A local existence result for an optimal control problem modeling the manoeuvring of an underwater vehicle

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Abstract

We prove a local existence result for the manoeuvrability control of a submarine. The problem is formulated as an optimal control problem with a nonlinear and highly coupled system of ODEs for the state law, a Lagrange type cost function, and nonlinear controls which take values on a convex and compact subset of $\mathbb{R}^3$. Finally, the existence of solution for this problem is obtained by applying a recent general existence result (see [P. Pedregal and J. Tiago, Existence results for optimal control problems with some special non-linear dependence on state and control, SIAM J. Control Opt., 48 n.2 (2009) 415-437]) which, however, requires some modifications to be used in our specific case.

keywords Submarine, manoeuvrability control, optimal control problem, relaxation, Young measures, existence theory.

AMS 49J15, 49M20, 78M05, 93C15.

1 Introduction

In this paper we turn over the existence of solution for the model of manoeuvrability control of a submarine which has been recently proposed in [1] (see also [2, 3, 4]). It corresponds to a real-life engineering problem so that all the hypotheses and ingredients that we will consider in the sequel are motivated by real (non-academic) requirements. To describe such model a state vector is defined

$$\mathbf{x} = (x, y, z, \phi, \theta, \psi, u, v, w, p, q, r) \in \Omega \subset \mathbb{R}^{12}, \tag{1}$$

where $X_{world} = (x, y, z; \phi, \theta, \psi)$ indicates the position and orientation of the submarine in the world fixed coordinate system, and $V_{body} = (u, v, w; p, q, r)$ is the vector of linear and angular velocities measured in the body coordinate system. Throughout this paper we follow the usual SNAME notation [2]. Permitted ranges of Euler angles are

$$-\pi < \phi < \pi, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad 0 < \psi < 2\pi, \tag{2}$$

so that

$$\Omega = \mathbb{R}^3 \times \left]-\pi, \pi\right[\times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\times \left]-0, 2\pi\right[\times \mathbb{R}^6.$$

The control vector is

$$\mathbf{u} = (\delta_b, \delta_s, \delta_r), \tag{3}$$

where δ_b and δ_s represent, respectively, the angle of the bow and stern coupled planes, and δ_r is deflection of rudder. These controls act on the system in linear and quadratic form. Therefore, it is convenient to consider the mapping

$$\Phi(\mathbf{u}) = \left(\mathbf{u}, \mathbf{u}^2\right) \equiv \left(\delta_b, \delta_s, \delta_r, \delta_b^2, \delta_s^2, \delta_r^2\right) \in \mathbb{R}^6.$$

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Admissible controls $\mathbf{u}$ are measurable functions that should lie in a certain set $K \subset \mathbb{R}^3$, which, in our case, is given by

$$K = [-a_1, a_1] \times [-a_2, a_2] \times [-a_3, a_3],$$

with $0 < a_1, a_2, a_3 < \pi/2$. Finally, the state law is described by a system of twelve ordinary differential equations

$$\mathbf{x}'(t) = Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)) \tag{4}$$

where

$$Q : \mathbb{R}^{12} \to \mathcal{M}^{12\times 6} \quad \text{and} \quad Q_0 : \mathbb{R}^{12} \to \mathbb{R}^{12}$$

will be described in Section 3. At this point, we just indicate that the right-hand side of (4) includes both kinematic and dynamic equations of motion (see [1, 2, 3, 4] for more details).

The manoeuvrability control problem for an underwater vehicle describes a situation where we want to reach (or to be very close to) a final state $\mathbf{x}^T$ in time T, while minimizing the use of control during the time interval $[0, T]$. Typically, this type of problem is formulated as an exact or approximate controllability problem. However, in many real situations, there is no need for *all* the components of the state variable to be close of a final target $\mathbf{x}^T$ at time T, but only *some of them* (see [1] for some examples). Up to the knowledge of the authors, there are no satisfactory results in the literature for these nonlinear controllability problems in which controls appear in a nonlinear form. Notice also that if a controllability problem has a solution, then such a solution may be obtained by using an optimal control formulation as we propose below.

The requirement about the minimum use of control can be understood as minimizing the typical cost

$$\int_0^T \|\mathbf{u}(t)\|^2\, dt \tag{5}$$

while the previous aspect on the final state $\mathbf{x}^T$ can be seen as minimizing

$$\begin{aligned} \frac{1}{2}\left\|\mathbf{x}(T) - \mathbf{x}^T\right\|^2 &= \frac{1}{2}\int_0^T \frac{d}{dt}\left\|\mathbf{x}(t) - \mathbf{x}^T\right\|^2 dt + \frac{1}{2}\left\|\mathbf{x}(0) - \mathbf{x}^T\right\|^2 \\ &= \int_0^T < \mathbf{x}(t) - \mathbf{x}^T,\ Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)) > dt + \frac{1}{2}\left\|\mathbf{x}(0) - \mathbf{x}^T\right\|^2. \end{aligned}$$

Hence, we consider the cost

$$\begin{aligned} &\int_0^T \left[< \mathbf{x}(t) - \mathbf{x}^T,\ Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)) > + \|\mathbf{u}(t)\|^2\right] dt \\ = &\int_0^T \left[c(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + c_0(\mathbf{x}(t))\right] dt \end{aligned}$$

where the vector c is given by

$$\begin{cases} c_i(\mathbf{x}) = \sum_{j=1}^{12}\left(\mathbf{x} - \mathbf{x}^T\right)_j Q_{ji}, & i = 1, 2, 3, \\ c_i(\mathbf{x}) = \sum_{j=1}^{12}\left(\mathbf{x} - \mathbf{x}^T\right)_j Q_{ji} + 1, & i = 4, 5, 6, \end{cases}$$

and

$$c_0(\mathbf{x}) = < \mathbf{x} - \mathbf{x}^T,\ Q_0(\mathbf{x}) > .$$

Notice that the above expression for vector c is a particular case of a more general expression

$$\begin{cases} \hat{c}_i(\mathbf{x}) = \sum_{j=1}^{12}\alpha_j\left(\mathbf{x} - \mathbf{x}^T\right)_j Q_{ji}, & i = 1, 2, 3, \\ \hat{c}_i(\mathbf{x}) = \sum_{j=1}^{12}\alpha_j\left(\mathbf{x} - \mathbf{x}^T\right)_j Q_{ji} + \beta_i, & i = 4, 5, 6, \end{cases}$$

where $\alpha_j \geq 0$, $1 \leq j \leq 12$, and $\beta_i \geq 0$, $4 \leq i \leq 6$ correspond to some penalty parameters which are introduced to weigh at convenience (5) and the fact that some components of the state vector being close to a final given target at time T. For some interesting (at the practical point of view) choices of these penalty parameters we refer to [1]. For simplicity and since it does not change mathematically the problem, in the sequel we have taken those parameters equal to one.

