) If $s_r M \sigma_\kappa(-x_c) = (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, $x_c = 0$, and $3s_r - 2s_w > 0$ then the origin is asymptotically stable.

e) If $s_r M \sigma_\kappa(-x_c) = (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, $x_c = 0$, and $3s_r - 2s_w < 0$ then the origin is unstable.

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Stability of other (possible) equilibrium points

- If $s_r M \sigma_\kappa(-x_c) > (n+\delta)\sqrt{1+\kappa^2 x_c^2}$, then there exists at least a positive equilibrium point and the origin is unstable.
- Let $s_r \geq s_w$:
  a) There exist $\overline{x}_c$ and $\overline{M} = M(\overline{x}_c) > 0$ such that for all $x_c > \overline{x}_c$ and $0 < M < \overline{M}$ the origin is the sole equilibrium point and it is globally stable.
  b) The positive equilibrium point arising in cases c.1) and d.1) previously presented attracts all trajectories starting from positive initial conditions.

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Computational results

- With our computational experiments:
  - We aim to validate the previous theoretical findings
  - We attempt to uncover insights not possible through analytical means
  - We introduce innovative tools to explore the model's properties
- Computational tests were conducted in Python, considering various parameter combinations, which we refer to as scenarios
- Each scenario corresponds to a particular shape of the map

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Scenarios

Scenario	M	x_c	κ	s_r	s_w	δ	n
1	1	-2	0.5	1	0.2	0.2	0.5
2	10	-2	0.5	1	0.2	0.2	0.5
3	1	2	0.5	1	0.2	0.2	0.5
4	25	4	0.8	0.2	0.2	0.2	0.5
5	25	4	0.8	0.25	0.2	0.2	0.5
6	2.5	5	0.8	1	0.2	0.2	0.5
7	10	2	0.5	1	0.2	0.2	0.5
8	3.00941946	4.76959597	0.70003226	1	0.2	0.2	0.5
9	6	6	0.8	1	0.2	0.2	0.5
10	10	10	0.2	0.7	0.1	0.2	0.5

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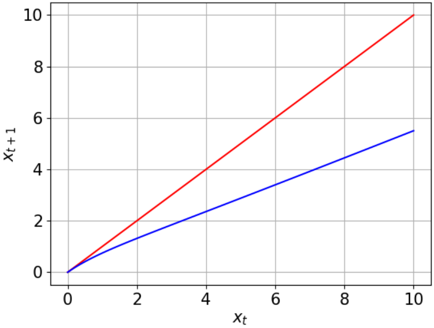
The map is concave with just the origin as equilibrium point



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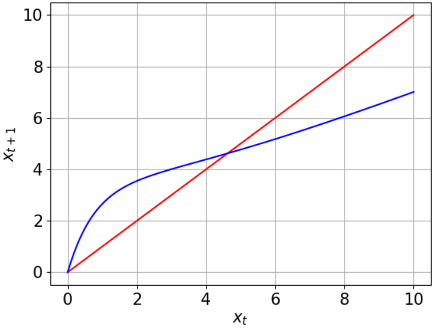
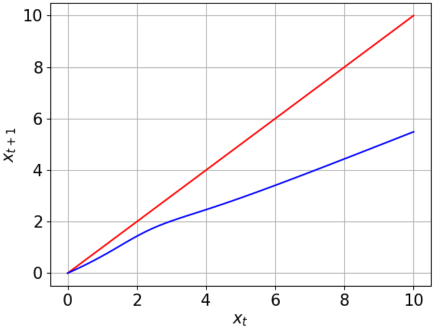
Scenario 2

The map is concave with the origin and a positive equilibrium point

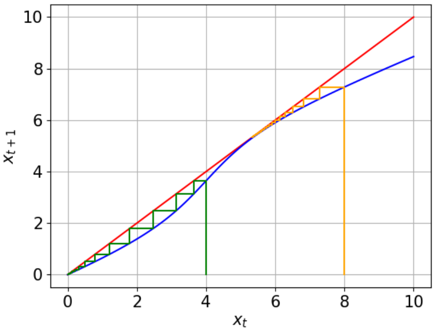


Scenario 3

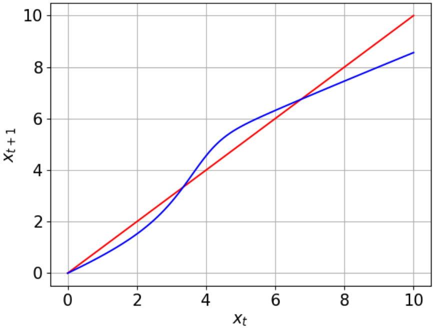
The map is convex/concave, with the origin as the only equilibrium point



