DISTRIBUTIONAL FRACTIONAL POWERS OF THE LAPLACEAN. RIESZ POTENTIALS

CELSO MARTÍNEZ, MIGUEL SANZ AND FRANCISCO PERIAGO

ABSTRACT. For different reasons it is very useful to dispose of a duality formula for the fractional powers of the Laplacean, namely, $((-\Delta)^{\alpha} u, \phi) = (u, (-\Delta)^{\alpha} \phi)$, $\alpha \in \mathbb{C}$, for ϕ belonging to a suitable functional space and u to its topological dual. Unfortunately, this formula makes no sense in the classical spaces of distributions. For this reason we introduce a new space of distributions where the above formula is established. Finally, we apply this distributional point of view of the fractional powers of the Laplacean to obtain some properties of the Riesz potentials in a wide class of spaces which contains the L^p−spaces.

1. INTRODUCTION

Throughout this paper we consider complex functions defined on $\mathbb{R}^n$. By $\mathcal{D}$ we denote the space of functions of class C^{∞} with compact support and by $\mathcal{S}$ the Schwartz space, both endowed with their usual topologies. Given a topological vectorial space Y, its topological dual will be denoted by Y'. If $T : D(T) \subset Y \to Y$ is a linear operator and $X \subset Y$ is a linear subspace of Y, by T_X we denote the operator in X with domain $D(T_X) = \{x \in X \cap D(T) : Tx \in X\}$ and defined by $T_X x = Tx$, for $x \in D(T_X)$. If $X = L^p$ we write T_p instead of T_{L^p}.

It is known that the restriction of the negative of the distributional Laplacean, $-\Delta$, to L^p−spaces is a non-negative operator. Hence, we can calculate its fractional powers in these spaces. However, as well as the duality identity

$$(\Delta u, \phi) = (u, \Delta\phi), \text{ for } \phi \in \mathcal{D} \text{ and } u \in \mathcal{D}'$$

makes sense to Δf, for a non classically differentiable function, it would be desirable that the fractional power of exponent α, $\operatorname{Re}\alpha > 0$, of this operator satisfies an analogous relation, namely

$$((-\Delta)^{\alpha} u, \phi) = (u, (-\Delta)^{\alpha} \phi)$$

for ϕ belonging to a suitable functional space $\mathcal{T}$ and u to its topological dual $\mathcal{T}'$. In the distributional space $\mathcal{T}'$ must be included the L^p−spaces, $1 \leq p \leq \infty$, and the fractional power $(-\Delta)^{\alpha}$ must be understood in the sense of the classical theory of fractional powers developed by A. V. Balakrishnan and H. Komatsu in Banach spaces and by C. Martínez, M. Sanz and V. Calvo in locally convex spaces. We solve this problem in section 3. It is not possible to take $\mathcal{T} = \mathcal{D}$ neither $\mathcal{T} = \mathcal{S}$. For this reason we introduce an appropriate functional space.

1991 *Mathematics Subject Classification.* Primary 47A60, 47A10, 47B06, 47F05, 42B20; Secondary 44A45, 26A33.

Key words and phrases. Fractional powers, Laplacean operator, Riesz potentials, Singular integrals.

Work partially supported by DGICYT, grant PB95-1103. Spain.

For a complete study of the theory of fractional powers and its applications we refer the reader to [1, 3, 4, 5, 8, 9, 10, 11, 12, 13, 18] , for instance. In section 2 we establish some specific properties of this theory, in locally convex spaces, that we need later.

In section 4 we apply this distributional point of view of the theory of fractional powers of $-\Delta$ to the study of the Riesz potentials. Given a complex number α such that $0 < \operatorname{Re}\alpha < \frac{n}{2}$, the Riesz potential R_α acting on a locally integrable on $\mathbb{R}^n$ function f is defined by

$$(R_\alpha f)(x) = \frac{\Gamma\left(\frac{n}{2}-\alpha\right)}{\pi^{n/2}2^{2\alpha}\Gamma(\alpha)} \int_{\mathbb{R}^n} |x-y|^{2\alpha-n} f(y)dy$$

whenever this convolution exists. This always happens if $f \in L^p$, $1 \leq p < \frac{n}{2\operatorname{Re}\alpha}$, since the function

$$\psi_\alpha(x) = |x|^{2\alpha-n} \quad , \; x \in \mathbb{R}^n, \, x \neq 0,$$

belongs to $L^1 + L^q$, for $\frac{n}{n-2\operatorname{Re}\alpha} < q \leq \infty$.

If we take the Fourier transform of the Riesz potential R_α, $0 < \operatorname{Re}\alpha < \frac{n}{2}$, we find that

$$(R_\alpha f)^\wedge(x) = (2\pi|x|)^{-2\alpha} \hat{f}(x) \text{ for } f \in \mathcal{S}.$$

On the other hand, since $((-\Delta) f)^\wedge(x) = (2\pi|x|)^2 \hat{f}(x)$, it is natural to think that a "good" definition of the fractional power of $(-\Delta)$ has to satisfy

$$((-\Delta)^\alpha f)^\wedge(x) = (2\pi|x|)^{2\alpha} \hat{f}(x)$$

for $f \in \mathcal{S}$ and $\operatorname{Re}\alpha \neq 0$. For this reason it is usual to identify the operator R_α with the fractional power $(-\Delta)^{-\alpha}$. However, the identity $R_\alpha f = (-\Delta)^{-\alpha} f$ has been only proved for $f \in \mathcal{S}$ and therefore, the identity $R_\alpha = (-\Delta)^{-\alpha}$ (as operators in L^p) has only a "formal" meaning.

In this paper we study the operator $(-\Delta)^{-\alpha}$ in the context of the classical theory of fractional powers and we obtain a relationship between this operator and R_α in the context of the duality $(\mathcal{T}, \mathcal{T}')$.

As a consequence of this we deduce some interesting properties of the operator $[R_\alpha]_p$. Moreover, our distributional point of view of Riesz potentials allow us to obtain some properties of R_α in other spaces.

Finally, we introduce the operator $B_{\alpha,\varepsilon}$, $\varepsilon > 0$, given by

$$(B_{\alpha,\varepsilon} f)(x) = \frac{1}{(4\pi)^\alpha \Gamma(\alpha)} \int_{\mathbb{R}^n} \left(\int_0^\infty e^{-\frac{\pi|y|^2}{t}} e^{-\frac{\varepsilon}{4\pi}t} t^{-\frac{n}{2}+\alpha-1} \, dt \right) f(x-y)dy$$

which is similar to the Bessel potential (where $\varepsilon = 1$). We prove that

$$s - \lim_{\varepsilon \to 0^+} [B_{\alpha,\varepsilon}]_p = [R_\alpha]_p \;\; , \; 1 < p < \frac{n}{2\operatorname{Re}\alpha}.$$

2. Previous results on fractional powers

In this section, X will be a sequentially complete locally convex space endowed with a directed family of seminorms $\mathfrak{P}$. The following definition was introduced in [13].

Definition 2.1. *We say that a closed linear operator* $A : D(A) \subset X \to X$ *is non-negative if* $]-\infty, 0[$ *is contained in the resolvent set* $\rho(A)$ *and the set* $\left\{\lambda (\lambda + A)^{-1} : \lambda > 0\right\}$ *is equicontinuous, i.e., if for all* $\mathfrak{p} \in \mathfrak{P}$ *there is a seminorm* $\mathfrak{p}_0(\mathfrak{p}) \in \mathfrak{P}$ *and a constant* $M = M(A, \mathfrak{p}) > 0$ *such that*

$$\mathfrak{p}\left(\lambda (A+\lambda)^{-1} x\right) \leq M \mathfrak{p}_0(x) \quad , \lambda > 0, \quad x \in X.$$

We denote by $D(A)$ the domain of A and by $R(A)$ the range of A. From now on, α will be a complex number such that $\operatorname{Re}\alpha > 0$.

It is not hard to show that if A is a non-negative operator then

$$\lim_{\mu \to 0^+} A^n \left[(A+\mu)^{-1}\right]^n x = x \text{ , for } x \in \overline{R(A)} \text{ and } n \in \mathbb{N}. \tag{1}$$

Consequently, $\overline{R(A)} = \overline{R(A^n)}$. This identity can be extended to exponents $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$.