To sum up, we can write the manoeuvrability control problem as

$$
(\mathrm{P}) \quad \begin{cases} \text{Minimize in } \mathbf{u}: & \int_0^T \left[c\left(\mathbf{x}\left(t\right)\right)\Phi\left(\mathbf{u}\left(t\right)\right)+c_0\left(\mathbf{x}\left(t\right)\right)\right]dt \\ \text{subject to} & \\ & \mathbf{x}'\left(t\right)=Q\left(\mathbf{x}\left(t\right)\right)\Phi\left(\mathbf{u}\left(t\right)\right)+Q_0\left(\mathbf{x}\left(t\right)\right), \quad 0<t<T \\ & \mathbf{x}\left(0\right)=\mathbf{x}^0\in\Omega \\ & \mathbf{x}\left(t\right)\in\Omega \quad \text{and} \quad \mathbf{u}\left(t\right)\in K,\ 0\le t\le T. \end{cases}
$$

The main goal of this paper is to prove the following local existence result.

Theorem 1.1. *For $T>0$, small enough, there exists an optimal solution of (P).*

We notice that the constraint on T is imposed to be able to guarantee that the state law is well-posed. The existence of T will be established during the proof of Theorem 1.1. As we will see later on, the fundamental question for this existence result is the relation between the vector c, the matrix Q, the mapping Φ and the set K. The role played by Q_0 is related to the existence and uniqueness of solution for the state law. As for the function c_0, as we will see in the proof of Theorem 1.1, it does not play an essential role. To prove Theorem 1.1 we will apply a very recent general existence result [5] which requires some modifications to adapt the specific structure of our model. Section 2 is devoted to present this general result (Theorem 2.1) with its corresponding changes. In Section 3 we will check that our model satisfies the hypotheses required by this last theorem.

Finally, we would like to emphasize that although the problem addressed in this work comes from an important real problem in naval industry, our main result is of theoretical type. The numerical resolution of (P) is of a major importance for engineering applications. We do not consider here such issue because it has already been considered in [1]. However, we should point out the importance of supporting numerical simulations of (P) with the theoretical result Theorem 1.1.

2 A general existence result for some specific optimal control problems

Throughout this section we follow the same ideas as in [5], but since our problem is slightly different from the one considered there and to make the paper easier for readers we include detailed statements and proofs.

To study the existence of solution for (P) we will turn ourself over the general optimal control problem of the type

$$
(\mathrm{CP}) \quad \begin{cases} \text{Minimize in } \mathbf{u}: & \int_0^T \left[c\left(\mathbf{x}\left(t\right)\right)\cdot\Phi\left(\mathbf{u}\left(t\right)\right)+c_0\left(\mathbf{x}\left(t\right)\right)\right]dt \\ \text{subject to} & \\ & \mathbf{x}'\left(t\right)=Q\left(\mathbf{x}\left(t\right)\right)\Phi\left(\mathbf{u}\left(t\right)\right)+Q_0\left(\mathbf{x}\left(t\right)\right), \quad 0<t<T \\ & \mathbf{x}\left(0\right)=\mathbf{x}^0\in\mathbb{R}^N \\ & \mathbf{u}\left(t\right)\in K,\ 0\le t\le T. \end{cases} \tag{6}
$$

where $K\subset\mathbb{R}^M$. The mappings

$$
\Phi(\mathbf{u})\in\mathbb{R}^S, \quad Q:\mathbb{R}^N\to\mathcal{M}^{N\times S}, \quad Q_0:\mathbb{R}^N\to\mathbb{R}^N, \quad c:\mathbb{R}^N\to\mathbb{R}^S
$$

should be such that the cost function is defined and takes finite values for admissible pairs $(\mathbf{x},\mathbf{u})$ and the state system is well-posed.

As we will see, the fundamental question for the existence result is the relation between the vector c, the matrix Q, the application Φ and set K. For a better understanding of such relations we consider additionally a $\mathcal{C}^1$ mapping

$$
\Psi: R^S\to R^{S-M}, \quad \Psi=(\psi_1,...,\psi_{S-M}), \quad (S>M), \tag{7}
$$

so that $\Phi(K)\subset\{\Psi=0\}$. This means that we are embedding the image space $\Phi(K)$ into a level surface (submanifold) defined by Ψ. Notice for example that for problem (P) where

$$
\Phi(\mathbf{u})=(u_1,u_2,u_3,(u_1)^2,(u_2)^2,(u_3)^2)\in\mathbb{R}^6
$$

we have

$$
\Psi(v)=((v_1)^2-v_4,(v_2)^2-v_5,(v_3)^2-v_6)\in\mathbb{R}^3.
$$

Also we define for every pair (c, Q) the set

$$\mathcal{N}(c,Q) = \left\{ v \in \mathbb{R}^S : Qv = 0, c \cdot v \leq 0 \right\}. \tag{8}$$

Similarly, we consider

$$\begin{gathered} \mathcal{N}(K,\Phi) = \\ \left\{ v \in \mathbb{R}^S : \text{ for each } \mathbf{u} \in K, \text{ either } \nabla\Psi(\Phi(\mathbf{u})) \cdot v = 0 \text{ or } \exists i \text{ s. t. } \nabla\psi_i(\Phi(\mathbf{u})) \cdot v > 0 \right\}, \end{gathered} \tag{9}$$

the set of "growth directions" of Ψ over $\Phi(K)$. We are now in conditions to state the existence result proved in [5] adapted to our frame.

Theorem 2.1. *Assume that the mapping Ψ as above is component-wise convex and $\mathcal{C}^1$. If for each $\mathbf{x} \in \mathbb{R}^N$, we have*

$$\mathcal{N}\left(c\left(\mathbf{x}\right), Q\left(\mathbf{x}\right)\right) \subset \mathcal{N}\left(K, \Phi\right), \tag{10}$$

then the corresponding optimal control problem (CP) has at least one solution.

