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On the existence and uniqueness of solutions for an incomplete second-order abstract Cauchy problem

by

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Abstract. We prove existence and uniqueness of classical solutions for an incomplete second-order abstract Cauchy problem associated with operators which have polynomially bounded resolvent. Some examples of differential operators to which our abstract result applies are also included.

1. Introduction. Let $(X, \|\cdot\|)$ be a complex Banach space and let $A : D(A) \subseteq X \to X$ be a closed linear operator. Consider the incomplete second-order abstract Cauchy problem

$$\text{(ACP)} \quad \begin{cases} u''(t) + Au(t) = 0 & \text{for } t > 0, \\ u(0) = u_0, \\ \sup_{t>0} \|u(t)\| < \infty. \end{cases}$$

This problem was studied by Balakrishnan in [2, Theorem 6.1] in the case that A is sectorial and densely defined (see also [1, Theorem 3.8.3] or [9, Theorem 6.3.2]). We recall that A is said to be *ω-sectorial* with $0 < \omega < \pi$ if the resolvent set $\varrho(A)$ of A contains a sector

$$S_\omega = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \omega\}$$

and if there exists a constant $C > 0$ such that

$$(1) \qquad \|(\lambda - A)^{-1}\| \le C|\lambda|^{-1} \quad \text{for all } \lambda \in S_\omega.$$

Many important elliptic differential operators are sectorial, especially when they are considered in L^p-spaces (see, for instance, [8, Chapter 3]). However, (1) fails for the same elliptic operators when they are considered in a space $C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, of Hölder continuous functions, as shown in

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[8, Example 3.1.33] for the case of the Laplacian. For these operators we only have an estimate such as

$$(2) \qquad \|(\lambda - A)^{-1}\| \le C|\lambda|^{\gamma} \quad \text{for all } \lambda \in S_\omega \text{ and some } -1 < \gamma < 0.$$

More generally, we say that a closed linear operator A has *polynomially bounded resolvent* if $\varrho(A)$ contains a sector S_ω for some $0 < \omega < \pi$, and there are constants $C > 0$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ such that

$$(3) \qquad \|(\lambda - A)^{-1}\| \le C(1 + |\lambda|)^n \quad \text{for all } \lambda \in S_\omega.$$

It is not difficult to show that if $\varrho(A)$ contains S_ω and A satisfies (2), then $0 \in \varrho(A)$ and (3) holds for $n = 0$.

The aim of this paper is to prove existence and uniqueness of classical solutions for (ACP) when A is an operator with polynomially bounded resolvent, possibly with non-dense domain. In Section 2 we provide a result that relates an analytic semigroup of growth order α to analytic C_0-semigroups on certain intermediate spaces. This is the key to the proof of our main theorem which is stated and proved in Section 3. Section 4 contains some examples of differential operators to which our abstract result can be applied.

Finally, we note that for initial values $x \in D(A^{n+1})$, the result of Theorem 3 can be deduced using the fractional powers of [7, Theorem 5.4] and the ideas developed in [5, Section 2] that are needed to prove [5, Remark 2.14]. However, we present here a direct method which enables us to prove the result for a set of initial data larger than $D(A^{n+1})$, as Example 1 shows. Note that the fractional powers of [7, Theorem 5.4] coincide with the operators C^b introduced in [15].

2. Analytic semigroups of growth order α. Semigroups of growth order α were introduced by G. Da Prato [3] in 1966 (see also [12, 14, 18]). It is well known that every semigroup of growth order α gives rise to a strongly continuous semigroup on an intermediate space. In this section we show that if the semigroup of growth order α is analytic, then the intermediate space can be chosen such that the corresponding C_0-semigroup is analytic as well. We first recall the basic definition.

Let $\alpha > 0$. A family $(T(t))_{t>0}$ of bounded linear operators on X is said to be a *semigroup of growth order* α if it satisfies

(A_1) $T(t+s) = T(t)T(s)$ for all $t, s > 0$,
(A_2) for every $x \in X$, the mapping $t \mapsto T(t)x$ is continuous on $]0, \infty[$,
(A_3) $\|t^\alpha T(t)\|$ is bounded as $t \to 0$,
(A_4) $T(t)x = 0$ for all $t > 0$ implies $x = 0$, and
(A_5) $X_0 = \bigcup_{t>0} T(t)X$ is dense in X.