Lemma 2.1. *The identity*

$$\overline{R(A)} = \overline{R(A^\alpha)} \text{ , } \operatorname{Re}\alpha > 0$$

holds.

Proof. It is known (see [8] and [12]) that if $0 < \operatorname{Re}\alpha < n$, n integer, then the fractional power A^α is given by

$$A^\alpha x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1} \left[(\lambda+A)^{-1} A\right]^n x \, d\lambda \text{ , } x \in D(A^n). \tag{2}$$

Moreover, in [12, Theorem 4.1] we proved that

$$D(A^\alpha) = \left\{x \in X : A^\alpha (1+A)^{-n} x \in D(A^n)\right\}$$

and

$$A^\alpha x = (1+A)^n A^\alpha (1+A)^{-n} x \text{ , for } x \in D(A^\alpha). \tag{3}$$

From (2) it follows that $R\left(A^\alpha \left[(1+A)^{-1}\right]^n\right) \subset \overline{R(A)}$. Hence, by (3) we conclude that $R(A^\alpha) \subset \overline{R(A)}$.

On the other hand, by additivity (see [13]) we find that $R(A^\alpha) \supset R(A^n)$ and consequently $\overline{R(A^\alpha)} = \overline{R(A)}$. ∎

Remark 2.1. *Balakrishnan and Komatsu defined the fractional power of exponent* α *of* A *as the closure of the operator given by (2). Therefore, the range of this fractional power is included in* $\overline{D(A) \cap R(A)}$, *which is a proper subspace of* $\overline{R(A)}$ *if* $D(A)$ *is non-dense. So, the property given in the previous lemma is a specific property of the concept of fractional power given by the authors in* [11, 13].

From (1) one deduces that if $R(A)$ is dense, then A is one-to-one. Moreover, the operator A_R has dense range in $\overline{R(A)}$ (we write A_R instead of $A_{\overline{R(A)}}$). As A_R is non-negative in $\overline{R(A)}$ (due to the fact that $(A+\lambda)^{-1}\left(\overline{R(A)}\right) \subset \overline{R(A)}$, $\lambda > 0$), it easily follows that A_R is one-to-one. It is also evident that if A is a one-to-one, non-negative operator (with non necessarily dense range), then A^{-1} is non-negative. In this case, the fractional power $A^{-\alpha}$ is given by $A^{-\alpha} = \left(A^{-1}\right)^\alpha$. This operator, $A^{-\alpha}$, is closed since A^α also is (see [12]).

Definition 2.2. *Given $n > \operatorname{Re}\alpha > 0$, $x = A^n y \in R(A^n)$, $y \in D(A^n)$, we define*

$$A_{-\alpha}x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty \lambda^{n-\alpha-1}(\lambda+A)^{-n}x\,d\lambda. \tag{4}$$

From (2) one deduces that $A_{-\alpha}A^n y = A^{n-\alpha}y$. Moreover, with the change $\lambda \to \lambda^{-1}$ in (4), it is very easy to show that if A is one-to-one, then $A_{-\alpha}x = A^{-\alpha}x$, for $x \in R(A^n)$. In this case, A^α is one-to-one and

$$(A^\alpha)^{-1} = A^{-\alpha}. \tag{5}$$

That is because $A^\alpha A^{-\alpha}x = x$, for $x \in R(A^n)$ and $A^{-\alpha}A^\alpha x = x$, for $x \in D(A^n)$. By (3) these identities also hold for $x \in D(A^{-\alpha})$ and $x \in D(A^\alpha)$, respectively.

Proposition 2.2. *$A_{-\alpha}$ is closable and its closure is given by*

$$\overline{A_{-\alpha}} = (A_R)^{-\alpha} \tag{6}$$

Consequently, if A is one-to-one, then $A^{-\alpha}$ is an extension of $\overline{A_{-\alpha}}$. The identity $A^{-\alpha} = \overline{A_{-\alpha}}$ holds if and only if $R(A)$ is dense.

Proof. Given $x = A^n y \in R(A^n)$ and $\mu > 0$ we get

$$\begin{aligned} A(\mu+A)^{-1}A_{-\alpha}x &= A_{-\alpha}A(\mu+A)^{-1}x \\ &= (A_R)_{-\alpha}A(\mu+A)^{-1}x \\ &= (A_R)^{-\alpha}A(\mu+A)^{-1}x \end{aligned}$$

and taking limits as $\mu \to 0$, as $(A_R)^{-\alpha}$ is closed, we conclude that $x \in D\left[(A_R)^{-\alpha}\right]$ and $(A_R)^{-\alpha}x = A_{-\alpha}x$. Hence, $A_{-\alpha}$ is closable and $(A_R)^{-\alpha}$ is an extension of $\overline{A_{-\alpha}}$.

Let now $x \in D\left[(A_R)^{-\alpha}\right]$ and $\mu > 0$. As $A^n(A+\mu)^{-n}x \in R[(A_R)^n]$ it follows that

$$\begin{aligned} A^n(\mu+A)^{-n}(A_R)^{-\alpha}x &= (A_R)^{-\alpha}A^n(\mu+A)^{-n}x \\ &= A_{-\alpha}A^n(\mu+A)^{-n}x \end{aligned}$$

and taking limits as $\mu \to 0$ we conclude that $x \in D\left(\overline{A_{-\alpha}}\right)$ and $\overline{A_{-\alpha}}x = (A_R)^{-\alpha}x$. This proves (6).

If A is one-to-one and $R(A)$ is not dense, by choosing $x \notin \overline{R(A)}$, it is evident that $\left(A^{-1}\right)^n\left(A^{-1}+1\right)^{-n}x \notin \overline{R(A)}$. By additivity one deduces that $A^{-\alpha}A^{-n+\alpha}(A+1)^{-n}x \notin \overline{R(A)}$. Consequently, $A^{-\alpha}$ is a proper extension of $(A_R)^{-\alpha}$. ∎

From (1) it is easy to show that if $x \in \overline{R(A)}$, then

$$\lim_{\mu\to0^+}\mu^n(A+\mu)^{-n}x = 0. \tag{7}$$

This result can be improved in this way:

Proposition 2.3. *The operators $A^\alpha(\mu+A)^{-\alpha}$ and $\mu^\alpha(\mu+A)^{-\alpha}$ are uniformly bounded for $\mu > 0$. Moreover, given $x \in X$, the following assertions are equivalent:*

(i) $x \in \overline{R(A)}$.
(ii) $\lim_{\mu\to0^+}\mu^\alpha(\mu+A)^{-\alpha}x = 0$.
(iii) $\lim_{\mu\to0^+}A^\alpha(\mu+A)^{-\alpha}x = x$.

Proof. Let us first note that as $(A+\mu)^{-1}$ is bounded, then so does $(A+\mu)^{-\alpha}$. Moreover, since $D\left[(A+\mu)^{\alpha}\right] = D\left(A^{\alpha}\right)$ (see [13]) one deduces that $D\left(A^{\alpha}\left(\mu+A\right)^{-\alpha}\right) = X$.

By additivity, we can restrict the proof of the first assertion to the case $0 < \operatorname{Re}\alpha < 1$. In this case, given $\mathfrak{p} \in \mathfrak{P}$, as

$$(\mu+A)^{-\alpha} x = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty} \lambda^{-\alpha}\left(\lambda+\mu+A\right)^{-1} x\, d\lambda \ , \ x \in X \tag{8}$$

we find that

$$\mathfrak{p}\left[(\mu+A)^{-\alpha} x\right] \leq \mu^{-\operatorname{Re}\alpha} c(\alpha) M \mathfrak{p}_0(x) \ \ , \ x \in X. \tag{9}$$

Hence, the operators $\mu^{\alpha}(\mu+A)^{-\alpha}$, $\mu > 0$, are uniformly bounded. In a similar way, from (2), $n=1$, one deduces that the operator $A^{\alpha} - (\mu+A)^{\alpha}$ can be extended to a bounded operator, T, on X which satisfies:

$$\mathfrak{p}(Tx) \leq \mu^{\operatorname{Re}\alpha} k(M,\alpha)\, \mathfrak{p}_1(x) \, , \ x \in X \text{ and } \mathfrak{p}_1 \in \mathfrak{P}. \tag{10}$$

From (9) and (10) it follows that

$$A^{\alpha}(\mu+A)^{-\alpha} - 1 = \left[A^{\alpha} - (\mu+A)^{\alpha}\right](\mu+A)^{-\alpha} \ \ , \ \mu > 0, \tag{11}$$

are uniformly bounded.