Remark 2.1. *Notice that in the statement of Theorem 2.1, we have dropped the strictly convexity of Ψ as it was asked in [5]. Also, we remark that it is sufficient to check condition (10) for all $\mathbf{x} \in \bar{\Omega} \subset \mathbb{R}^N$ where $\bar{\Omega}$ is the set including all the possible solutions $\mathbf{x}$ of the initial value problem in (6) obtained for any admissible control in K.*

An essential tool to the proof of this result is the verification of the assumption

Assumption 2.1. *With the same notations as above, denote by $L = \Phi(K)$ and let $\Lambda = co(L)$ be the convex hull of L. Assume that for each fixed $\mathbf{x} \in \mathbb{R}^N$ and $\xi \in Q(\mathbf{x})\Lambda + Q_0(\mathbf{x}) \subset \mathbb{R}^N$, any solution $\tilde{m}$ of the problem*

$$\begin{cases} \textit{Minimize in } m \in \Lambda : & c(\mathbf{x}) \cdot m + c_0(\mathbf{x}) \\ \textit{subject to} & \\ & \xi = Q(\mathbf{x})m + Q_0(\mathbf{x}) \end{cases} \tag{11}$$

satisfies $\tilde{m} \in L$.

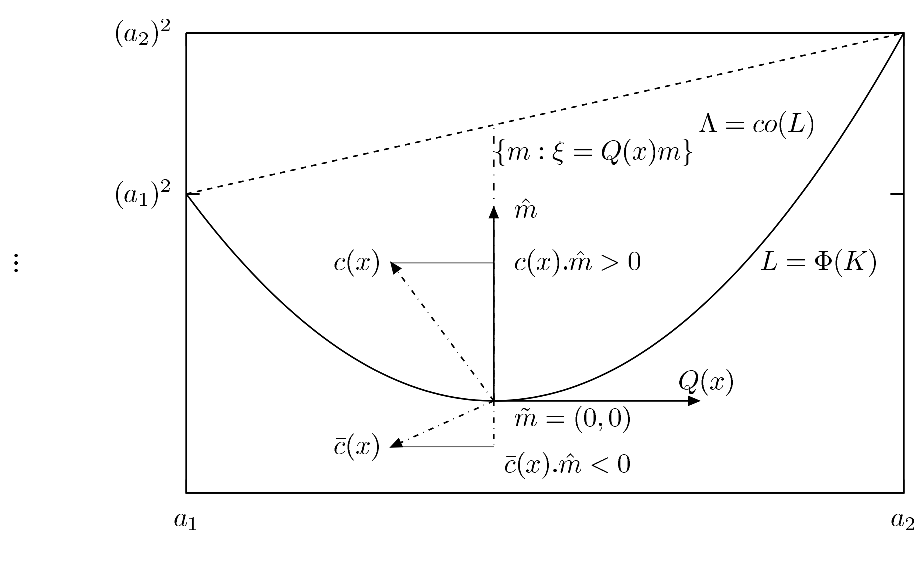

Figure 1: Illustration of Assumption 2.1 for the case $N = M = 1$, $\Phi = (\mathbf{u}, \mathbf{u}^2)$ and $K = [a_1, a_2]$.

This hypothesis has a very simple geometrical meaning, as we show in Figure 1 for the simple case were $N = M = 1$, $K = [a_1, a_2]$ and $\Phi(\mathbf{u}) = (\mathbf{u}, \mathbf{u}^2)$. The set L is part of the parameterized curve by Φ and Λ is its convex hull. For simplicity, in this example the fixed ξ and x have been chosen in such a way that $Q_0(x)$ and $c_0(x)$ are both nulls and $\{m \in \Lambda : \xi = Q(x)m\}$ intersects L at the origin, precisely at $\tilde{m} = (0,0)$. It is easy to see that for these x and ξ, the vectors c for which Assumption 2.1 is satisfied are those such that

$$c(x) \cdot \hat{m} > c(x) \cdot \tilde{m} = 0$$

for any $\hat{m} \in \{m \in \Lambda : \xi = Q(x)m\}$. Such condition is not satisfied, for example, for a vector $\bar{c}(x)$ such that

$$\bar{c}(x) \cdot \hat{m} < \bar{c}(x) \cdot \tilde{m} = 0.$$

This assumption allows us to proceed through a relaxation process using Young measures (as in [5, 6, 7, 8, 9]) and conclude that there is a Dirac-type solution of the relaxed problem which corresponds to a solution of the original problem.

Before starting the proof of the existence result, let us first consider the following Lemma.

Lemma 2.1. *Let Ψ be as in Theorem 2.1. If c, Q, Φ and K in (CP) are such that condition (10) is satisfied, then Assumption 2.1 holds.*

Proof. We want to see that for every fixed $\mathbf{x} \in \mathbb{R}^N$ and $\xi \in Q(\mathbf{x})\Lambda + Q_0(\mathbf{x})$ the minimizer of $c(\mathbf{x}) \cdot v + c_0(\mathbf{x})$ over the set of vectors $v \in \Lambda$ verifying the restriction $\xi = Q(\mathbf{x})v + Q_0(\mathbf{x})$ can only be in L, where both L and Λ are as in Assumption 2.1.

Suppose that $v_0 \in L$ and $v_1 \in \Lambda$ both belong to the manifold

$$\{\xi = Q(\mathbf{x})v + Q_0(\mathbf{x})\}$$

but they verify

$$c(\mathbf{x})v_1 + c_0(\mathbf{x}) \leq c(\mathbf{x})v_0 + c_0(\mathbf{x}).$$

As Ψ is component-wise convex and $L \subset \{\Psi = 0\}$, we have $\Lambda = co(L) \subset \{\Psi \leq 0\}$. Hence,

$$\Psi(v_0) = 0,\ \Psi(v_1) \leq 0,\ c \cdot v_1 \leq c \cdot v_0, \text{ and } Qv_1 = Qv_0\ (= \xi - Q_0).$$

Therefore it is obvious that $v = v_1 - v_0 \in \mathcal{N}(c(\mathbf{x}), Q(\mathbf{x}))$. Due to condition (10), $v \in \mathcal{N}(K, \Phi)$. Accordingly to the definition of $\mathcal{N}(K, \Phi)$ either $\nabla\psi_i(v_0)v > 0$ for some i or $\nabla\Psi(v_0)v = 0$. Suppose we are in the first situation. Because of the convexity of Ψ,

$$\psi_i(v_1) - \psi_i(v_0) - \nabla\psi_i(v_0)v \geq 0 \Leftrightarrow$$

$$\psi_i(v_1) \geq \nabla\psi_i(v_0)v > 0.$$

But this is impossible because $\psi_i(v_1) > 0$ cannot happen for a vector in Λ.