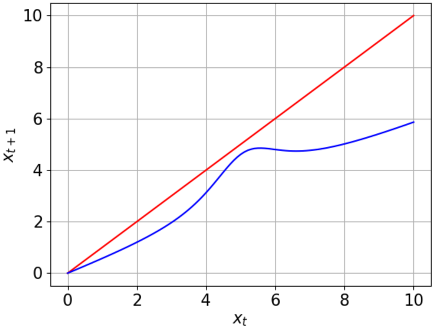
The map is convex/concave, with the origin and a positive equilibrium point that is also tangent to the bisector of the first quadrant



The map is convex/concave, with the origin and two positive equilibrium points

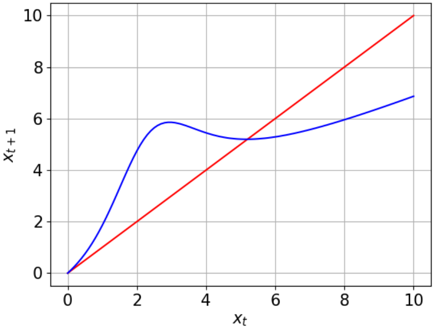


The map is convex/concave/convex, with the origin as the only equilibrium point

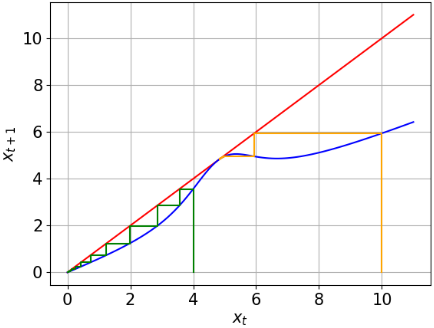


Scenario 7

The map is convex/concave/convex, with the origin and a positive equilibrium point



The map is convex/concave/convex, with the origin and a positive equilibrium point that is tangent to the bisector of the first quadrant



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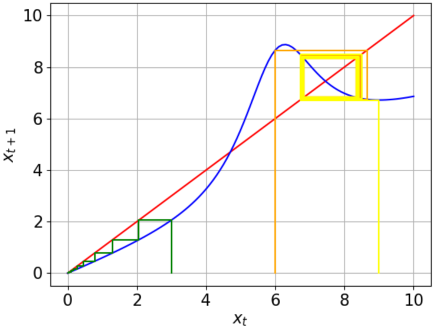
The map is convex/concave/convex, with the origin and two positive equilibrium points



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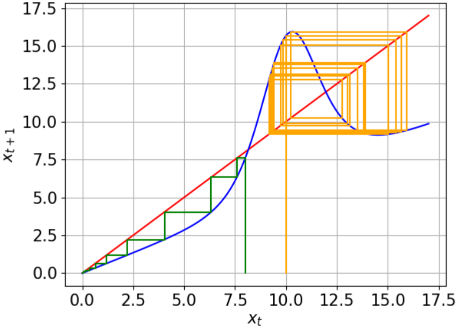

Chaotic analysis

- The map in scenarios 9 and 10 is convex/concave/convex, with the origin and two positive equilibrium points
- We name x_M the local maximum
- We name x_m the local minimum
- We plot bifurcation diagrams starting from x_M and x_m

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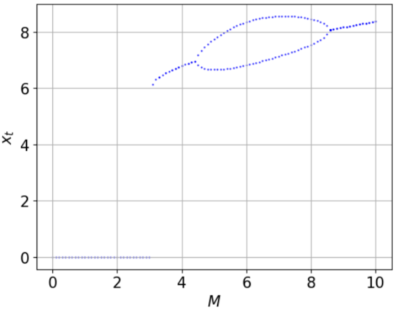
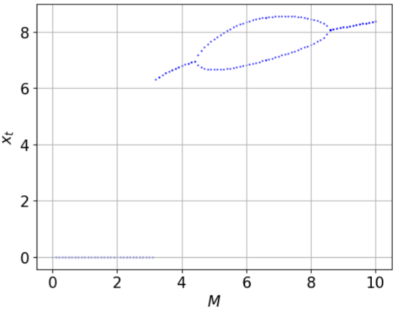
Starting from x_M (left) and x_m (right)