Note that for any such family $(T(t))_{t>0}$, the limit

$$\nu_0 = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|$$

exists and belongs to $[-\infty, \infty)$. Hence, for every $\nu > \nu_0$, there exists $M_\nu \geq 1$ such that

$$\|t^\alpha T(t)\| \leq M_\nu e^{\nu t} \quad \text{for all } t > 0. \tag{4}$$

The infinitesimal generator A_0 of a semigroup $(T(t))_{t>0}$ of growth order α is defined as

$$A_0 x = \lim_{t\to 0} \frac{1}{t} (T(t)x - x)$$

with domain $D(A_0) = \{x \in X : \lim_{t\to 0} t^{-1}(T(t)x - x) \text{ exists}\}$. As noted in [12], the operator A_0 is closable. Its closure $A = \overline{A}_0$ is called the *complete infinitesimal generator* of $(T(t))_{t>0}$.

The *continuity set* of $(T(t))_{t>0}$ is the set

$$\Omega = \{x \in X : T(t)x - x \to 0 \text{ as } t \to 0,\ t > 0\}. \tag{5}$$

Clearly, X_0 is dense in Ω. Moreover, we have $D(A^{n+1}) \subseteq \Omega$, where n is the integer part of α (see [12, Lemma 3.3]).

Let $\nu > \nu_0$. Then by (4), for every $x \in \Omega$, the function $t \mapsto \|e^{-\nu t} T(t)x\|$ is bounded on $[0, \infty[$. It is well known that

$$N(x) = \sup_{t>0} \|e^{-\nu t} T(t)x\| \quad \text{for all } x \in \Omega$$

defines a norm on Ω and that the space $(\Omega, N(\cdot))$ is a Banach space. Note that $N(x) \geq \|x\|$ for all $x \in \Omega$. Hence, Ω is densely and continuously embedded into X.

Since the semigroup $T(\cdot)$ is strongly continuous on $]0, \infty[$, the operators $T(t)$ with $t > 0$ leave Ω invariant. Therefore, we can consider the restriction of $(T(t))_{t>0}$ to Ω. We set

$$U(t) = T(t)|_\Omega \quad \text{for all } t > 0$$

and $U(0) = I_\Omega$. By [10, Theorem 2.2], the operator family $(U(t))_{t\geq 0}$ forms a strongly continuous semigroup on Ω satisfying $N(U(t)x) \leq e^{\nu t} N(x)$ for all $t \geq 0$ and $x \in \Omega$. Moreover, its generator B is the part of A_0 in Ω, that is, $D(B) = \{x \in D(A_0) : A_0 x \in \Omega\}$ and $Bx = A_0 x$ for all $x \in D(B)$. As the next lemma shows, the operator B is also the part of A in Ω.

LEMMA 1. *Let $x \in D(A)$ be such that $Ax \in \Omega$. Then $x \in D(A_0)$.*

Proof. By [12, Lemma 3.1], the function $t \mapsto T(t)x$, $t > 0$, is differentiable with

$$\frac{d}{dt} T(t)x = A_0 T(t)x = T(t)Ax \quad \text{for all } t > 0.$$

Since $Ax \in \Omega$, the derivative $\frac{d}{dt}T(\cdot)x$ is continuous on $[0,\infty[$. It follows that

$$\lim_{t\to 0} T(t)x = \lim_{t\to 0}\Big(T(1)x - \int_t^1 T(s)Ax\,ds\Big) = T(1)x - \int_0^1 T(s)Ax\,ds = y.$$

Since $T(t)y = \lim_{r\to 0} T(t)T(r)x = \lim_{r\to 0} T(t+r)x = T(t)x$ for all $t > 0$, property (A_4) yields $y = x$. Hence $x \in \Omega$ and

$$\lim_{t\to 0}\frac{1}{t}(T(t)x - x) = \lim_{t\to 0}\frac{1}{t}\int_0^t T(s)Ax\,ds = Ax.$$

In particular, $x \in D(A_0)$. ∎

Following [16], if the semigroup $(T(t))_{t>0}$ of growth order α has an extension to a sector S_δ with $0 < \delta \le \pi/2$ such that

(A_1') $T(t+s) = T(t)T(s)$ for all $t, s \in S_\delta$,

(A_2') the mapping $t \mapsto T(t)$ is analytic on S_δ,

(A_3') for each $0 < \varepsilon < \delta$, there exist constants $M_\varepsilon \ge 1$ and $\nu \in \mathbb{R}$ such that

$$\|t^\alpha T(t)\| \le M_\varepsilon e^{\nu \operatorname{Re} t} \quad \text{for all } t \in \overline{S}_{\delta-\varepsilon} \setminus \{0\},$$

then the family $(T(t))_{t\in S_\delta}$ is called an analytic semigroup of growth order α.