Suppose now that $\nabla\Psi(v_0)v = 0$. Again by convexity of each component of Ψ, we have

$$\Psi(v_1) - \Psi(v_0) - \nabla\Psi(v_0)v \geq 0,$$

that is,

$$0 = \Psi(v_0) \leq \Psi(v_1) \leq 0.$$

Hence, as $v_1 \in \Lambda = (\Lambda \setminus L) \cup L$ and

$$\Lambda \setminus L \subset \{\Psi(v) \leq 0, \quad \exists i \text{ s.t. },\ \psi_i(v) < 0\}$$

we conclude that $v_1 \in L$ and Assumption 2.1 holds. □

We can now prove Theorem 2.1.

Proof. We begin by the relaxation of (CP) using Young measures associated with sequences of admissible controls. Consider the problem

$$(RP) \quad \text{Minimize in } \mu = \{\mu_t\}_{t \in (0,T)} : \quad \tilde{I}(\mu) = \int_0^T [\int_K c(\mathbf{x}(t)) \cdot \Phi(\lambda) d\mu_t(\lambda)] + c_0(\mathbf{x}(t))dt$$

subject to

$$\mathbf{x}'(t) = \int_K Q(\mathbf{x}(t))\Phi(\lambda)d\mu_t(\lambda) + Q_0(\mathbf{x}(t))$$

and

$$\text{supp}(\mu_t) \subset K, \quad \mathbf{x}(0) = \mathbf{x}^0 \in \mathbb{R}^N.$$

Notice that the theory of Young measures ([6, 7, 8, 9]) allows us to conclude that this formulation is, in particular, well posed, as having $\mathbf{u} \in L^\infty([0,T],K)$ for K bounded implies (see [10]) that the associated Young measures $\{\mu_t\}_t$ belongs to

$$\mathcal{Y}^p((0,T),P(K)) =$$
$$\left\{\mu = \{\mu_t\}_{t\in(0,T)} : \int_0^T \int_K \|\lambda\|^p d\mu_t(\lambda)dt < \infty,\ \mu_t \in P(K)\right\} \quad \text{for every } p > 1,$$

where $P(K)$ is the space of probability measures supported in K. The existence of an optimal measure for this problem is immediately established by applying the existence result in [6] for the particular case where K is bounded.

In addition, (RP) can be rewritten by taking advantage of the moment structure of the cost density and the state equation. If we consider the set

$$\Lambda = \{m \in \mathbb{R}^S :\ m = \int_K \Phi(\lambda)d\nu(\lambda), \nu \in P(K)\},$$

then for each Young measure $\mu = \{\mu_t\}_t$ we can associate a function in $L^\infty([0,T],\Lambda)$ given by

$$m(t) = \int_K \Phi(\lambda)d\mu_t(\lambda).$$

This relation is not one-to-one but we can also associate at least one Young measure to each function in $L^\infty([0,T],\Lambda)$. The set Λ is very special. Indeed, notice that L defined above as $L = \Phi(K)$ is part of Λ as it corresponds to generalized moments associated to Dirac-type Young measures. Moreover, in [11] it was shown that when K is a compact and convex set we have

$$\Lambda = \overline{co(L)} = co(L)$$

so that Λ is a convex, compact set, defined as

$$\Lambda = co(\Phi(K)).$$

This considerations allow us to conclude that the relaxed problem (RP) is equivalent to the optimal control problem

$$(LP) \quad \text{Minimize in } m \in \Lambda : \quad \int_0^T c(\mathbf{x}(t)) \cdot m(t) + c_0(\mathbf{x}(t))dt$$

subject to

$$\mathbf{x}'(t) = Q(\mathbf{x}(t))m(t) + Q_0(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}^0,$$

whose optimal solution (for the existence of such a solution see [12]) corresponds to a Young measure which is an optimal solution (not necessarily unique) of (RP). Next, we will characterize this optimal solution, say $\tilde{m}(.)$ of (LP). To that purpose consider the function

$$\varphi(\mathbf{x},\xi) =$$
$$\begin{cases} \min_{m\in\Lambda}\{c(\mathbf{x}) \cdot m + c_0(\mathbf{x})\ : \xi = Q(\mathbf{x})m + Q_0(\mathbf{x})\} & \text{if } \xi \in Q(\mathbf{x})\Lambda + Q_0(\mathbf{x}) \\ +\infty & \text{else.} \end{cases}$$

This density function is the typical integrand of the cost which defines the equivalent variational problem (VP)

$$\text{Minimize in } \mathbf{x}(t) : \quad \int_0^T \varphi(\mathbf{x}(t), \mathbf{x}'(t))dt$$

subject to $\mathbf{x}(0) = \mathbf{x}^0$, $\mathbf{x}(t) \in AC([0,T],\mathbb{R}^N)$. The equivalence between problems (VP) and (LP) is well known and can be found in [12, 13] and in more recent works under a similar framework [7, 14]. Accordingly, there is a solution for (VP), let us say $\tilde{\mathbf{x}}(.)$, whose connection to $\tilde{m}(.)$ is established through the relation

$$\begin{aligned} \varphi(\tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}'(t)) &= \min_{m\in\Lambda}\{c(\tilde{\mathbf{x}}(t)) \cdot m(t) + c_0(\tilde{\mathbf{x}}(t) :\ \tilde{\mathbf{x}}'(t) = Q(\tilde{\mathbf{x}}(t))m(t) + Q_o(\tilde{\mathbf{x}}(t))\} \\ &= c(\tilde{\mathbf{x}}(t)) \cdot \tilde{m}(t) + c_0(\tilde{\mathbf{x}}(t)) \quad a.e. \quad t \in (0,T). \end{aligned}$$

This means that for almost every t, $\tilde{m}(t)$ is the minimizer of

$$\{c(\tilde{\mathbf{x}}(t)) \cdot m(t) + c_0(\tilde{\mathbf{x}}(t) : \ \tilde{\mathbf{x}}'(t) = Q(\tilde{\mathbf{x}}(t))m(t) + Q_o(\tilde{\mathbf{x}}(t))\}.$$

By Lemma 2.1,

$$\tilde{m}(t) \in L = \Phi(K)$$

so that there is a Dirac-type Young measure μ solution of (RP), associated to $\tilde{m}$. As a consequence, (CP) has an optimal solution $\mathbf{u} \in L^\infty([0,T], K)$ such that $\mu = \{\delta_{\mathbf{u}(t)}\}_{t\in(0,T)}$. □