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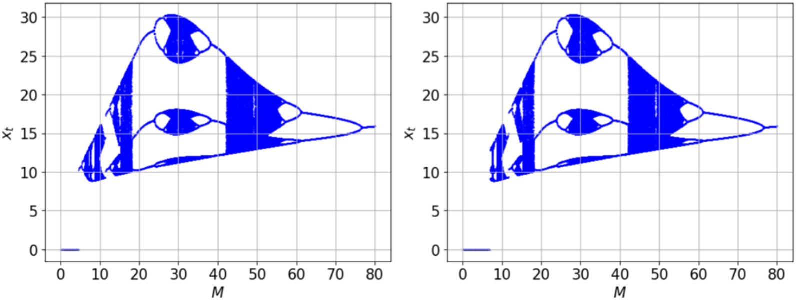
Starting from x_M (left) and x_m (right)



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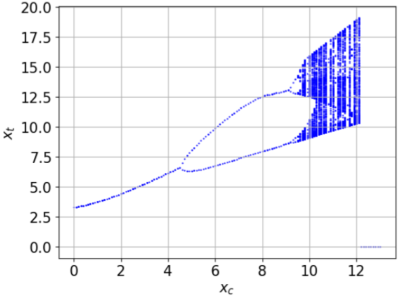
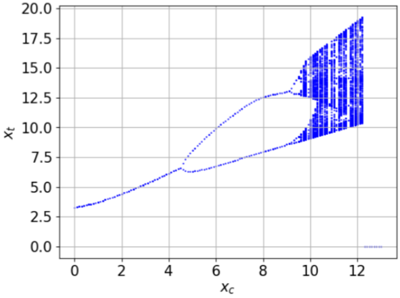
x_c-bifurcation diagram for scenario 10

Starting from x_M (left) and x_m (right)





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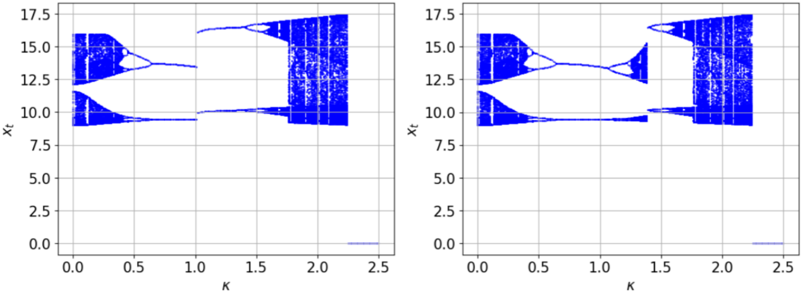
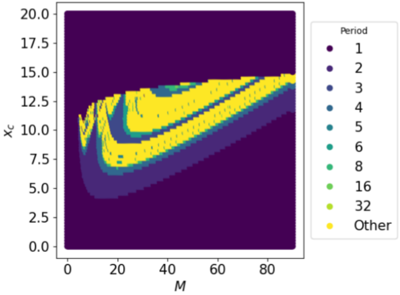
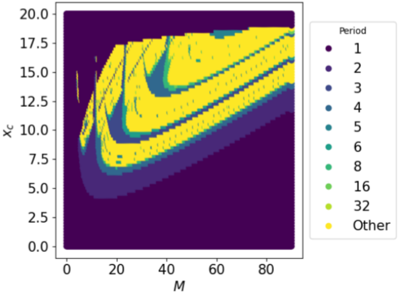
κ-bifurcation diagram for scenario 10

Starting from x_M (left) and x_m (right)



Starting from x_M (left) and x_m (right)





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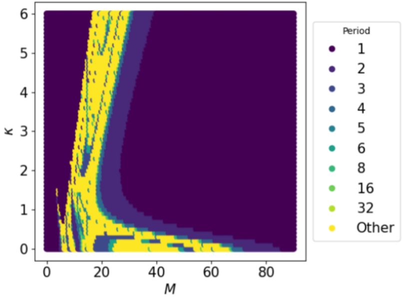
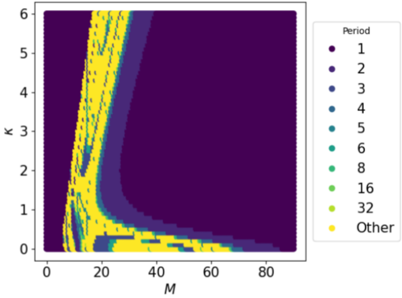
Starting from x_M (left) and x_m (right)