Let $(T(t))_{t\in S_\delta}$ be an analytic semigroup of growth order α on X, with generator A_0 and complete infinitesimal generator $A = \overline{A}_0$. It follows from the above that there exists a Banach space that is densely and continuously embedded in X, on which A generates a strongly continuous semigroup. The aim of this section is to show that there exists a Banach space, also densely and continuously embedded in X, on which A generates a strongly continuous *analytic* semigroup.

The continuity set Ω of $(T(t))_{t\in S_\delta}$ is given by (5). Let $0 < \varepsilon \le \delta$. We define the *angular continuity set* Ω_ε of $(T(t))_{t\in S_\delta}$ by

$$\Omega_\varepsilon = \{x \in X : T(t)x - x \to 0 \text{ as } t \to 0,\ t \in \overline{S}^0_{\delta-\varepsilon}\},$$

where $\overline{S}^0_{\delta-\varepsilon} = \overline{S}_{\delta-\varepsilon} \setminus \{0\}$ and $\overline{S}^0_0 =]0,\infty[$. In particular, $\Omega_\delta = \Omega$. By (A_2') and the definition of the sets Ω_ε, it is clear that

$$X_0 \subseteq \Omega_{\varepsilon_1} \subseteq \Omega_{\varepsilon_2} \subseteq \Omega \subseteq X \quad \text{for all } 0 < \varepsilon_1 < \varepsilon_2 < \delta.$$

Let $0 < \varepsilon \le \delta$ and $\nu > 0$ be as in property (A_3'). Given $x \in \Omega_\varepsilon$, we have

$$N_\varepsilon(x) = \sup_{t\in \overline{S}^0_{\delta-\varepsilon}} \|e^{-\nu t}T(t)x\| < \infty.$$

This follows from the fact that $T(t)x \to x$ as $t \to 0$, together with the estimate (A_3'). It is not difficult to see that the mapping $x \mapsto N_\varepsilon(x)$ defines a norm on Ω_ε and that the space $(\Omega_\varepsilon, N_\varepsilon(\cdot))$ is a Banach space. Note that if $0 < \varepsilon_1 < \varepsilon_2 \le \delta$, then $N_{\varepsilon_1}(x) \ge N_{\varepsilon_2}(x) \ge \|x\|$ for all $x \in \Omega_{\varepsilon_1}$. Hence, Ω_{ε_1} is continuously embedded in Ω_{ε_2} as well as in X.

Fix $0 < \varepsilon < \delta$. Since the semigroup $T(\cdot)$ is strongly continuous on $\overline{S}^0_{\delta-\varepsilon}$, the operator $T(t)$ with $t \in \overline{S}^0_{\delta-\varepsilon}$ leaves Ω_ε invariant. Hence, we can consider the restriction of $T(t)$ to Ω_ε. We set

$$U_\varepsilon(t) = T(t)|_{\Omega_\varepsilon} \quad \text{for all } t \in \overline{S}^0_{\delta-\varepsilon}, \quad U_\varepsilon(0) = I|_{\Omega_\varepsilon}.$$

Proposition 2. *The operator family $\{U_\varepsilon(t) : t \in \overline{S}_{\delta-\varepsilon}\}$ forms an analytic C_0-semigroup on Ω_ε satisfying $N_\varepsilon(U_\varepsilon(t)) \le e^{\nu \operatorname{Re} t}$ for all $t \in \overline{S}_{\delta-\varepsilon}$. Its generator is the part of A in Ω_ε.*

Proof. Clearly, the operators $U_\varepsilon(t)$ with $t \in \overline{S}_{\delta-\varepsilon}$ are linear operators on Ω_ε with

$$N_\varepsilon(U_\varepsilon(t)x) = \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \|e^{-\nu s}T(s)T(t)x\| = \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \|e^{\nu t}e^{-\nu(s+t)}T(s+t)x\|$$
$$\le e^{\nu \operatorname{Re} t} \sup_{s\in t+\overline{S}^0_{\delta-\varepsilon}} \|e^{-\nu s}T(s)x\| \le e^{\nu \operatorname{Re} t} N_\varepsilon(x)$$

for all $x \in \Omega_\varepsilon$.