Let us now prove that (iii) implies (i). It is evident that (iii) implies that $x \in \overline{R(A^{\alpha})} = \overline{R(A)}$, according to Lemma 2.1.

To prove that (i) implies (ii) let us suppose that $x = Ay \in R(A)$ and that $0 < \operatorname{Re}\alpha < 1$. From (8) we obtain that

$$\mathfrak{p}\left[(\mu+A)^{-\alpha} x\right] \leq h_0(\alpha) M \mathfrak{p}_0(x) + h_1(\alpha)(M+1)\, \mathfrak{p}_2(y)\, , \tag{12}$$

where $\mathfrak{p}_0, \mathfrak{p}_2 \in \mathfrak{P}$.

Therefore, $\lim_{\mu\to 0^+} \mu^{\alpha}(\mu+A)^{-\alpha} x = 0$. As $\mu^{\alpha}(\mu+A)^{-\alpha}$ are uniformly bounded for $\mu > 0$, by additivity, this property also holds for $x \in \overline{R(A)}$ and $\operatorname{Re}\alpha \geq 1$.

Let us finally prove that (ii) implies (iii). If $0 < \operatorname{Re}\alpha < 1$, by applying the operator $\mu^{1-\alpha}(\mu+A)^{-1+\alpha}$ we find that $\lim_{\mu\to 0^+} \mu(\mu+A)^{-1} x = 0$. Therefore $x \in \overline{R(A)}$, since $x = A(\mu+A)^{-1} x + \mu(\mu+A)^{-1} x$. By (11), (12) and (10) it easily follows that (iii) holds for $x \in R(A)$ and by density this property also holds for $x \in \overline{R(A)}$. Finally, if $\operatorname{Re}\alpha \geq 1$ we take $m \in \mathbb{N}$ such that $\beta = \operatorname{Re}\frac{\alpha}{m} < 1$ and from

$$A^{\alpha}(\mu+A)^{-\alpha} x - x = \left[1 + \sum_{1\leq j\leq m-1} A^{j\beta}(\mu+A)^{-j\beta}\right]\left[A^{\beta}(\mu+A)^{-\beta} x - x\right]$$

one deduces that $\lim_{\varepsilon\to 0^+} A^{\alpha}(\mu+A)^{-\alpha} x = x$. ∎

Proposition 2.4. *The operator* $\overline{A_{-\alpha}}$ *satisfies*

$$s - \lim_{\mu\to 0^+} (A+\mu)^{-\alpha} = \overline{A_{-\alpha}}. \tag{13}$$

Proof. Let us denote by $T = s - \lim_{\mu \to 0^+} (A+\mu)^{-\alpha}$. If $x \in D(T)$, then

$$\lim_{\mu \to 0^+} \mu^\alpha (A+\mu)^{-\alpha} x = 0$$

and, by Proposition 2.3 we conclude that $x \in \overline{R(A)}$. Also by Proposition 2.3, taking into account that $\overline{R(A)} = \overline{R(A_R)}$ and $(A_R+\mu)^{-\alpha} x = (A+\mu)^{-\alpha} x$, we have

$$\lim_{\mu \to 0^+} (A_R)^\alpha (A_R+\mu)^{-\alpha} x = x.$$

Therefore, as $(A_R)^\alpha$ is closed, we deduce that $Tx \in D[(A_R)^\alpha]$ and $(A_R)^\alpha Tx = x$. Hence, by (5) it follows that $x \in D\left[(A_R)^{-\alpha} = \overline{A_{-\alpha}}\right]$ and $\overline{A_{-\alpha}} x = Tx$.

Conversely, if $x \in D\left(\overline{A_{-\alpha}}\right)$ then $x = (A_R)^\alpha y$, for $y \in D[(A_R)^\alpha]$. By Proposition 2.3 we obtain

$$\lim_{\mu \to 0^+} (A+\mu)^{-\alpha} x = \lim_{\mu \to 0^+} (A_R+\mu)^{-\alpha} (A_R)^\alpha y = y.$$

Therefore, $x \in D(T)$ and the proof is complete. ∎

According to [3], if A has dense domain and range and $\tau \in \mathbb{R}$, the imaginary power $A^{i\tau}$ is the closure of the closable operator

$$A_{i\tau} x = A^{1+i\tau} y \ , \ x = Ay \in D(A) \cap R(A). \tag{14}$$

It is evident that $A^{i\tau}$ and $(A+1)^{-1}$ commute.

Proposition 2.5. *Let A be a non-negative operator with dense domain and range and $\tau \in \mathbb{R}$. Then $A^{\alpha+i\tau}$ is an extension of $A^{i\tau} A^\alpha$.*

Proof. Let $n > \operatorname{Re}\alpha$ be a positive integer. Given $x \in D(A^\alpha)$ such that $A^\alpha x \in D(A^{i\tau})$, by (14) and additivity we have

$$\begin{aligned} A(A+1)^{-n} A^{i\tau} A^\alpha x &= A^{i\tau} A A^\alpha (A+1)^{-n} x \\ &= A^{1+i\tau} A^\alpha (A+1)^{-n} x \\ &= A A^{\alpha+i\tau} (A+1)^{-n} x \end{aligned}$$

and, as A is one-to-one,

$$(A+1)^{-n} A^{i\tau} A^\alpha x = A^{\alpha+i\tau} (A+1)^{-n} x.$$

The identity (3) now implies that $x \in D(A^{\alpha+i\tau})$ and $A^{\alpha+i\tau} x = A^{i\tau} A^\alpha x$. ∎

As a straightforward consequence of this proposition we find that if $A^{i\tau}$ is bounded, then $D(A^{\alpha+i\tau}) = D(A^\alpha)$.

We conclude this section with a result which states that the restriction to subspaces commutes with the fractional powers.

Proposition 2.6. *Let Y be a sequentially complete locally convex space and $A : D(A) \subset Y \to Y$ be a non-negative operator. Let $X \subset Y$ be a linear subspace of Y with the same topological properties of Y (but not necessarily a topological subspace of Y) and let us suppose that the restricted operator A_X is non-negative in X. If there exists a positive integer $n > \operatorname{Re}\alpha$ such that*

$$A^\alpha x = (A_X)^\alpha x, \text{ for all } x \in D[(A_X)^n], \tag{15}$$

then

$$[A^\alpha]_X = (A_X)^\alpha. \tag{16}$$

In particular, if we suppose that the topology of X satisfies the following property:

(p): *If $(x_n)_{n\in\mathbb{N}} \subset X$ converges, in the topology of X, to x_0 and also converges in the induced topology by Y to x_1, then $x_0 = x_1$.*

Then (16) holds.

Proof. It is evident that $(1 + A_X)^{-n}x = (1 + A)^{-n}x$, for all $x \in X$. Therefore, given $x \in D([A^\alpha]_X)$ we have

$$(A_X)^\alpha (1 + A_X)^{-n}x = A^\alpha(1 + A)^{-n}x = (1 + A)^{-n}A^\alpha x.$$

Taking into account that $A^\alpha x \in X$ one deduces that $(A_X)^\alpha (1 + A_X)^{-n}x \in D[(A_X)^n]$. Hence, by (3) we conclude that $x \in D[(A_X)^\alpha]$ and $(A_X)^\alpha x = A^\alpha x$.

In a similar way, given $x \in D[(A_X)^\alpha]$, as

$$A^\alpha(1 + A)^{-n}x = (A_X)^\alpha (1 + A_X)^{-n}x = (1 + A_X)^{-n} (A_X)^\alpha x$$

it follows easily that $x \in D(A^\alpha)$ and $A^\alpha x = (A_X)^\alpha x \in X$.