Remark 2.2. *Note that if, in addition, Φ is component-wise one to one, convex and strictly convex for at least one component over K, and c, c_0 are such that the cost function in (LP) is convex with respect to m, then the solution of (CP) is unique. Indeed, suppose that $\mathbf{u}_1(.)$ and $\mathbf{u}_2(.)$ are different optimal solutions of (CP). Then $\mu_1 = \{\delta_{\mathbf{u}_1(t)}\}_t$ and $\mu_2 = \{\delta_{\mathbf{u}_2(t)}\}_t$ are optimal solutions of (RP). As Φ is component-wise one to one, the corresponding generalized moments defined by $m_1(t) = \Phi(\mathbf{u}_1(t))$ and $m_2(t) = \Phi(\mathbf{u}_2(t))$ are different optimal solutions of (LP). Hence, if the integrand in (LP) is convex with respect to m, then we have that $m = \lambda m_1 + (1-\lambda)m_2$, $\lambda \in]0,1[$, is also an optimal solution of (LP), and consequently $m \in L$. But since $L = \Phi(K)$ and Φ is strictly convex for some component i, m does not belong to L. A contradiction. Therefore we must have $\mathbf{u}_1 = \mathbf{u}_2$. A very simple and interesting case in which the cost function in (LP) is convex corresponds to c and c_0 constant functions.*

3 Proof of Theorem 1.1

In this section we will apply the first part of Theorem 2.1 to the optimal control problem (P). In fact, some numerical simulations (see [1]) suggest that the solution of (P) is not unique. We proceed in several steps:

3.1 Step 1: the matrices Q and Q_0

We start by paying some attention to the matrices Q and Q_0 of the control system, as it is fundamental to verify the well-posedness character of the state law and condition (10) of Theorem 2.1. We recall the notation introduced in Section 1 where we have set

$$\mathbf{x} = (x, y, z, \phi, \theta, \psi, u, v, w, p, q, r) \in \Omega \subset \mathbb{R}^{12},$$

with $X_{world} = (x, y, z; \phi, \theta, \psi)$ and $V_{body} = (u, v, w; p, q, r)$. Using this notation, accordingly to what we have seen also in Section 1 and using the data in [1] we know that Q is given by

$$Q = \begin{pmatrix} 0_{6\times 6} \\ M^{-1}F(V_{body}) \end{pmatrix}$$

where the matrix M is given by

$$M = \begin{pmatrix} m - \frac{\varrho}{2}L^3X'_{\dot{u}} & 0 & 0 & 0 & mZ_G & -mY_G \\ 0 & m - \frac{\varrho}{2}L^3Y'_{\dot{v}} & 0 & -mZ_G - \frac{\varrho}{2}L^4Y'_{\dot{p}} & 0 & mX_G - \frac{\varrho}{2}L^4Y'_{\dot{r}} \\ 0 & 0 & 0 & m - \frac{\varrho}{2}L^3Z'_{\dot{w}} & -mX_G - \frac{\varrho}{2}L^4Z'_{\dot{q}} & mY_G \\ 0 & -mZ_G - \frac{\varrho}{2}L^4K'_{\dot{v}} & mY_G & I_x - \frac{\varrho}{2}L^5K'_{\dot{p}} & -I_{xy} & -I_{xz} - \frac{\varrho}{2}L^5K'_{\dot{r}} \\ mZ_G & 0 & -mX_G - \frac{\varrho}{2}L^4M'_{\dot{w}} & -I_{xy} & I_y - \frac{\varrho}{2}L^5M'_{\dot{q}} & -I_{yz} \\ -mY_G & mX_G - \frac{\varrho}{2}L^4N_{\dot{v}} & 0 & -I_{xz} - \frac{\varrho}{2}L^5N'_{\dot{p}} & -I_{yz} & I_z - \frac{\varrho}{2}L^5N'_{\dot{r}} \end{pmatrix}$$

and $F = (G,\ H)$, $G, H \in \mathcal{M}^{6\times 3}$, with

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\varrho}{2}l^2(Y'_{\delta_r} + Y'_{\delta_r\eta}(\eta - \frac{1}{C})C)u^2 \\ \frac{\varrho}{2}l^2(Z'_{\delta_b})u^2 & \frac{\varrho}{2}l^2(Z'_{\delta_s} + Z'_{\delta_s\eta}(\eta - \frac{1}{C})C)u^2 & 0 \\ 0 & 0 & \frac{\varrho}{2}l^3(K'_{\delta_r})u^2 \\ \frac{\varrho}{2}l^3(M'_{\delta_b})u^2 & \frac{\varrho}{2}l^3(M'_{\delta_s} + M'_{\delta_s\eta}(\eta - \frac{1}{C})C)u^2 & 0 \\ 0 & 0 & \frac{\varrho}{2}l^3(N'_{\delta_r} + N'_{\delta_r\eta}(\eta - \frac{1}{C})C)u^2 \end{pmatrix}$$

and

$$H = \begin{pmatrix} \frac{\rho}{2} l^2 (X'_{\delta_b \delta_b}) u^2 & \frac{\rho}{2} l^2 (X'_{\delta_s \delta_s}) u^2 & \frac{\rho}{2} l^2 (X'_{\delta_r \delta_r}) u^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, considering the dimensionless hydrodynamic coefficients in [1, Appendix] we obtain

$$Q(\mathbf{x}) = u^2 \begin{pmatrix} & & 0_{6\times 6} & & & \\ Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ 0 & 0 & Q_{23} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ 0 & 0 & Q_{43} & 0 & 0 & 0 \\ Q_{51} & Q_{52} & Q_{53} & 0 & 0 & 0 \\ 0 & 0 & Q_{63} & 0 & 0 & 0 \end{pmatrix}$$

$$= (x_7)^2 \begin{pmatrix} & & 0_{6\times 6} & & & \\ -0.0056307 & -0.0056219 & 0.0002292 & -0.0028418 & -0.0011310 & -0.0037067 \\ 0 & 0 & -0.0001291 & 0 & 0 & 0 \\ 1.527832 & 1.4903911 & -0.0617573 & -0.0001656 & -0.0000659 & -0.0002160 \\ 0 & 0 & 0.0001049 & 0 & 0 & 0 \\ -0.0162938 & -0.0162684 & 0.0006631 & 0 & 0 & 0 \\ 0 & 0 & -0.0002773 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

We remark that Q, the 12×6 matrix of the coefficients interacting with the control, only depends on the surge velocity. Such particularity allows us to verify condition (10) quite easily, as we will see below.