The definition implies that the family $U_\varepsilon(\cdot)$ has the semigroup property $U_\varepsilon(t+s) = U_\varepsilon(t)U_\varepsilon(s)$ for all $t, s \in \overline{S}_{\delta-\varepsilon}$.

Let $x \in \Omega_\varepsilon$. From (A_3') it follows that the function $t \mapsto e^{-\nu t}T(t)x$ is uniformly $\|\cdot\|$-continuous on $\overline{S}_{\delta-\varepsilon}$. Hence

$$N_\varepsilon(U_\varepsilon(t)x - x) = \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \|e^{-\nu s}T(s)(T(t)x - x)\|$$
$$= \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \|e^{-\nu(s+t)}T(s+t)x - e^{-\nu s}T(s)x + (e^{\nu t} - 1)e^{-\nu(s+t)}T(s+t)x\|$$
$$\le \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \|e^{-\nu(s+t)}T(s+t)x - e^{-\nu s}T(s)x\| + |e^{\nu t} - 1|\, N_\varepsilon(x) \to 0$$

as $t \to 0$, $t \in \overline{S}^0_{\delta-\varepsilon}$. This means that $U_\varepsilon(\cdot)$ is strongly continuous on $\overline{S}_{\delta-\varepsilon}$.

Take $\theta \in (-(\delta-\varepsilon), \delta-\varepsilon)$. By the above, the operator family $(U_\varepsilon(e^{i\theta}t))_{t\ge0}$ forms a strongly continuous semigroup on Ω_ε. We show next that its generator B_θ is the part of $e^{i\theta}A$ in Ω_ε, that is, B_θ is given by $B_\theta x = e^{i\theta}Ax$ for all $x \in D(B_\theta) = \{x \in D(A) : Ax \in \Omega_\varepsilon\}$.

First, recall that $\|x\| \le N_\varepsilon(x)$ for all $x \in \Omega_\varepsilon$. Hence, if $x \in D(B_\theta)$ then

$$\left\| \frac{1}{t}\,(T(e^{i\theta}t)x - x) - B_\theta x \right\| \le N_\varepsilon\left(\frac{1}{t}\,(U_\varepsilon(e^{i\theta}t)x - x) - B_\theta x \right) \to 0$$

as $t \to 0$. Since by [19, Theorem 1], $(T(e^{i\theta}t))_{t>0}$ is a semigroup of growth order α whose complete infinitesimal generator is $e^{i\theta}A$, this shows that $x \in D(A)$ and $B_\theta x = e^{i\theta}Ax$. Hence, B_θ is contained in the part of $e^{i\theta}A$ in Ω_ε.

Conversely, let $x \in D(A)$ be such that $Ax \in \Omega_\varepsilon$. As Ω_ε is contained in the continuity set of the semigroup $(T(e^{i\theta}t))_{t>0}$ of growth order α, it follows by [19, Theorem 1] and Lemma 1 that $\|t^{-1}(T(e^{i\theta}t)x - x) - e^{i\theta}Ax\| \to 0$ as $t \to 0$. Since $Ax \in \Omega_\varepsilon$, [12, Lemma 3.1] shows that the function $t \mapsto T(e^{i\theta}t)x$ is continuously differentiable in $[0,\infty[$ with $\frac{d}{dt}T(e^{i\theta}t)x = e^{i\theta}T(e^{i\theta}t)Ax$ for all $t \geq 0$. Here, we set $T(0) = I_X$. This gives $T(e^{i\theta}t)x - x = \int_0^t e^{i\theta}T(e^{i\theta}r)Ax\,dr$ for all $t \geq 0$. Then

$$
\begin{aligned}
&N_\varepsilon\left(\frac{1}{t}\,(U_\varepsilon(e^{i\theta}t)x - x) - e^{i\theta}Ax\right)\\
&\quad= \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \left\|e^{-\nu s}T(s)\left[\frac{1}{t}\,(T(e^{i\theta}t)x - x) - e^{i\theta}Ax\right]\right\|\\
&\quad= \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \left\|\frac{1}{t}\int_0^t e^{-\nu s}e^{i\theta}T(s+e^{i\theta}r)Ax\,dr - \frac{1}{t}\int_0^t e^{-\nu s}e^{i\theta}T(s)Ax\,dr\right\|\\
&\quad\leq \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \frac{1}{t}\int_0^t \|e^{-\nu(s+e^{i\theta}r)}T(s+e^{i\theta}r)Ax - e^{-\nu s}T(s)Ax)\|\,dr\\
&\qquad+ \sup_{s\in\overline{S}^0_{\delta-\varepsilon}} \frac{1}{t}\int_0^t |e^{\nu e^{i\theta}r} - 1|\,\|e^{-\nu(s+e^{i\theta}r)}T(s+e^{i\theta}r)Ax\|\,dr\\
&\quad\leq \sup_{\substack{s,r\in\overline{S}^0_{\delta-\varepsilon}\\ |s-r|\leq t}} \|e^{-\nu s}T(s)Ax - e^{-\nu r}T(r)Ax)\| + \sup_{0\leq r\leq t} |e^{\nu e^{i\theta}r} - 1|N_\varepsilon(Ax) \to 0
\end{aligned}
$$