If X satisfies the property (p), then it is evident that (15) holds. Therefore (16) also holds. ∎

3. Distributional fractional powers of $-\Delta$

From now on, if Y is a vectorial space included in the general space of distributions, we will denote by Δ_Y the restriction to Y of the distributional Laplacean, i.e., Δ_Y will be the operator defined on the domain

$$D(\Delta_Y) = \{u \in Y : \Delta u \in Y\}$$

as $\Delta_Y u = \Delta u$, for $u \in D(\Delta_Y)$. If $Y = L^p$, we will write Δ_p instead of Δ_{L^p}.

Proposition 3.1. *Neither $-\Delta_{\mathcal{D}}$ nor $-\Delta_{\mathcal{S}}$ are non-negative.*

Proof. Let $\phi : \mathbb{R}^n \to]0, \infty[$, $\phi \in \mathcal{D}$, non-identically vanishing. Given $\lambda > 0$, if the operator $\lambda - \Delta_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ was surjective, the function

$$\begin{aligned} \left[(\lambda - \Delta_{\mathcal{D}})^{-1}\phi\right](x) &= \int_0^\infty e^{-\lambda t} (K_t * \phi)(x)\ dt \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty e^{-\lambda t} K_t(x-y)\, dt\right) \phi(y)\, dy \end{aligned}$$

would vanish outside a compact set. However, for all $x \in \mathbb{R}^n$ this function is positive and therefore $(\lambda - \Delta_{\mathcal{D}})^{-1}\phi \notin C_0^\infty$. Here we have denoted by K_t the heat kernel, i.e.,

$$K_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t},\ x \in \mathbb{R}^n\ ,\ t > 0.$$

On the other hand, also by means of the Fourier transform, it is very easy to show that, for $\lambda > 0$, the operator $\lambda - \Delta_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ is bijective and its inverse $(\lambda - \Delta_{\mathcal{S}})^{-1}$ is continuous. Moreover, if $\phi \in \mathcal{S}$ and $x \in \mathbb{R}^n$ then

$$\left[(\lambda - \Delta_{\mathcal{S}})^{-1}\phi\right]^\wedge(x) = \frac{1}{\lambda + 4\pi^2 |x|^2} \hat{\phi}(x).$$

Consequently, if $-\Delta_{\mathcal{S}}$ was a non-negative operator, given $\alpha \in \mathbb{C}$ such that $0 < \operatorname{Re}\alpha < 1$, by (2) we would obtain

$$\begin{aligned}
\left[(-\Delta_{\mathcal{S}})^{\alpha}\phi\right]^{\wedge}(x) &= \left(\frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\lambda^{\alpha-1}(-\Delta_{\mathcal{S}})(\lambda-\Delta_{\mathcal{S}})^{-1}\phi\, d\lambda\right)^{\wedge}(x) \\
&= \frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\lambda^{\alpha-1}\left[(-\Delta_{\mathcal{S}})(\lambda-\Delta_{\mathcal{S}})^{-1}\phi\right]^{\wedge}(x)\; d\lambda \\
&= \left(\frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\lambda^{\alpha-1}\frac{4\pi^2|x|^2}{\lambda+4\pi^2|x|^2}\, d\lambda\right)\overset{\wedge}{\phi}(x) \\
&= \left(4\pi^2|x|^2\right)^{\alpha}\overset{\wedge}{\phi}(x),
\end{aligned}$$

where we have used the fact that the Fourier transform is a continuous operator from $\mathcal{S}$ to itself and that the convergence in the usual topology of $\mathcal{S}$ implies uniform convergence.

Therefore, the function $\left(4\pi^2|x|^2\right)^{\alpha}\overset{\wedge}{\phi}(x)$ would belong to $\mathcal{S}$, which in general, is not true. ∎

This proposition justifies the introduction of a new space, to study the Laplacean, instead of the spaces $\mathcal{D}$ or $\mathcal{S}$.

Definition 3.1. *We denote by $\mathcal{T}$ the space of functions $\phi : \mathbb{R}^n \to \mathbb{C}$ of class C^{∞} such that any partial derivative belongs to $L^1 \cap L^{\infty}$. We endow this space with the natural topology defined by the seminorms*

$$|\phi|_m = max\left\{\left\|D^{\beta}\phi\right\|_1,\ \left\|D^{\beta}\phi\right\|_{\infty} : \beta \in \mathbb{N}^n,\ |\beta| \leq m\right\},\ \phi \in \mathcal{T},\ m \in \mathbb{N}.$$

Remark 3.1. *It is very easy to show that $\mathcal{T}$ endowed with the increasing countable family of seminorms $\{|\cdot|_m : m \in \mathbb{N}\}$ is a Fréchet space. However, this space is non-normable. If it was normable, there would be an index m_0 and a constant $k_{m_0} \geq 0$ such that $|\phi|_{m_0+1} \leq k_{m_0}|\phi|_{m_0}$ for all $\phi \in \mathcal{T}$. Thus, if we take a multi-index β such that $|\beta| = m_0 + 1$ and a function $\psi \in \mathcal{T}$ with $D^{\beta}\psi$ non-identically vanishing, for $\phi(x) = \psi(rx)$ ($r > 1$ constant) we have*

$$r^{m_0+1}\left\|D^{\beta}\psi\right\|_{\infty} = \left\|D^{\beta}\phi\right\|_{\infty} \leq k_{m_0}|\phi|_{m_0} \leq k_{m_0}r^{m_0}|\psi|_{m_0},$$

and taking limits when $r \to \infty$ we conclude that $\left\|D^{\beta}\psi\right\|_{\infty} = 0$, which is a contradiction.

Remark 3.2. *It is evident that $\mathcal{T} \subset L^p$ $(1 \leq p \leq \infty)$ and also that $\mathcal{S} \subset \mathcal{T}$. Moreover, it is very easy to show that the induced topology of $\mathcal{S}$ is weaker than the usual topology of this space, and that $\mathcal{D}$ is dense in $(\mathcal{T},\ |\cdot|_m,\ m \in \mathbb{N})$. It is also easy to prove that, for all multi-index β, $\lim_{|x|\to\infty} D^{\beta}\phi(x) = 0$ for $\phi \in \mathcal{T}$.*

Lemma 3.2. *If $f \in L^p$ $(1 \leq p \leq \infty)$ and $\phi \in \mathcal{T}$, then the convolution $f * \phi$ exists, belongs to C^{∞} and satisfies that $D^{\beta}(f * \phi) = f * D^{\beta}\phi$, for all multi-index β. In particular, the convolution $R_{\alpha}\phi = \psi_{\alpha} * \phi$ is well-defined if $0 < \operatorname{Re}\alpha < \frac{n}{2}$ and $\phi \in \mathcal{T}$. Moreover, $R_{\alpha}\phi \in C^{\infty}$. If $f \in L^1$, then $f * \phi \in \mathcal{T}$ and the operator $\phi \to f * \phi$ is continuous in $\mathcal{T}$.*

Proof. The proof of this result is an immediate consequence of the Hölder and the Young inequalities. The convolution $R_{\alpha}\phi = \psi_{\alpha} * \phi$ exists since the function ψ_{α} can be decomposed as: $\psi_{\alpha} = \mu_{\alpha} + v_{\alpha}$, where $\mu_{\alpha} = \psi_{\alpha}\chi_{B(0,1)} \in L^1$ $(\chi_{B(0,1)}$

denotes the characteristic function of the ball of radius 1, centered at the origin) and $v_\alpha = \psi_\alpha - \mu_\alpha \in L^q$, $\frac{n}{n-2\operatorname{Re}\alpha} < q \leq \infty$. ∎

Theorem 3.3. *The operator $\Delta_{\mathcal{T}}$, restriction to the space $\mathcal{T}$ of the Laplacean, is continuous and it is also the infinitesimal generator of the heat semigroup, which is a contractive semigroup of class C_0. Consequently, $-\Delta_{\mathcal{T}}$ is a non-negative operator.*

Proof. It is evident that $\Delta_{\mathcal{T}}$ is continuous. On the other hand, as $K_t \in L^1$ and $\|K_t\|_1 = 1$, from the preceding lemma one deduces that $P_t\phi = K_t * \phi \in \mathcal{T}$, for all $\phi \in \mathcal{T}$, and

$$|P_t\phi|_m \leq \|K_t\|_1 \, |\phi|_m = |\phi|_m \ , \ m = 0, 1, 2, \dots$$

Now, by the approximations to the identity theorem we conclude that $\mathcal{T}\text{-}\lim_{t\to 0} P_t\phi = \phi$.