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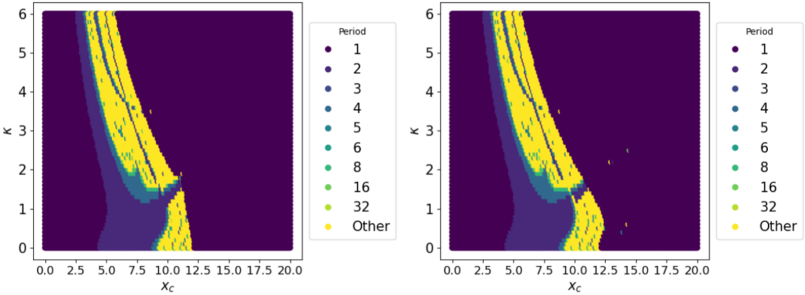
x_c vs. κ diagram for scenario 10

Starting from x_M (left) and x_m (right)



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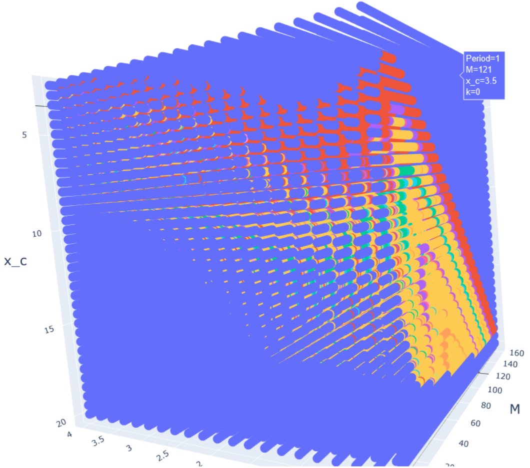
3-D diagram for scenario 10 - I

Starting from x_m



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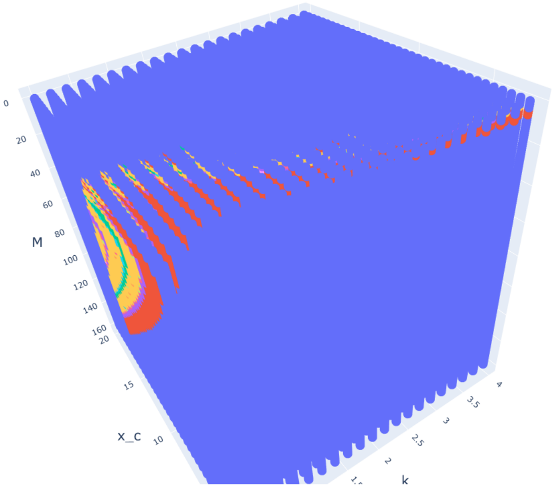
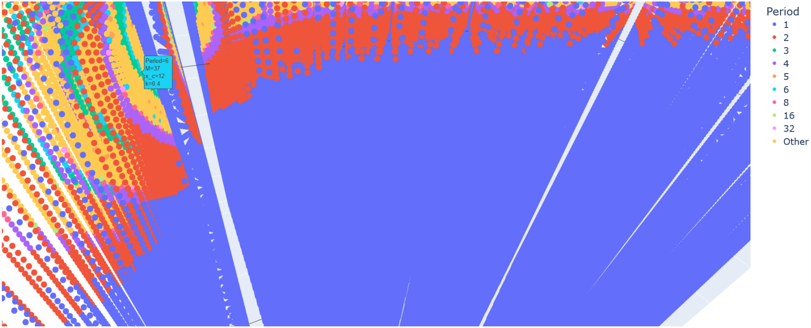
3-D diagram for scenario 10 - II

Starting from x_M



3-D diagram for scenario 10 - III

A magnified view starting from x_M



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Conclusions

- We explored a growth model using the κ-logistic function, which accounts for rare events, leading to a generalized sigmoidal production function.
- This model can handle both concave and non-concave production functions, relevant for economies at various development stages or encountering rare events.
- Theoretical findings were confirmed by computational experiments demonstrating the system's behavior under significant scenarios.
- Future work will focus on deeper analysis of basins of attraction and applying the κ-logistic function to multi-dimensional growth models.

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