as $t \to 0$ because the function $s \mapsto e^{-\nu s}T(s)Ax$ is uniformly $\|\cdot\|$-continuous in $\overline{S}_{\delta-\varepsilon}$ and the function $s \mapsto e^{\nu e^{i\theta}s}$ is uniformly continuous on any compact interval $[0,\tau]$. Hence, $x \in D(B_\theta)$ and $B_\theta x = e^{i\theta}Ax$.

We have shown that for every $\theta \in (-(\delta-\varepsilon), \delta-\varepsilon)$, the operator $e^{i\theta}A|_{\Omega_\varepsilon}$ with domain $D(e^{i\theta}A|_{\Omega_\varepsilon}) = \{x \in D(A) : Ax \in \Omega_\varepsilon\}$ is the generator of the C_0-semigroup $(U_\varepsilon(e^{i\theta}t))_{t\geq 0}$ on Ω_ε. But this means that the operator $A|_{\Omega_\varepsilon}$ with domain $D(A|_{\Omega_\varepsilon}) = \{x \in D(A) : Ax \in \Omega_\varepsilon\}$ is the generator of an analytic C_0-semigroup on Ω_ε. This semigroup is given by $(U_\varepsilon(t))_{t\in\overline{S}_{\delta-\varepsilon}}$. ■

We note that the strong continuity of the semigroups $(U_\varepsilon(t))_{t\in\overline{S}_{\delta-\varepsilon}}$ implies that the spaces Ω_ε are in fact densely and continuously embedded in each other with increasing ε and in X, since $\bigcup_{t>0} U_\varepsilon(t)\Omega_\varepsilon \subseteq X_0 \subseteq \Omega_\varepsilon$ and $\bigcup_{t>0} U_\varepsilon(t)\Omega_\varepsilon$ is $N_\varepsilon(\cdot)$-dense in Ω_ε.

3. Existence and uniqueness of solutions. Suppose A is a densely defined, closed linear operator on the complex Banach space X, satisfying (3) for some $0 < \omega < \pi/2$, $C \geq 1$ and $n > -1$. Note that we explicitly assume

$\omega < \pi/2$, that is, we do not require that $\varrho(A)$ contains a half plane. A straightforward argument using the power series expansion of the resolvent $(\lambda - A)^{-1}$ of A in $\lambda \in S_\omega$ shows that there exists a ball B_d of radius d centred at zero such that $B_d \subseteq \varrho(A)$ and

$$\|(\lambda - A)^{-1}\| \le C(1 + |\lambda|)^n \quad \text{for all } \lambda \in B_d \cup S_\omega.$$

Hence, we can define fractional powers $(-A)^b$ with $b \in \mathbb{C}$, as in [15].

Let $0 < b < \pi/(2(\pi - \omega))$ and put $\varrho = \pi/2 - b(\pi - \omega)$. By [15, Proposition 2.12], the fractional power $-(-A)^b$ is the complete infinitesimal generator of an analytic semigroup $\{T_b(t) : t \in S_\varrho\}$ of growth order $(n+1)/b$. More precisely, $T_b(\cdot)$ is a family of bounded linear operators on X satisfying

(i) $T_b(t + s) = T_b(t)T_b(s)$ for all $t, s \in S_\varrho$,
(ii) the mapping $t \mapsto T_b(t)$ is analytic in the sector S_ϱ,
(iii) the operators $T_b(t)$ with $t \in S_\varrho$ are injective,
(iv) there exists $C_b > 0$ such that for every $t \in S_\varrho$,

$$\|T_b(t)\| \le C_b(\operatorname{Re} t - |\operatorname{Im} t| \tan(b(\pi - \omega)))^{-(n+1)/b}, \tag{6}$$

(v) the set $X_b = \bigcup_{t>0} T_b(t)X$ is dense in X.