A simple calculation shows that, for $t > 0$, $s > 0$,

$$P_t P_s\phi = K_t * (K_s * \phi) = (K_t * K_s) * \phi = K_{t+s} * \phi = P_{t+s}\phi.$$

Hence, we conclude that $\{P_t : t > 0\}$ is a contractive semigroup of class C_0.

Let A be its infinitesimal generator. We shall prove that $A = \Delta_{\mathcal{T}}$. Given $t_0 > 0$ and $\phi \in D(A)$ we have

$$\mathcal{T} - \lim_{\delta\to 0} \left[\delta^{-1}\left(P_{t_0+\delta}\phi - P_{t_0}\phi\right) - P_{t_0}A\phi\right] = 0$$

and hence,

$$\frac{\partial}{\partial t}\,|_{t=t_0} \left[\left(P_t\phi\right)(x)\right] = \left(P_{t_0}A\phi\right)(x) \ \text{, for all } x \in \mathbb{R}^n.$$

On the other hand, as the function $u(x,t) = (P_t\phi)(x)$ is a solution of the heat equation,

$$\frac{\partial}{\partial t}\,|_{t=t_0} \left[\left(P_t\phi\right)(x)\right] = \left(\Delta P_{t_0}\phi\right)(x) = \left(P_{t_0}\Delta\phi\right)(x).$$

So, we deduce that $P_{t_0}\Delta\phi = P_{t_0}A\phi$, and taking limits when $t_0 \to 0$ we conclude that $\Delta\phi = A\phi$.

In a similar way as in Banach spaces, it is not hard to show (see [20, Th. 1, p. 240]) that if A is the infinitesimal generator of an equicontinuous semigroup of class C_0, then $-A$ is a non-negative operator.

Finally, we shall prove that $D(A) = \mathcal{T}$. To do this it is sufficient to prove that $1 - \Delta_{\mathcal{T}}$ is a one-to-one operator. In effect: for every $\phi \in \mathcal{T}$ there exists $\psi \in D(A)$ such that $(1 - A)\psi = (1 - \Delta_{\mathcal{T}})\phi$, since $1 - A$ is surjective, due to the fact that $-A$ is non-negative. Since $\psi \in D(A)$,

$$(1 - A)\psi = (1 - \Delta_{\mathcal{T}})\psi = (1 - \Delta_{\mathcal{T}})\phi$$

and as $1 - \Delta_{\mathcal{T}}$ is one-to-one we conclude that $\phi = \psi \in D(A)$.

To prove that $1 - \Delta_{\mathcal{T}}$ is a one-to-one operator it is sufficient to take Fourier transforms since, if $(1 - \Delta_{\mathcal{T}})\phi = 0$ then

$$[(1 - \Delta)\phi]^\wedge(x) = \left(1 + 4\pi^2|x|^2\right)\overset{\wedge}{\phi}(x) = 0, \ \text{for all } x \in \mathbb{R}^n$$

and hence $\overset{\wedge}{\phi} = 0$, that implies $\phi = 0$. ∎

Remark 3.3. *By means of Fourier transforms it is also very easy to show that* $\Delta_{\mathcal{T}}$ *is a one-to-one operator. However, this operator has non-dense range. To prove this, let us consider the linear form* $u : \phi \to \int_{\mathbb{R}^n} \phi(x)\, dx$ *which is continuous and non-identically vanishing. However,* $(u, \Delta\phi) = 0$ *for all* $\phi \in \mathcal{T}$*, due to the density of* $\mathcal{D}$ *in* $\mathcal{T}$*.*

Let us now consider the topological dual space of $\mathcal{T}$ that we shall denote by $\mathcal{T}'$. Note that as the topology that $\mathcal{T}$ induces on $\mathcal{S}$ is weaker than the usual topology of this space, we find that if $u \in \mathcal{T}'$ then u can be identified as a tempered distribution. Moreover, as $\mathcal{S}$ is dense in $\mathcal{T}$, u is completely determined by its restriction to the space $\mathcal{S}$.

We endow $\mathcal{T}'$ with the topology of uniform convergence on bounded sets of $\mathcal{T}$, i.e., the topology defined by the seminorms

$$|u|_B = \sup_{\phi \in B} |(u, \phi)| \ , \ u \in \mathcal{T}' \ , \ B \subset \mathcal{T} \ \text{ a bounded set.}$$

In $\mathcal{T}'$ the two main requirements that we need hold: the negative of the Laplacean is a non-negative operator and the spaces L^p $(1 \le p \le \infty)$ are included in $\mathcal{T}'$.

Remark 3.4. *Since* $\mathcal{T}$ *is non-normable, no countable family of bounded sets exists such that every bounded set in* $\mathcal{T}$ *is contained in this family. Hence,* $\mathcal{T}'$ *is non-metrizable. However, by the Banach-Steinhaus theorem (see* [16, p. 86]*), this space is sequentially complete. So, we have a non-trivial example of a sequentially complete locally convex space where it will be very useful to apply the theory of fractional powers developed in* [12, 13]*.*

Proposition 3.4. *For all* $1 \le p \le \infty$*,* $L^p \subset \mathcal{T}'$ *and the induced topology of* L^p *is weaker than the usual topology of this space.*

Proof. Let us consider $f \in L^p$, $B \subset \mathcal{T}$ a bounded set and let us denote by

$$k = \sup_{\phi \in B} \left\{ \|\phi\|_1 \, , \ \|\phi\|_\infty \right\},$$

which is finite since B is a bounded set. From the Hölder inequality, if q is the conjugate exponent of p, it follows that

$$\sup_{\phi \in B} \left| \int_{\mathbb{R}^n} f(x)\, \phi(x) \ dx \right| \le \sup_{\phi \in B} \left[\|\phi\|_q \, , \ \|f\|_p \right] \le k \, \|f\|_p \, ,$$

and thus $f \in \mathcal{T}'$ and $|f|_B \le k \, \|f\|_p$. ∎

Derivation and convolution in $\mathcal{T}'$. Given $u \in \mathcal{T}'$ and β a multi-index, the distributional derivative $D^\beta u$ can be extended to an element (that we also denote by $D^\beta u$) that belongs to the dual space $\mathcal{T}'$ and which is defined by

$$\left(D^\beta u, \phi \right) = \left(u, (-1)^{|\beta|} \, D^\beta \phi \right) \ , \ \phi \in \mathcal{T}.$$

In particular, the Laplacean operator in $\mathcal{T}'$, $\Delta_{\mathcal{T}'}$, acts as

$$\left(\Delta_{\mathcal{T}'} u, \phi \right) = (u, \Delta\phi) \ , \ \phi \in \mathcal{T}.$$

For the convolution, given $f \in L^1$ and $u \in \mathcal{T}'$, we define the convolution $u * f$ as the linear form

$$\phi \to \left(u, \widetilde{f} * \phi \right) \ , \ \phi \in \mathcal{T},$$

where $\tilde{f}(x) = f(-x)$. From Lemma 3.2 it follows that $u * f \in \mathcal{T}'$.

Theorem 3.5. *$-\Delta_{\mathcal{T}'}$ is a continuous and non-negative operator but it is not a one-to-one operator.*

Proof. Given $B \subset \mathcal{T}$, a bounded set, and $u \in \mathcal{T}'$, as the set $E = \{\Delta\phi : \phi \in B\}$ is also bounded, from

$$|\Delta_{\mathcal{T}'}u|_B = \sup_{\phi\in B} |(\Delta_{\mathcal{T}'}u, \phi)| = \sup_{\phi\in B} |(u, \Delta\phi)| = |u|_E$$

it follows that $\Delta_{\mathcal{T}'}$ is continuous.

Let us now consider $\lambda > 0$ and $u \in \mathcal{T}'$. It is very easy to prove that the linear form $v : \psi \to \left(u, (\lambda - \Delta_{\mathcal{T}})^{-1}\psi\right)$ is continuous and that $(\lambda - \Delta_{\mathcal{T}'})\, v = u$. Therefore, $\lambda - \Delta_{\mathcal{T}'}$ is surjective.