As for Q_0, it is given by

$$Q_0 = \begin{pmatrix} \mathcal{T}(\phi, \theta, \psi) V_{body} \\ M^{-1} F_0(V_{body}, \phi, \theta, \psi) \end{pmatrix} \in \mathbb{R}^{12},$$

where $\mathcal{T}$ is the transformation matrix in the kinematic equations

$$(X_{world})' = \mathcal{T}(\phi, \theta, \psi) V_{body}$$

defined by

$$\mathcal{T} = \begin{pmatrix} J_1(\phi, \theta, \psi) & 0_{3\times 3} \\ 0_{3\times 3} & J_2(\phi, \theta, \psi) \end{pmatrix}$$

with

$$J_1(\phi, \theta, \psi) = \begin{pmatrix} \cos\psi \cos\theta & -\sin\psi \cos\theta + \cos\psi \sin\theta \sin\phi & \sin\psi \sin\phi + \cos\psi \cos\phi \sin\theta \\ \sin\psi \cos\theta & \cos\psi \cos\phi + \sin\phi \sin\theta \sin\psi & -\cos\psi \sin\phi + \sin\theta \sin\psi \cos\phi \\ -\sin\theta & \cos\theta \sin\phi & \cos\theta \cos\phi \end{pmatrix}$$

and

$$J_2(\phi, \theta, \psi) = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi / \cos\theta & \cos\phi / \cos\theta \end{pmatrix}.$$

Concerning F_0, it is defined in [1] through the ordinary differential system of six equations

$$M V'_{body} = F_0(V_{body}, \phi, \theta, \psi) + F(V_{body}) \Phi(\mathbf{u})$$

so that it corresponds to the terms which do not depend on the controls. To obtain Q_0 we write F_0 with the data given in [1] and multiply it by M^{-1}, just as we have done for Q. Using the state notation

$$\mathbf{x} = (x_j), \quad \bar{F}_0(\mathbf{x}) = ((\bar{F}_0)_j) = M^{-1} F_0(\mathbf{x}), \quad 1 \le j \le 6,$$

we obtain

$$
\begin{aligned}
(\bar{F}_0)_1 = {} & 0.21\sin x_4\cos x_5 + 5.593x_{12}\,|x_{12}| - 10.68x_{12}^2 - 7.234x_{11}x_{12} + 2.905x_{10}x_{12} - 0.93x_8x_{12} - 0.11x_7x_{12} \\
& -19.65x_{11}\,|x_{11}| + 5.658x_{11}^2 + 0.015x_{10}x_{11} - 1.809x_9x_{11} + 0.61x_7x_{11} + 7.252x_{10}\,|x_{10}| - 0.4x_{10}^2 + 0.14x_9x_{10} \\
& -2.477x_8x_{10} + 0.21x_7x_{10} - 0.0085\sqrt{x_9^2+x_8^2}\,|x_9| - 0.0022x_7\,|x_9| - 0.0056x_8\sqrt{x_9^2+x_8^2} + 0.0074x_9^2 \\
& -0.015x_8x_9 - 0.022x_7x_9 + 0.012x_8\,|x_8| + 0.22x_8^2 + 0.013x_7x_8 - 0.0012x_7^2 - 0.014x_7 + 0.2
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_2 = {} & 0.032\sin x_4\cos x_5 + 4.918x_{12}\,|x_{12}| - 1.028x_{11}x_{12} - 0.21x_7x_{12} + 0.064x_{10}x_{11} + 1.101x_{10}\,|x_{10}| \\
& +0.41x_9x_{10} - 0.0073x_7x_{10} - 0.023x_8\sqrt{x_9^2+x_8^2} - 0.061x_8x_9 + 0.0017x_8\,|x_8| \\
& -0.01x_7x_8 + 2.4985\times 10^{-7}x_7^2 - 5.6213\times 10^{-5}x_7 + 0.0012
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_3 = {} & -0.43\sin x_5 - 57.56\sin x_4\cos x_5 - 1508.x_{12}\,|x_{12}| + 5212.x_{12}^2 + 1951.x_{11}x_{12} + 98.94x_{10}x_{12} + 884.9x_8x_{12} \\
& +30.0x_7x_{12} + 5149.x_{11}\,|x_{11}| + 108.1x_{11}^2 - 4.058x_{10}x_{11} - 0.047x_9x_{11} - 166.7x_7x_{11} - 1956.x_{10}\,|x_{10}| \\
& +107.8x_{10}^2 - 38.2x_9x_{10} + 667.5x_8x_{10} - 57.7x_7x_{10} + 2.215\sqrt{x_9^2+x_8^2}\,|x_9| + 0.59x_7\,|x_9| + 1.501x_8\sqrt{x_9^2+x_8^2} \\
& +4.2833\times 10^{-4}x_9^2 + 3.913x_8x_9 + 6.062x_7x_9 - 3.109x_8\,|x_8| - 54.46x_8^2 - 3.376x_7x_8 + 0.088x_7^2 + 0.099x_7 - 2.205
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_4 = {} & -0.098\sin x_4\cos x_5 - 2.562x_{12}\,|x_{12}| + 3.317x_{11}x_{12} + 0.051x_7x_{12} - 0.0069x_{10}x_{11} - 3.325x_{10}\,|x_{10}| \\
& -0.065x_9x_{10} - 0.098x_7x_{10} + 0.0025x_8\sqrt{x_9^2+x_8^2} + 0.0066x_8x_9 - 0.0053x_8\,|x_8| - 0.0057x_7x_8 \\
& -7.5427\times 10^{-7}x_7^2 + 1.697\times 10^{-4}x_7 - 0.0038
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_5 = {} & 0.62\sin x_4\cos x_5 + 16.2x_{12}\,|x_{12}| - 56.57x_{12}^2 - 20.96x_{11}x_{12} - 9.622x_8x_{12} - 0.32x_7x_{12} - 56.86x_{11}\,|x_{11}| \\
& -1.157x_{11}^2 + 0.044x_{10}x_{11} + 1.76x_7x_{11} + 21.01x_{10}\,|x_{10}| - 1.157x_{10}^2 + 0.41x_9x_{10} - 7.167x_8x_{10} + 0.62x_7x_{10} \\
& -0.025\sqrt{x_9^2+x_8^2}\,|x_9| - 0.0064x_7\,|x_9| - 0.016x_8\sqrt{x_9^2+x_8^2} - 0.042x_8x_9 - 0.065x_7x_9 + 0.033x_8\,|x_8| \\
& +0.59x_8^2 + 0.036x_7x_8 - 9.4993\times 10^{-4}x_7^2 - 0.0011x_7 + 0.024
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_6 = {} & 0.0037\sin x_4\cos x_5 + 2.308x_{12}\,|x_{12}| - 0.12x_{11}x_{12} - 0.079x_7x_{12} - 1.91x_{10}x_{11} + 0.12x_{10}\,|x_{10}| + 0.0063x_9x_{10} \\
& -0.0073x_7x_{10} - 0.0043\,|x_8|\,\sqrt{x_9^2+x_8^2} - 3.2111\times 10^{-4}x_8\sqrt{x_9^2+x_8^2} - 0.071x_8x_9 + 1.9811\times 10^{-4}x_8\,|x_8| \\
& -0.0042x_7x_8 + 2.8285\times 10^{-8}x_7^2 - 6.3637\times 10^{-6}x_7 + 1.412\times 10^{-4}
\end{aligned}
$$