We write $\Omega_b(A)$ and $\Omega_{b,\varepsilon}(A)$ with $0 < \varepsilon \le \varrho$ to denote the continuity set and the angular continuity sets of $T_b(\cdot)$, respectively. In the applications, the continuity sets play a very important role so that it is interesting to obtain lower and upper bounds for these sets. In addition to the inclusions given in Section 2, we have

$$D(A^{n+1}) \subseteq \Omega_{b,\varepsilon}(A) \quad \text{for all } 0 < \varepsilon \le \varrho.$$

This follows from the fact that the holomorphic $(-A)^{-(n+1)}$-regularised semigroup $(W_b(t))_{t\in S_\varrho}$ generated by $-(-A)^{n+1}(-A)^b(-A)^{-(n+1)}$ (see [7, Theorem 5.4 and Proposition 5.3]) is given by $W_b(t) = T_b(t)(-A)^{-(n+1)}$ for all $t \in S_\varrho$, and $W_b(0) = (-A)^{-(n+1)}$. By [6, Definition 21.3], for every $0 < \varepsilon < \varrho$, $W_b(\cdot)$ is strongly continuous on $\overline{S}_{\varrho-\varepsilon}$. Hence $D(A^{n+1}) \subseteq \Omega_{b,\varepsilon}(A)$.

Let $0 < \varepsilon \le \varrho$. From the estimate (6), it follows that

$$\|T_b(t)x\| \le C_b\left(\frac{\cos(b(\pi - \omega))}{\cos(\pi/2 - \varepsilon)}\right)^{(n+1)/b} |t|^{-(n+1)/b}\|x\| \quad \text{for all } t \in \overline{S}^0_{\varrho-\varepsilon}. \tag{7}$$

Hence we may choose $\nu = 0$ and obtain $N_{b,\varepsilon}(x) = \sup_{t\in\overline{S}^0_{\varrho-\varepsilon}} \|T_b(t)x\|$ as the norm on $\Omega_{b,\varepsilon}(A)$.

By $U_{b,\varepsilon}(\cdot)$ we denote the analytic C_0-semigroup of contractions on $\Omega_{b,\varepsilon}(A)$ as given by Proposition 2. That is, $U_{b,\varepsilon}(t) = T_b(t)|_{\Omega_{b,\varepsilon}(A)}$ for all $t \in \overline{S}^0_{\varrho-\varepsilon}$, and $U_{b,\varepsilon}(0) = I_{\Omega_{b,\varepsilon}(A)}$.

If the operator A is non-densely defined and satisfies (2), then we consider the part A_D of A in the Banach space $(X_D = \overline{D(A)}, \|\cdot\|)$, that is, the operator $A_D : D(A_D) \subseteq X_D \to X_D$ with domain $D(A_D) = \{x \in D(A) : Ax \in X_D\}$, defined as $A_D x = Ax$ for $x \in D(A_D)$. The operator A_D is densely defined and satisfies (3) with $n = 0$. Hence, we can construct fractional powers of A_D and the semigroups generated by them. We denote by $\Omega_b(A_D)$ and $\Omega_{b,\varepsilon}(A_D)$ the associated continuity sets.

We now turn our attention to (ACP) for the operator A above. By a solution of (ACP) we mean a $\|\cdot\|$-bounded function $u \in C^2(]0,\infty[;X) \cap C(]0,\infty[;D(A))$ such that $u''(t) + Au(t) = 0$ for all $t > 0$, and $\lim_{t\to 0} u(t) = u_0$.

Our main result reads as follows.

THEOREM 3. (i) *If A is densely defined and satisfies* (3), *then* (ACP) *has a unique solution for all $u_0 \in \Omega_{1/2}(A)$.*

(ii) *If A is non-densely defined and satisfies* (2), *then* (ACP) *has a unique solution for all $u_0 \in \Omega_{1/2}(A_D)$.*

Proof. (i) From [15, Lemma 1.4], it follows that $(-A)^{1/2}(-A)^{1/2}x = -Ax$ for all $x \in D(A^{2n+4})$. By [15, Lemma 2.10], $\bigcup_{t>0} T_{1/2}(t)X \subseteq D(A^\infty)$. Hence, the function $u(t) = T_{1/2}(t)u_0$ is a solution of (ACP).