On the other hand, let $u \in \mathcal{T}'$ such that $(\lambda - \Delta_{\mathcal{T}'})\, u = 0$. Then, if $\phi \in \mathcal{T}$

$$((\lambda - \Delta_{\mathcal{T}'})\, u, \phi) = (u, (\lambda - \Delta_{\mathcal{T}})\, \phi) = 0,$$

and thus (as $R(\lambda - \Delta_{\mathcal{T}}) = \mathcal{T}$, due to the fact that $-\Delta_{\mathcal{T}}$ is a non-negative operator) $u = 0$.

If we take a bounded set, $B \subset \mathcal{T}$, and $u \in \mathcal{T}'$, since $-\Delta_{\mathcal{T}}$ is a non-negative operator, the set $F = \left\{\mu\,(\mu - \Delta_{\mathcal{T}})^{-1}\phi : \phi \in B,\ \mu > 0\right\}$ is also bounded and thus, for $\lambda > 0$,

$$\begin{aligned} \left|\lambda\,(\lambda - \Delta_{\mathcal{T}'})^{-1} u\right|_B &= \sup_{\phi\in B} \left|\left(\lambda\,(\lambda - \Delta_{\mathcal{T}'})^{-1} u, \phi\right)\right| \\ &= \sup_{\phi\in B} \left|\left(u, \lambda\,(\lambda - \Delta_{\mathcal{T}})^{-1}\phi\right)\right| \le |u|_F\,. \end{aligned}$$

We now conclude that $-\Delta_{\mathcal{T}'}$ is a non-negative operator.

Finally, as the constant functions belong to the space $\mathcal{T}'$ and obviously their Laplacean is null we find that $\Delta_{\mathcal{T}'}$ is not a one-to-one operator. ∎

In the next theorem we point out a dual relationship between the operators $(-\Delta_{\mathcal{T}})^{\alpha}$ and $(-\Delta_{\mathcal{T}'})^{\alpha}$.

Theorem 3.6. *If $\phi \in \mathcal{T}$, $u \in \mathcal{T}'$ and $\operatorname{Re}\alpha > 0$, then the duality formula*

$$((-\Delta_{\mathcal{T}'})^{\alpha}\, u, \phi) = (u, (-\Delta_{\mathcal{T}})^{\alpha}\, \phi)$$

holds.

Proof. Let $m > \operatorname{Re}\alpha > 0$ be a positive integer, $\phi \in \mathcal{T}$ and $u \in \mathcal{T}'$. Since $\Delta_{\mathcal{T}'}$ is continuous

$$\begin{aligned} \frac{\Gamma(\alpha)\,\Gamma(m-\alpha)}{\Gamma(m)}\,((-\Delta_{\mathcal{T}'})^{\alpha}\, u, \phi) &= \left(\int_0^{\infty} \lambda^{\alpha-1}\left[(-\Delta_{\mathcal{T}'})\,(\lambda - \Delta_{\mathcal{T}'})^{-1}\right]^m u\, d\lambda,\ \phi\right) \\ &= \int_0^{\infty} \left(\lambda^{\alpha-1}\left[(-\Delta_{\mathcal{T}'})\,(\lambda - \Delta_{\mathcal{T}'})^{-1}\right]^m u,\ \phi\right) d\lambda \\ &= \int_0^{\infty} \left(u,\ \lambda^{\alpha-1}\left[(-\Delta_{\mathcal{T}})\,(\lambda - \Delta_{\mathcal{T}})^{-1}\right]^m \phi\right) d\lambda \\ &= \left(u,\ \int_0^{\infty} \lambda^{\alpha-1}\left[(-\Delta_{\mathcal{T}})\,(\lambda - \Delta_{\mathcal{T}})^{-1}\right]^m \phi\, d\lambda\right) \\ &= \frac{\Gamma(\alpha)\,\Gamma(m-\alpha)}{\Gamma(m)}\,(u,\ (-\Delta_{\mathcal{T}})^{\alpha}\,\phi)\,, \end{aligned}$$

where the first and the last identity follow from (2), the second one is a consequence of the fact that the convergence in $\mathcal{T}'$ implies weak convergence; the third one can be justified by the duality relations between $(\lambda - \Delta_{\mathcal{T}'})^{-1}$ and $(\lambda - \Delta_{\mathcal{T}})^{-1}$ and, finally, the fourth one is an immediate consequence of the continuity of u. ∎

4. Riesz Potentials

In this section we shall obtain a relationship between the Riesz potentials and the fractional powers of the negative of the Laplacean operator in the spaces $\mathcal{T}$ and $\mathcal{T}'$. As a consequence of this, we shall deduce some interesting properties of the operator R_α.

Lemma 4.1. *If* $0 < \operatorname{Re}\alpha < \frac{n}{2}$ *and* $\phi \in \mathcal{T}$, *then*

$$(-\Delta_{\mathcal{T}})^{n-\alpha}\phi = R_\alpha(-\Delta)^n\phi = (-\Delta)^n R_\alpha\phi. \tag{17}$$

Proof. By applying (2)

$$(-\Delta_{\mathcal{T}})^{n-\alpha}\phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty \lambda^{n-\alpha-1}\left[(\lambda-\Delta_{\mathcal{T}})^{-1}\right]^n(-\Delta_{\mathcal{T}})^n\phi\, d\lambda.$$

On the other hand, as $(\lambda - \Delta_{\mathcal{T}})^{-1}$ is the Laplace transform of the heat semigroup, P_t, it easily follows (see [20, p. 242]) that

$$\left[(\lambda-\Delta_{\mathcal{T}})^{-1}\right]^n\psi = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}(K_t*\psi)\,dt\ ,\ \psi\in\mathcal{T}.$$

If $\psi = (-\Delta_{\mathcal{T}})^n\phi$, as the $\mathcal{T}$-convergence implies pointwise-convergence

$$\begin{aligned}&\left((-\Delta_{\mathcal{T}})^{n-\alpha}\phi\right)(x)\\ =\ &\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty\lambda^{n-\alpha-1}\left(\int_0^\infty t^{n-1}e^{-\lambda t}(K_t*\psi)(x)\,dt\right)d\lambda,\end{aligned}$$

for all $x\in\mathbb{R}^n$, where, if we interchange the order of integration

$$\left((-\Delta_{\mathcal{T}})^{n-\alpha}\phi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(K_t*\psi)(x)\ dt,$$

since $\frac{1}{\Gamma(n-\alpha)}\int_0^\infty \lambda^{n-\alpha-1}e^{-\lambda t}\ d\lambda = t^{\alpha-n}$. Note that we can apply the Tonelli-Hobson theorem due to $0<\operatorname{Re}\alpha<\frac{n}{2}$ and

$$|(K_t*\psi)(x)| \le \|K_t\|_\infty\|\psi\|_1 = \frac{1}{(4\pi t)^{n/2}}\|\psi\|_1,$$

$$|(K_t*\psi)(x)| \le \|K_t\|_1\|\psi\|_\infty = \|\psi\|_\infty.$$

In a similar fashion

$$\int_0^\infty t^{\alpha-1}(K_t*\psi)(x)\ dt = \int_{\mathbb{R}^n}\left(\int_0^\infty t^{\alpha-1}K_t(y)\ dt\right)\psi(x-y)\,dy,$$

since

$$\int_0^\infty t^{\alpha-1}K_t(y)\ dt = \frac{\Gamma\left(\frac{n}{2}-\alpha\right)}{2^{2\alpha}\pi^{n/2}}|y|^{2\alpha-n}.$$

This proves the first identity. The second one follows from Lemma 3.2. ∎

It is known that if $1<p<\infty$, then $-\Delta_p$ is non-negative, with dense domain and range.

Proposition 4.2. *If* $1 \leq p < \frac{n}{2\,\mathrm{Re}\,\alpha}$, *then* $R_\alpha f \in \overline{R(\Delta_{\mathcal{T}'})}$ *for all* $f \in L^p$. *Moreover*

$$(-\Delta_{\mathcal{T}'})^{n-\alpha} f = (-\Delta)^n R_\alpha f. \tag{18}$$

Proof. Let us first prove that $L^p \subset \overline{R(\Delta_{\mathcal{T}'})}$, $1 \leq p < \infty$.