Notice that in fact $\bar{F}_0$ does not depend on (x_1, x_2, x_3), but for simplicity we will still consider Q_0 as a vector function from $\mathbb{R}^{12}$ to $\mathbb{R}^{12}$ which is described by

$$
Q_0(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} J_1(x_4,x_5,x_6) & 0_{3\times 3} \\ 0_{3\times 3} & J_2(x_4,x_5,x_6) \end{pmatrix} \begin{pmatrix} x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix} \\ \\ \bar{F}_0(\mathbf{x}) \end{pmatrix}
$$

where J_1, J_2 and $\bar{F}_0$ are as above.

3.2 Step 2: local existence and uniqueness of solutions for the state law

Let us now show that it is possible to find a time interval $I = [0, T]$ for which the initial value problem

$$\text{(IVP)} \begin{cases} \mathbf{x}'(t) = Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)), & 0 < t < T \\ \mathbf{x}(0) = \mathbf{x}^0 \in \Omega \end{cases}$$

is well posed in the sense that for every control function $\mathbf{u} \in L^\infty(0, T; K)$ there is a unique solution. We start by recalling the classical theory on this subject and therefore we rewrite (IVP) in the standard way

$$\begin{cases} \mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t)), & 0 < t < T \\ \mathbf{x}(0) = \mathbf{x}^0 \in \Omega \subset \mathbb{R}^N, \end{cases} \tag{13}$$

with $\mathbf{f} : I \times \Omega \to \mathbb{R}^N$, $N = 12$ in our case. As is well-known (see for instance [15, Appendix C]), if $\mathbf{f}$ satisfies conditions (H1)-(H4) below, then we can ensure the existence and uniqueness of a maximal solution for (13).

(H1) For each $\mathbf{x} \in \Omega$, the function $\mathbf{f}(\cdot, \mathbf{x}) : I \to \mathbb{R}^N$ is measurable,

(H2) for each $t \in I$, the function $\mathbf{f}(t, \cdot) : \Omega \to \mathbb{R}^N$ is continuous,

(H3) $\mathbf{f}$ is locally Lipschitz on $\mathbf{x}$, that is, for each $\mathbf{x}^0 \in \Omega$ there are a real number $\rho > 0$ and a locally integrable function

$$\alpha : I \to \mathbb{R}^+$$

such that the ball $B_\rho(\mathbf{x}^0)$ of radius ρ centered at $\mathbf{x}^0$ is contained in Ω and

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \le \alpha(t)\,\|\mathbf{x} - \mathbf{y}\|$$

for each $t \in I$ and $\mathbf{x}, \mathbf{y} \in B_\rho(\mathbf{x}^0)$, and

(H4) $\mathbf{f}$ is locally integrable on t, that is, for each $\mathbf{x}^0 \in \Omega$ there exists a locally integrable function $\beta : I \to \mathbb{R}^+$ such that

$$\left\|\mathbf{f}\left(t, \mathbf{x}^0\right)\right\| \le \beta(t) \quad \text{a. e. } t \in I.$$

Our next task is to check that (H1)-(H4) hold in our particular case. For any $\mathbf{u} \in L^\infty(\mathbb{R}; K)$, since the control variable $\mathbf{u}$ appears in linear and quadratic form, it is clear that the function

$$\mathbf{f}(t, \mathbf{x}) = Q(\mathbf{x})\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}) \tag{14}$$

is measurable with respect to t for each fixed $\mathbf{x} \in \Omega$. In addition, looking at the particular form of (14), it is clear that for each t, the function $\mathbf{x} \to \mathbf{f}(t, \mathbf{x})$ is continuous. With respect to conditions (H3) and (H4), again the form in which the controls appear let us conclude that (H4) is satisfied. As for the local Lipschitz condition (H3), since $\mathbf{f} = (f_1, \cdots, f_{12})$ is a vector function, we should check that condition for each component. Due to the constraints (2) and taking into account that the first six components of $\mathbf{f}$ only include the transformation matrix between body and world references frames, we have that $f_1, \cdots, f_6 \in C^\infty(\Omega)$ and therefore they are locally Lipschitz with respect to $\mathbf{x}$. As for the remaining $f_7, \cdots, f_{11}$, we notice that these components include by one side, polynomial terms and terms in the form of absolute value, all of them locally Lipschitz, and terms with the structure of $x_j\sqrt{x_j^2 + x_k^2}$ and $|x_j|\sqrt{x_j^2 + x_k^2}$, where $\mathbf{x} = (x_1, \cdots, x_{12})$, by the other. Since these functions are bounded and its derivatives have bounded discontinuities, they are also locally Lipschitz (see [16, p. 126 and p. 154]). Finally, the product of locally Lipschitz functions is also a locally Lipschitz function.