Assume that there is another solution v of (ACP). Since $0 \in \varrho(A)$, the operator $(-A)^{-(n+2)}$ is bounded. Hence, we may consider the function ψ given by $\psi(t) = (-A)^{-(n+2)}v(t)$ for all $t > 0$, and the vector $\psi_0 = (-A)^{-(n+2)}u_0$. Clearly ψ is a solution of (ACP) for the initial value ψ_0. Moreover, ψ is a solution of the corresponding abstract Cauchy problem in the Banach space $(D(A^{n+2}), \|\cdot\|_{n+2})$, where $\|\cdot\|_{n+2}$ stands for the graph norm $\|x\|_{n+2} = \|x\| + \|A^{n+2}x\|$ for all $x \in D(A^{n+2})$.

As mentioned above, we have the inclusion $D(A^{n+2}) \subseteq \Omega_{1/2,\varepsilon}$. Since the Banach spaces $D(A^{n+2})$ and $\Omega_{1/2,\varepsilon}$ are both continuously embedded in X, it follows by the Closed Graph Theorem that this inclusion is continuous. Hence, ψ is a solution of the abstract Cauchy problem considered in the Banach space $(\Omega_{1/2,\varepsilon}, N_{1/2,\varepsilon}(\cdot))$. Moreover, by Proposition 2, the part of A in $\Omega_{1/2,\varepsilon}$ is a sectorial operator. Therefore we may apply Balakrishnan's Theorem [2, Theorem 6.1] on sectorial operators to conclude that

$$\psi(t) = U_{1/2,\varepsilon}(t)\psi_0 = (-A)^{-(n+2)}T_{1/2}(t)u_0 \quad \text{for all } t > 0.$$

Since the operator $(-A)^{-(n+2)}$ is injective, this means $v = u$.

(ii) In the Banach space $X_D = \overline{D(A)}$, consider the problem

$$(\mathrm{ACP}_D) \begin{cases} u''(t) + A_D u(t) = 0 & \text{for } t > 0, \\ u(0) = u_0, \\ \sup_{t>0}\|u(t)\| < \infty. \end{cases}$$

By (i), the function $u_D(t) = T^D_{1/2}(t)u_0$ is the unique solution of (ACP_D). Here $T^D_{1/2}(\cdot)$ denotes the semigroup associated with $-(-A_D)^{1/2}$. Clearly u_D is also a solution of (ACP). Let $v :]0,\infty[\to D(A)$ be another solution of (ACP). Since $v(t) \in D(A)$ for all $t > 0$, it follows that $v'(t) = \lim_{h\to 0} t^{-1}(v(t+h) - v(t)) \in X_D$ for all $t > 0$ and, similarly, that $v''(t) \in X_D$ for all $t > 0$. As $v(\cdot)$ solves (ACP), this implies $v(t) \in D(A_D)$ for all $t > 0$, and therefore v is a solution of (ACP_D). Hence, $v = u_D$. ∎

REMARK 1. As mentioned in the introduction, Theorem 3 with initial datum $u_0 \in D(A^{n+1})$ can be deduced from Theorem 5.4 of [7] and the ideas needed in the proof of Remark 2.14 of [5]. However, $D(A^{n+1})$ is, in general, strictly contained in $\Omega_{1/2}(A)$ as the following example shows.

EXAMPLE 1. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be complex Banach spaces. Suppose A_1 is an operator in X_1 with polynomially bounded resolvent and such that $-(-A_1)^{1/2}$ is the complete generator of an analytic semigroup of growth order α for some $\alpha > 0$. Denote by $T_1(\cdot)$ this semigroup associated with $-(-A_1)^{1/2}$. Let A_2 be an unbounded, densely defined, sectorial operator in X_2 such that $0 \in \varrho(A_2)$. Then the fractional power $-(-A_2)^{1/2}$ is the generator of an equibounded analytic C_0-semigroup, say $T_2(\cdot)$. Consider the Banach space $X = X_1 \times X_2$ endowed with the norm

$$\|x\| = \max\{\|x_1\|_1, \|x_2\|_2\} \quad \text{for all } x = (x_1, x_2) \in X$$

and the operator A in X with domain $D(A) = D(A_1) \times D(A_2)$ and defined by

$$A(x_1, x_2) = (A_1x_1, A_2x_2) \quad \text{for all } (x_1, x_2) \in D(A).$$

Then $-A$ is an operator with polynomially bounded resolvent and $-(-A)^{1/2}$ is the complete infinitesimal generator of the analytic semigroup $T(\cdot) = T_1(\cdot) \times T_2(\cdot)$ of growth order α. Since the continuity set of $T_2(\cdot)$ is equal to X_2 and A_2 is unbounded, the continuity set of $T(\cdot)$ strictly contains $D(A^k)$ for all $k \geq 1$.