If $1 < p < \infty$, we know that L^p is the L^p-closure of $R(\Delta_p)$ which by Proposition 3.4 is included in the $\mathcal{T}'-$closure. Thus, $L^p \subset \overline{R(\Delta_{\mathcal{T}'})}$. Moreover, as L^p is dense in L^1 one also deduces that $L^1 \subset \overline{R(\Delta_{\mathcal{T}'})}$.

Let $f \in L^p$. By means of the Young inequality, the condition $1 \leq p < \frac{n}{2\,\mathrm{Re}\,\alpha}$ implies that $R_\alpha f$ can be decomposed as: $R_\alpha f = g + h$, $g \in L^p$, $h \in L^r$ and $r > 0$ such that $\frac{1}{r} < \frac{1}{p} - \frac{2\,\mathrm{Re}\,\alpha}{n}$. Therefore $R_\alpha f \in \overline{R(\Delta_{\mathcal{T}'})}$.

By Theorem 3.6 and (17), the proof of (18) can be reduced to prove that given $f \in L^p$, then

$$\int_{\mathbb{R}^n} (R_\alpha f)(x)\phi(x)dx = \int_{\mathbb{R}^n} f(x)(R_\alpha \phi)(x)dx \text{ , for all } \phi \in \mathcal{T}$$

and this identity easily follows from the Tonelli-Hobson theorem. ∎

In the following result, Δ_R denotes the restriction of the distributional Laplacean to $\overline{R(\Delta_{\mathcal{T}'})}$.

Theorem 4.3. *If* $1 \leq p < \frac{n}{2\,\mathrm{Re}\,\alpha}$, *then* $L^p \subset D\left[(-\Delta_R)^{-\alpha}\right]$ *and*

$$(-\Delta_R)^{-\alpha} f = R_\alpha f \text{ , for all } f \in L^p. \tag{19}$$

Proof. Let $f \in L^p$. By applying $(\lambda - \Delta_{\mathcal{T}'})^{-n}$ $(\lambda > 0)$ to both sides of (18) and taking into account that $(-\Delta_{\mathcal{T}'})^{n-\alpha}$ commutes with this operator we obtain

$$\begin{aligned}(\lambda - \Delta_{\mathcal{T}'})^{-n} (-\Delta_{\mathcal{T}'})^{n-\alpha} f &= (-\Delta_{\mathcal{T}'})_{-\alpha} (-\Delta_{\mathcal{T}'})^n (\lambda - \Delta_{\mathcal{T}'})^{-n} f \\ &= (\lambda - \Delta_{\mathcal{T}'})^{-n} (-\Delta_{\mathcal{T}'})^n R_\alpha f.\end{aligned}$$

Since f and $R_\alpha f$ belong to $\overline{R(-\Delta_{\mathcal{T}'})}$, taking limits as $\lambda \to 0$ we conclude that $f \in D\left(\overline{(-\Delta_{\mathcal{T}'})_{-\alpha}}\right)$ and $\overline{(-\Delta_{\mathcal{T}'})_{-\alpha}} f = R_\alpha f$. Finally, from (6) one deduces (19). ∎

Corollary 4.4 (Additivity). *If* $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,(\alpha + \beta) < \frac{n}{2p}$, *then*

$$R_\beta R_\alpha f = R_{\alpha+\beta} f \text{ , for all } f \in L^p \tag{20}$$

Proof. Let $f \in L^p$. The existence of $R_{\alpha+\beta} f$ and $R_\alpha f$ is evident. Moreover, from Theorem 4.3 one deduces that $R_\alpha f = (-\Delta_R)^{-\alpha} f$ and $R_{\alpha+\beta} f = (-\Delta_R)^{-\alpha-\beta} f$.

As we have already seen in the proof of Proposition 4.2, $R_\alpha f = g + h$, $g \in L^p$ and $h \in L^r$, for all $r > 0$ such that $\frac{1}{r} < \frac{1}{p} - \frac{2\,\mathrm{Re}\,\alpha}{n}$. It is clear that $R_\beta g$ exists and, if we take $\frac{1}{r} > \frac{2\,\mathrm{Re}\,\beta}{n}$ also exists $R_\beta h$. Hence, there exists $R_\beta R_\alpha f$.

Theorem 4.3 implies that $R_\beta R_\alpha f = (-\Delta_R)^{-\beta} R_\alpha f$. From the additivity of the fractional powers we now deduce (20). ∎

Corollary 4.5. *Let* $X \subset L^1 + L^p$ $(1 \leq p < \infty)$ *be a sequentially complete locally convex space which satisfies property (p) in Proposition 2.6 with* $Y = \mathcal{T}'$. *Then, if the operator* $-\Delta_X$ *is non-negative, the identity*

$$(-\Delta_X)^{-\alpha} = [R_\alpha]_X \text{ , } 0 < \mathrm{Re}\,\alpha < \frac{n}{2p} \tag{21}$$

holds. In particular,

$$(-\Delta_p)^{-\alpha} = [R_\alpha]_p \,. \tag{22}$$

Proof. This result is an immediate consequence of Theorem 4.3 and Proposition 2.6. ∎

Remark 4.1. *The identity (21) can be applied in some interesting spaces such as*

$$X = L^r + L^s \ , \ 1 \le r \le s \le p$$

with its usual norm, or

$$X = \{f \in L^{r_1} + L^{s_1} : \Delta f \in L^{r_2} + L^{s_2}\} \ , 1 \le r_k \le s_k \le p \ , \ k = 1, 2$$

with the graph norm.

Another consequence of (21) is that R_α is one-to-one in $L^1 + L^p$.

As a consequence of the general properties of the fractional powers of $-\Delta_p$ we deduce the following results:

Corollary 4.6. *The following properties hold:*

(i): *If $\alpha, \beta \in \mathbb{C}$ such that $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta < \frac{n}{2p}$, then*

$$D\left([R_\beta]_p\right) \subset D\left([R_\alpha]_p\right).$$

(ii): *If $1 < p < \frac{n}{2\operatorname{Re}\alpha}$ and $\beta \in \mathbb{C}$ such that $\operatorname{Re}\alpha = \operatorname{Re}\beta$, then*

$$D\left([R_\alpha]_p\right) = D\left([R_\beta]_p\right). \tag{23}$$

Proof. The first assertion follows from (22) and the additivity of the fractional powers.

On the other hand, it is known (see [15]) that if $1 < p < \infty$ and $\tau \in \mathbb{R}$, then $(-\Delta_p)^{i\tau}$ is bounded. Therefore, from Proposition 2.5 one deduces (23). ∎

Following [7] we introduce the notion of ω-sectoriality. Given $\omega \in \,]0, \pi]$, we say that a closed linear operator $A : D(A) \subset X \to X$ is ω-sectorial if the spectrum of A satisfies that

$$\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\} \cup \{0\}$$

and the operators $z(z - A)^{-1}$ are uniformly bounded for $z \notin S_\omega$. Kato and Hille proved (see [6, p. 384] and [7]) that if A is ω-sectorial, $0 < \omega < \frac{\pi}{2}$, then $-A$ is the infinitesimal generator of an analytic semigroup of amplitude $\frac{\pi}{2} - \omega$. Conversely, if $\{T(z) : z \in S_\tau \setminus \{0\}\}$, $0 < \tau \le \frac{\pi}{2}$, is an analytic semigroup and $-A$ is its infinitesimal generator, then A is $\left(\frac{\pi}{2} - \tau + \varepsilon\right)$-sectorial, for $0 < \varepsilon < \tau$.

It is known (see [2]) that if $1 \le p < \infty$, then the operator Δ_p is the infinitesimal generator of the heat semigroup, which is analytic. Hence, $-\Delta_p$ is $\left(\frac{\pi}{2} - \delta + \varepsilon\right)$-sectorial, for $\delta = \arctan \frac{1}{ne}$ and $0 < \varepsilon < \delta$. If $1 < p < \infty$, by means of the Mihlin multiplier theorem it can be proved (see [15]) that $-\Delta_p$ is ε-sectorial for all $\varepsilon > 0$.

