Topological sequence entropy and topological dynamics of tree maps

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Plan

Definitions and preliminary results

Topological dynamics

Definition and notations

- *A topological dynamical system is a pair* (X, f)*, where* X *is a compact metric space and* f *is a continuous map from* X *to itself.*

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- *For $x \in X$, we called orbit of x under f the set $Orb_f(x) =: \{f^n(x), n \in \mathbb{Z}_+\}$ where $f^n$the n-th iterate of f ; and f^0 is the identity of X.*

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- *A non empty subset A of X is called invariant (resp. strongly invariant) if $f(A) \subset A$ (resp. $f(A) = A$).*

Limit and minimal sets

Definition $\omega-$limit set

- *The $\omega-$limit set of the point x is the subset :*

$\omega_f(x) := \bigcap_{n\in\mathbb{N}} \overline{\{f^k(x), k \geq n\}}$, *equivalently*

$\omega_f(x) = \{y \in X : \exists (n_i)_i \subset \mathbb{N}, n_i \to +\infty, \lim_{i\to+\infty} d(f^{n_i}(x), y) = 0\}$

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• We denote by $P(f)$ and $\Omega(f)$, the sets of periodic points, almost periodic points, recurrent points and the non-wandering points of f. We denote by $\Lambda(f) := \bigcup_{x\in X} \omega_f(x)$.

chain recurrent points

- For $\epsilon > 0$, an ϵ-chain from x to y under f is a finite sequence $(x_i)_{0\leq i\leq n}$ such that $x_0 = x$ and $x_n = y$ and $d(f(x_{i-1}), x_i) < \epsilon$ for any $1 \leq i \leq n$. We denote by $CR(x, f)$ the set of points of X for which for any $\epsilon > 0$ there exist an ϵ-chain under f from x to y.
- A point $x \in X$ is *chain recurrent* if $x \in CR(x, f)$. We denote by $CR(f)$ the set of chain recurrent points.
  • We have the following inclusion, which are strict in general :

$$P(f) \subset \Lambda(f) \subset \Omega(f) \subset CR(f)$$

Example

• Let f be the identity map of $[0,1]$ and $(x_n)_{n\geq 0}$ a sequence of $[-1,1]$ such that $\lim_{n\to+\infty} x_n = 0$, and for some arrangement of the index of $(x_n)_{n\geq 0}$ $(y_n = \sum_{k=0}^{n} x_n)_{n\geq 0}$ is a dense sequence of $[0,1]$.

• For any $\epsilon > 0$ and for n large enough, we can find an ϵ-chain from 0 to 1 under f.

- Let $n \in \mathbb{N}$ so that $|y_{k+1} - y_k| < \epsilon$, for any $k \geq n$ and $|y_n| < \epsilon$, since $(y_k)_{k\geq 0}$ is a dense sequence of $[0,1]$, we can find $m > n$ so that $|y_m - 1| \leq \epsilon$.
- The set $\{0, y_n, y_{n+1}, \ldots y_m, 1\}$ ϵ-chain from 0 to 1 under f.
- Although the dynamics of f is trivial, we can find (In the same manner as 0 and 1) an ϵ-chain from x to y under f, for any $x, y \in [0,1]$.

Topological Sequence Entropy

• For two open covers $\mathcal{U}, \mathcal{V} = \{V_i,\ i \in J\}$, the common refinement is $\mathcal{U} \vee \mathcal{V} = \{U_i \cap V_j \mid i \in I,\ j \in J\}$. We denote by $N(\mathcal{U})$ the minimum number of open sets in $\mathcal{U}$ that cover X.

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• Let $A = (a_i)_{i \geq 0}$ be an increasing sequence of integers and $\mathcal{U}$ an open cover of X. The sequence topological entropy of T with respect to A and $\mathcal{U}$ is defined as :

$$h_A(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \ln \left(N \left(\bigvee_{0 \leq i \leq n-1} T^{-a_i}(\mathcal{U}) \right) \right).$$

• The topological sequence entropy of T with respect to A is :

$$h_A(T) = \sup\{h_A(T,\mathcal{U}) \mid \mathcal{U} \text{ open cover of } X\}.$$

• The topological sequence entropy of T is :

$$h_\infty(T) = \sup\{h_A(T) \mid A \text{ increasing sequence of positive integers}\}.$$

Topological entropy and topological sequence entropy

• Note that if A is the sequence of non-negative integers, then $h_A(T) = h(T)$, where $h(T)$ is the the topological entropy of T.

• A dynamical system is said to be null whenever $h_\infty(T) = 0$.

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• Note that if A is the sequence of non-negative integers, then $h_A(T) = h(T)$, where $h(T)$ is the the topological entropy of T.

• A dynamical system is said to be null whenever $h_\infty(T) = 0$.