4. Applications to partial differential equations. In this section, we give a few concrete examples of differential operators which satisfy (2) or (3) and, consequently, to which Theorem 3 can be applied.

Let $0 < \alpha < 1$, $m \in \mathbb{N}$, and Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. In the space $C^\alpha(\overline{\Omega})$ of Hölder continuous functions consider the operator $B : D(B) \subseteq C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ given by

$$Bu(x) = \sum_{|\beta|\leq 2m} a_\beta(x) D^\beta u(x) \quad \text{for all } x \in \overline{\Omega},$$

with domain $D(B) = \{u \in C^{2m+\alpha}(\overline{\Omega}) : D^\beta u|_{\partial\Omega} = 0 \text{ for all } |\beta| \leq m-1\}$. Here, β is a multiindex in $(\mathbb{N}\cup\{0\})^n$, $|\beta| = \sum_{j=1}^n \beta_j$ and $D^\beta = \prod_{j=1}^n \left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)^{\beta_j}$. We assume that the coefficients $a_\beta : \overline{\Omega} \to \mathbb{C}$ of B satisfy the following conditions:

(a) $a_\beta(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2m$,
(b) $a_\beta \in C^\alpha(\overline{\Omega})$ for all $|\beta| \le 2m$, and
(c) there is a constant $M > 0$ such that

$$M^{-1}|\xi|^{2m} \le \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \le M|\xi|^{2m} \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } x \in \overline{\Omega}.$$

In [17, Satz 1] it is proved that for $\sigma > 0$ sufficiently large, the operator $A = -(B+\sigma)$ satisfies (2) with $\gamma = \alpha/(2m) - 1$ and $\pi/2 < \omega < \pi$. Note that A is not densely defined since $D(A) = D(B) \subseteq C_0^\alpha(\overline{\Omega}) = \{u \in C^\alpha(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. So, Theorem 3(ii) applies to A.

As $-A$ satisfies the conditions of [13], we can also construct fractional powers and the semigroups generated by them as given there. It is not difficult to see that $\Omega_{1/2}(A_D)$ coincides with the set $\Omega_{1/2}(-A)$ of [13]. Moreover, we have the following upper and lower bounds for $\Omega_{1/2}(-A)$. By [13, Theorem 3.9(iii) and (vii)],

$$D((-A)^b) \subseteq \Omega_{1/2}(-A) \subseteq X_D \quad \text{for all } b > 1 + \gamma = \frac{\alpha}{2m},$$

and setting $C_{0,0}^{1+\alpha}(\overline{\Omega}) = \{u \in C^{1+\alpha}(\overline{\Omega}) : D^\beta u|_{\partial\Omega} = 0 \text{ for all } |\beta| \le 1\}$, by [4, Satz 3.3 a)], we have

$$C_{0,0}^{1+\alpha}(\overline{\Omega}) \subseteq D((-A)^b) \quad \text{for all } \frac{\alpha}{2m} < b < \frac{1}{2m}.$$

Note that for $b > \alpha/(2m)$, the fractional powers $(-A)^b$ defined in [13] coincide with the ones introduced in [4] and [17]. Hence, since $X_D \subseteq C_0^\alpha(\overline{\Omega})$, we have

$$C_{0,0}^{1+\alpha}(\overline{\Omega}) \subseteq \Omega_{1/2}(-A) \subseteq C_0^\alpha(\overline{\Omega}).$$

As a class of operators with polynomially bounded resolvent we mention the generators of integrated semigroups. Let $\alpha \ge 0$. If A is the densely defined generator of an α-times integrated semigroup $S^\alpha(\cdot)$ satisfying $\|S^\alpha(t)\| \le Mt^\beta e^{\omega t}$ for some constants $M \ge 1, \omega \ge 0, \beta \ge 0$, and all $t \ge 0$, then it can be proved (see [11]) that for all $\sigma > 0$ the operator $A - \omega - \sigma$ satisfies (3), in general with $0 < \omega \le \pi/2$. Concrete examples of differential operators that are generators of integrated semigroups can be found in [1, Chapter 8].

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