Corollary 4.7 (Sectoriality). *If $1 < p < \infty$ and $0 < \alpha < \frac{n}{2p}$, then $[R_\alpha]_p$ is $\varepsilon-$sectorial for all $\varepsilon > 0$. Moreover*

$$\sigma\left([R_\alpha]_p\right) = [0, \infty[\,.$$

Consequently, $-[R_\alpha]_p$ is the infinitesimal generator of an analytic semigroup of amplitude $\frac{\pi}{2}$.

If $0<\varepsilon<\delta=\arctan\frac{1}{ne}$ *and* $0<\alpha<\min\left\{\frac{n}{2},\ \frac{\pi}{\pi/2-\delta+\varepsilon}\right\}$, *then* $-[R_\alpha]_1$ *is a non-negative operator.*

Proof. It is known (see [7, Th. 2]) that if A is $\omega-$sectorial and $0<\alpha<\frac{\pi}{\omega}$, then A^α is $\alpha\omega-$sectorial. On the other hand, from the identity $z(z+A)^{-1}=A\left(z^{-1}+A\right)^{-1}$ it follows that if A is a one-to-one, $\omega-$sectorial operator, then A^{-1} is also $\omega-$sectorial. Hence, by (22) we deduce the properties of sectoriality of $[R_\alpha]_p$.

The identity $\sigma\left([R_\alpha]_p\right)=[0,\infty[$ follows from (22) and the spectral mapping theorem for fractional powers (see, for instance [12]). Finally, from [7] one deduces that $-[R_\alpha]_p$ is the infinitesimal generator of an analytic semigroup of amplitude $\frac{\pi}{2}$. ∎

Remark 4.2. *Note that* $-[R_\alpha]_1$ *does not generate any strongly* C_0-*semigroup since its domain is not dense.*

Corollary 4.8 (Multiplicativity). *If* $1<p<\infty$, $0<\alpha<\frac{n}{2p}$ *and* $\beta\in\mathbb{C}$ *such that* $0<\alpha\operatorname{Re}\beta<\frac{n}{2p}$, *then*

$$\left([R_\alpha]_p\right)^\beta=[R_{\alpha\beta}]_p$$

If $0<\varepsilon<\delta=\arctan\frac{1}{ne}$, $0<\alpha<\min\left\{\frac{n}{2},\frac{\pi}{\pi/2-\delta+\varepsilon}\right\}$ *and* $\beta\in\mathbb{C}$ *such that* $0<\alpha\operatorname{Re}\beta<\frac{n}{2}$, *then*

$$\left([R_\alpha]_1\right)^\beta=[R_{\alpha\beta}]_1$$

Proof. The proof is an immediate consequence of (22) and the multiplicativity of the fractional powers (see, for instance [19] and [14]). ∎

Given $\alpha\in\mathbb{C}_+$ and $\varepsilon>0$ we consider the function

$$G_{\alpha,\varepsilon}(x)=\frac{1}{(4\pi)^\alpha\,\Gamma(\alpha)}\int_0^\infty e^{-\frac{\pi|x|^2}{t}}e^{-\frac{\varepsilon}{4\pi}t}t^{-\frac{n}{2}+\alpha-1}\,dt\ ,\ x\in\mathbb{R}^n\ ,\ x\neq0.$$

It is easy to check that $G_{\alpha,\varepsilon}\in L^1$ and that its Fourier transform is $\hat{G}_{\alpha,\varepsilon}(x)=\left(\varepsilon+4\pi^2|x|^2\right)^{-\alpha}$ (see, for instance [17, p. 131]).

The Bessel potential of degree α acting on a locally integrable on $\mathbb{R}^n$ function f is defined by the convolution $B_{\alpha,\varepsilon}f=G_{\alpha,\varepsilon}*f$, if this convolution exists. As a consequence of the Young inequality, the operator

$$\begin{array}{rccc}[B_{\alpha,\varepsilon}]_p: & L^p & \rightarrow & L^p\\ & f & \mapsto & G_{\alpha,\varepsilon}*f\end{array}$$

is bounded.

Theorem 4.9. *If* $1\leq p<\infty$, $\varepsilon>0$ *and* $\operatorname{Re}\alpha>0$, *then*

$$(\varepsilon-\Delta_p)^{-\alpha}=[B_{\alpha,\varepsilon}]_p\,.$$

Moreover, if $1<p<\frac{n}{2\operatorname{Re}\alpha}$, *then*

$$s-\lim_{\varepsilon\rightarrow0^+}[B_{\alpha,\varepsilon}]_p=[R_\alpha]_p\,. \tag{24}$$

The operator $[R_\alpha]_1$ *is a proper extension of* $s-\lim_{\varepsilon\rightarrow0^+}[B_{\alpha,\varepsilon}]_1$.

Proof. By means of Fourier transforms

$$\left[(\varepsilon-\Delta_p)^{-\alpha} f\right]^{\wedge}(x)=\left(\varepsilon+4\pi^2|x|^2\right)^{-\alpha}\hat{f}(x)\ , \ a.e.\ x\in\mathbb{R}^n,\ f\in\mathcal{S}. \tag{25}$$

Since $(\varepsilon-\Delta_p)^{-\alpha}$ and $[B_{\alpha,\varepsilon}]_p$ are both bounded, by density, from (25) one deduces that $(\varepsilon-\Delta_p)^{-\alpha}=[B_{\alpha,\varepsilon}]_p$.

Finally, (24) is an immediate consequence of Propositions 2.2 and 2.4, taking into account (22). ∎

References

[1] A. V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them,* Pacific J. Math. **10** (1960), 419-437.

[2] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Springer-Verlag, Berlin Heidelberg, 1967.

[3] G. Dore and A. Venni, *On the closedness of the sum of two closed operators,* Math. Z. **196** (1987), 189-201.

[4] H. O. Fattorini, *The Cauchy Problem,* Encyclopedia of Mathematics and its Applications, **18**, Addison-Wesley, Massachussets, 1983.

[5] S. Guerre, *Some remarks on complex powers of* $(-\Delta)$ *and UMD spaces*, Illinois J. Math. **35** (1991), 401-407.

[6] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups,* Amer. Math. Soc. Colloq. Publ. **31**, Providence, 1957.

[7] T. Kato, *Note on fractional powers of linear operators,* Proc. Japan Acad. **36** (1960), 94-96.

[8] H. Komatsu, *Fractional powers of operators,* Pacific J. Math. **19** (1966), 285-346.

[9] H. Komatsu, *Fractional powers of operators, II. Interpolation spaces,* Pacific J. Math. **21** (1967), 89-111.

[10] H. Komatsu, *Fractional powers of operators, III. Negative powers,* J. Math. Soc. Japan **21** (1969), 205-220.

[11] C. Martínez and M. Sanz, *Fractional powers of non-densely defined operators,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **18** (1991), 443-454.

[12] C. Martínez and M. Sanz, *Spectral mapping theorem for fractional powers in locally convex spaces,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24** (1997), 685-702.

[13] C. Martínez, M. Sanz and V. Calvo, *Fractional powers of non-negative operators in Fréchet spaces,* Internat. J. Math. Math. Sci. **12** (1989), 309-320.

[14] C. Martínez, M. Sanz and L. Marco, *Fractional powers of operators,* J. Math. Soc. Japan, **40** (1988), 331-347.

[15] J. Prüss and H. Sohr, *Imaginary powers of elliptic second order differential operators in* L^p*−spaces,* Hiroshima Math. J. **23** (1993), 161-192.

[16] H. H. Schaeffer, *Topological Vector Spaces*, Springer-Verlag. New York, 1971.

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, New Yersey, 1970.

[18] H. Triebel, *Interpolation Theory, Function Spaces and Differential Operators,* North-Holland Amsterdam-New York, 1978.

[19] J. Watanabe, *On some properties of fractional powers of linear operators,* Proc. Japan Acad. **37** (1961), 273-275.

[20] K. Yosida, *Functional Analysis,* Springer-Verlag. New York, 1980.

Departament de Matemàtica Aplicada, Universitat de València, 46100 Burjassot, València, Spain.

E-mail address: martinel@uv.es, sanzma@uv.es, periago@uv.es.