• Observe that $h(T) \leq h_\infty(T)$ and this equality can be strict in general.

Tree and Dendrite

- A *continuum* is a compact connected metric space.
- An *arc* I (resp. a *circle*) is any space homeomorphic to the compact interval $[0,1]$ (resp. to the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : \ |z| = 1\}$).
- A *tree* T is a continuum which can be written as the union of finitely many arcs such that any two arcs are either disjoint or intersect only in one or both of their endpoints. Moreover T does not contains any circle.
- A *dendrite* is a locally connected continuum which contains no simple closed curve.

• Note that each arc is a tree and any tree is a dendrite.

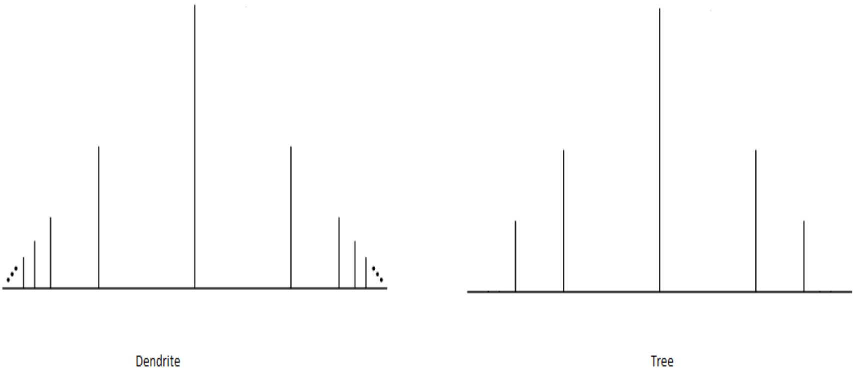

Figure – Example of a tree and a dendrite

Main result

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Theorem 1, J.S.Canovas (2007)

Let X be an arc and let $f : X \to X$ be continuous with zero topological entropy. Then

$$h_\infty(f|_{\omega(f)}) = h_\infty(f|_{\Omega(f)}) = h_\infty(f|_{CR(f)}) = 0.$$

We generalise Theorem 1 for the case of tree and we have :

Theorem 1, J.S.Canovas and D.Aymen (2023)

Let X be a tree and let $f : X \to X$ be continuous with zero topological entropy. Then

$$h_\infty(f|_{\omega(f)}) = h_\infty(f|_{\Omega(f)}) = h_\infty(f|_{CR(f)}) = 0.$$

- The proof is based on the following 4 results :

Main result

Let f be a tree map with zero topological entropy, we have the following :

- For any $x \in CR(f) \setminus P(f)$ the set $\omega_f(x)$ is infinite. (T. Sun, M. Xie and J. Zhao (2006))

- For any $x \in X$, we can find a decreasing sequence (with respect to the inclusion) of periodic arcs $(J_n)_{n\geq 0}$ with period m_n such that $\omega_f(x) \subset J_n \cup f(J_n) \cup ....f^{m_n-1}(J_n)$. (A.Ghassen (2015))

Main result

- Let $f : X \to X$ be a tree map such that $h(f) = 0$.
  Let $x \in CR(f) \setminus P(f)$ and let J be a periodic arc (with period m) such that $\omega_f(x) \subset J \cup f(J) \cup ....f^m(J)$. Then, the following statements hold :
  - $x \in J \cup f(J) \cup ....f^m(J)$.
  - For any $n \geq 0$, $f^{-n}(x) \cap CR(f) \subset J \cup f(J) \cup ....f^m(J)$.

- Let $f : X \to X$ be a tree map and let $g = f_{|CR(f)}$ be the restriction of f on $CR(f)$.

$$h(f) = 0 \Leftrightarrow (CR(f), g) \text{ is equicontinuous at any periodic point.}$$

• A dynamical system $(X, f))$ is said to be equicontinuous at a point $x \in X$ if the set of maps $F = \{f^n, \ n \geq 0\}$ is equicontinuous at x.
• With this 4 results we will prove The main Theorem.

Section 3

counter example in the case of dendrite

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To conclude, we will give an example of a map f on a dendrite X such that $h(f) = 0$, $\Omega(f) = P(f)$ and $h_\infty(f|_{\Omega(f)}) > 0$(in particular $h_\infty(f|_{CR(f)}) > 0$). The idea is to consider a pointwise periodic homeomorphism g on a given compact set Y such that $h_\infty(g) > 0$ and then extend the homeomorphism to a continuous self-map on the whole dendrite.