Therefore we may state that for each $\mathbf{x}^0 \in \Omega$ and $\mathbf{u} \in L^\infty(\mathbb{R}; K)$ there exists a maximal time $T(\mathbf{x}^0, \mathbf{u})$ and a unique maximal solution of (IVP) defined on $\left[0,\ T(\mathbf{x}^0, \mathbf{u})\right]$. In fact, looking at the proof of the mentioned existence result (see [15]), we can see that $T(\mathbf{x}^0, \mathbf{u})$ depends on both $\alpha(t) = \alpha(\mathbf{u}(t))$ and $\beta(t) = \beta(\mathbf{u}(t))$ in the sense that

$$\int_0^t \alpha(\tau)\,d\tau < 1 \quad \forall t \in [0, T(\mathbf{x}^0, \mathbf{u})]$$

and

$$\int_0^t \rho\alpha(\tau) + \beta(\tau)\,d\tau < \rho \quad \forall t \in [0, T(\mathbf{x}^0, \mathbf{u})].$$

Since Φ is continuous on the compact set K and taking into account the particular structure of matrices Q and Q_0, we can choose $\alpha(t)$ and $\beta(t)$ such that (H1)-(H4) are satisfied simultaneously for all $\mathbf{u} \in L^\infty(\mathbb{R}_+; K)$ and consequently we can choose T (uniformly in $\mathbf{u}$) such that problem (IVP) has a unique solution in $I = [0, T]$, with $T = T(\mathbf{x}^0)$, for every $\mathbf{u} \in L^\infty(I; K)$.

Remark 3.1. *As showed above, in order to ensure the well-posedness character of the state law in $[0, T]$ we must choose T small enough because the Lipschitz condition has a local nature. This result is in full agreement with the numerical simulations obtained in [1]. Indeed, in addition to constraints (2), we can also add some constraints on the state variables (x, y, z) due to the bounded nature of the ocean. In such case, as several numerical experiments in [1] showed, when the submarine is forced to move down in the ocean (depth change manoeuvre), there is also a displacement in the y-component. Thus, if the submarine is close to littoral (y bounded), then for T large enough the vehicle may intersect it and therefore the solution $\mathbf{x}$ of the state law will not remain under the required bounds for such time interval.*

3.3 Step 3: checking condition (10) in Theorem 2.1

We need to describe for every $\mathbf{x} \in \mathbb{R}^{12}$ (and corresponding pair $(c(\mathbf{x}), Q(\mathbf{x}))$) the set

$$\mathcal{N}(c(\mathbf{x}), Q(\mathbf{x})) = \left\{ v \in \mathbb{R}^6 : Q(\mathbf{x})v = 0, c(\mathbf{x}) \cdot v \le 0 \right\},$$

and check that such set is contained in

$$\mathcal{N}(K, \Phi) =$$

$$\left\{ v = (v_1, \cdots, v_6) \in \mathbb{R}^6 : \text{ for each } \mathbf{u} \in K, \text{ either } \nabla\Psi(\Phi(\mathbf{u}))v = 0 \text{ or there is } i \text{ with } \nabla\Psi_i(\Phi(\mathbf{u}))v > 0 \right\},$$

where Q is given by (12).

Let us first find the solution of $Qv = 0$. From now on we assume that $x_7 \neq 0$. The singular case $x_7 = 0$ is not physically admissible as it corresponds to surge velocity u which is strictly positive in this model [1]. Hence we have

$$\begin{cases} v_3 = 0 \\ v_6 = -\frac{1}{Q_{16}}(Q_{11}v_1 + Q_{12}v_2 + Q_{14}v_4 + Q_{15}v_5) \\ v_6 = -\frac{1}{Q_{36}}(Q_{31}v_1 + Q_{32}v_2 + Q_{34}v_4 + Q_{35}v_5) \\ v_2 = -\frac{Q_{51}}{Q_{52}}v_1. \end{cases}$$

Thus

$$\begin{cases} \cdots \\ \frac{1}{Q_{16}}(Q_{11} - \frac{Q_{51}}{Q_{52}}Q_{12})v_1 + \frac{Q_{14}}{Q_{16}}v_4 + \frac{Q_{15}}{Q_{16}}v_5 = \\ \frac{1}{Q_{36}}(Q_{31} - \frac{Q_{51}}{Q_{52}}Q_{32})v_1 + \frac{Q_{34}}{Q_{36}}v_4 + \frac{Q_{35}}{Q_{36}}v_5 \\ \cdots \end{cases}$$

but

$$\frac{Q_{14}}{Q_{16}} = 0.7666667 = \frac{Q_{34}}{Q_{36}}$$

and

$$\frac{Q_{15}}{Q_{16}} = 0.3051282 = \frac{Q_{35}}{Q_{36}}$$

so that

$$\begin{cases} \cdots \\ (Q_{11} - \frac{Q_{51}}{Q_{52}}Q_{12})v_1 = (Q_{31} - \frac{Q_{51}}{Q_{52}}Q_{32})v_1 \\ \cdots \end{cases} .$$

Since

$$Q_{11} - \frac{Q_{51}}{Q_{52}}Q_{12} = 0 \neq 0.0348637 = Q_{31} - \frac{Q_{51}}{Q_{52}}Q_{32},$$

we have

$$Qv = 0 \Leftrightarrow \begin{cases} v_1 = 0 \\ v_2 = 0 \\ v_3 = 0 \\ v_6 = -\frac{1}{Q_{16}}(Q_{14}v_4 + Q_{15}v_5) = \\ -\frac{1}{Q_{36}}(Q_{34}v_4 + Q_{35}v_5). \end{cases}$$

Before completing the characterization of $\mathcal{N}(c, Q)$ notice that the function Ψ used in describing $\mathcal{N}(K, \Phi)$ is given by

$$\Psi(m) = (m_1^2 - m_4, m_2^2 - m_5, m_3^2 - m_6), \quad m = (m_1, \cdots, m_6),$$

so that Ψ is obviously $\mathcal{C}^1$ and convex. Moreover,

$$\nabla\Psi(m) = \begin{pmatrix} 2m_1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2m_2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2m_3 & 0 & 0 & -1 \end{pmatrix}$$

Hence, for v such that $Qv = 0$ we obtain

$$\nabla\Psi(m) \cdot v = -\begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix}.$$

This means that if a vector v (in the manifold $Qv = 0$) satisfies

$$v_4 = v_5 = v_6 = 0$$

or at least one of these three components is negative, then it belongs to $\mathcal{N}(K, \Phi)$.

As we have seen above, we have the relation

$$v_6 = -\frac{1}{Q_{16}}(Q_{14}v_4 + Q_{15}v_5) = -\frac{1}{Q_{36}}(Q_{34}v_4 + Q_{35}v_5).$$

Hence, if both v_4 and v_5 are positive or null, we have v_6 necessarily negative or also null. Consequently

$$\mathcal{N}(c, Q) \subset \mathcal{N}(K, \Phi),$$

and applying Theorem 2.1 the proof is complete.

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