Let $I = [0, 1] \times \{0\}$. For $n \geq 1$, let

$$Y_n := \left\{ \left(x_{i,n}, \frac{1}{n} \right), \left(x_{i,n}, \frac{-1}{n} \right) : i \in \{1, 2, ..., 2^{n-1}\} \right\},$$

where $x_{i,n} := \frac{2i-1}{2^n}$ for $n \geq 1$ and $1 \leq i \leq 2^{n-1}$. Observe that

$$\lim_{n \to +\infty} Y_n = I$$

and $d(x_{i,n}, x_{i+1,n}) \leq \frac{1}{2^{n-1}}$ for $1 \leq i \leq 2^{n-1} - 1$.

counter example in the case of dendrite, the system (Y, g)

Let $Y := I \cup \left(\bigcup_{n\geq 1} Y_n\right)$, which clearly is a compact subset of $\mathbb{R}^2$. We define $g : Y \to Y$ by

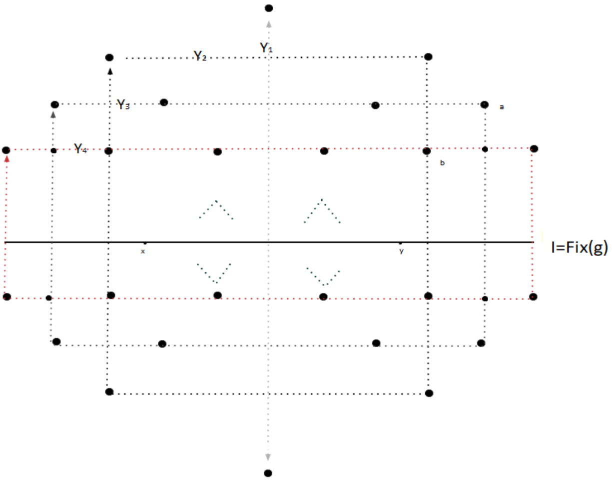
(i) $g(y) = y$, if $y \in I$.

(ii) For any $n \geq 1$ the subset Y_n is a periodic orbit such that :

- $g((x_{i,n}, \frac{1}{n})) = (x_{i+1,n}, \frac{1}{n})$ if $1 \leq i \leq 2^{n-1} - 1$.
- $g((x_{2^{n-1},n}, \frac{1}{n})) = (x_{2^{n-1},n}, \frac{-1}{n})$.
- $g((x_{i,n}, \frac{-1}{n})) = (x_{i-1,n}, \frac{-1}{n})$ if $2 \leq i \leq 2^{n-1}$.
- $g((x_{1,n}, \frac{-1}{n})) = (x_{1,n}, \frac{1}{n})$.

• g is a pointwise periodic homeomorphism (i.e $P(f) = Y$)

The system (Y, g)



counter example in the case of dendrite, the system (Y, g)

• For any $x, y \in Y = P(f)$, with $x \neq y$, we have $d(g^n(x), g^n(y)) \geq \epsilon > 0$ (since x, y are distinct periodic points), then the pair (x, y) is distal, Therefore the system (Y, g) is distal.

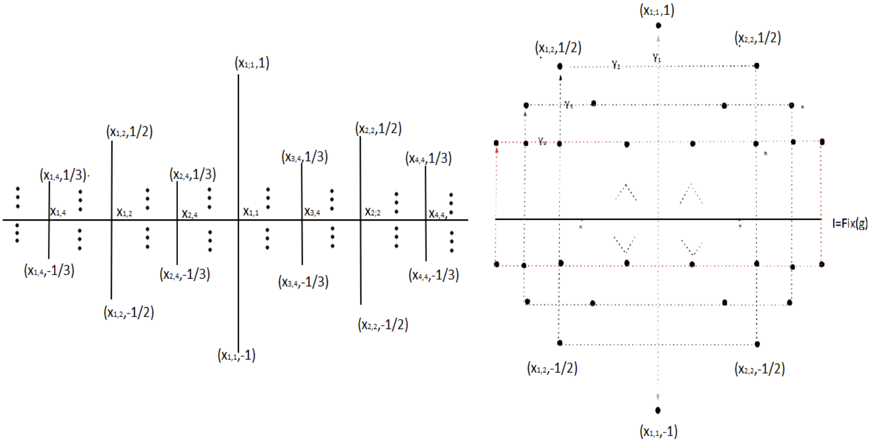
• For any $x \in I \subset Fix(g)$, we can find $a_n \in Y_n$ and some $m_n \in \mathbb{N}$ so that $(a_n)_{n\geq 0}$ converges to x and $d(f^{m_n}(x), f^{m_n}(a_n)) = d(x, f^{m_n}(a_n)) \geq \eta$ for some $\eta > 0$ hence the system (Y, g) is not equicontinuous.

counter example in the case of dendrite, the system (Y, g)

• We conclude that (Y, g) is a distal none equicontinuous system. By (J. Qiu and J. Zhao (2020)), $h_\infty(g) > 0$.

• We will extend the space Y into a dendrite X and g into a map $f : X \to X$.

The extended dendrite X



Thank you for your time and your attention