Optimal shape and position of the support for the internal exact control of a string*

Francisco Periago†

Abstract

In this paper, we consider the problem of optimizing the shape and position of the support ω of the internal exact control of minimal $L^2\left(0,T;L^2\left(\omega\right)\right)$ - norm for the 1-D wave equation. A relaxation for this problem is found and the minimizers of the relaxed problem are characterized through first-order optimality conditions.

Keywords: Optimal shape design, exact controllability, wave equation, relaxation, optimality conditions.

1 Introduction

The very important in practise problem of modelling control mechanisms for the stabilization or exact controllability of systems governed by partial differential equations has attracted the interest of many researchers both at the theoretical level and for its applications to real-life engineering problems. From a practical point of view, it is quite natural to introduce a constraint on the size of these controls. Therefore, the issue of choosing the best shape and position of those is very important in practise. We refer for instance to [5, 6, 7, 12, 13] for some recent results in the context of optimal stabilization for distributed parameter systems.

In this paper, the problem is addressed of optimizing the shape and location of the support of the internal exact control for the 1-D wave equation. Precisely, let $\Omega\subset\mathbb{R}$ be a bounded interval and let ω be an open subset of Ω. Given a time $T>0$ and two functions $\left(y^0,y^1\right)\in H_0^1\left(\Omega\right)\times L^2\left(\Omega\right)$, the problem of internal exact control for the wave equation refers to the existence of a control function $h_\omega=h_\omega\left(t,x\right)\in L^2\left(0,T;L^2\left(\Omega\right)\right)$ such that $\mathrm{supp}(h_\omega)\subset[0,T]\times\overline{\omega}$ and for which the solution y of the

*Work supported by projects MTM2007-62945 from Ministerio de Educación y Ciencia and 00675/PI/04 from Fundación Séneca (Murcia, Spain).

†Departamento de Matemática Aplicada y Estadística, ETSI Industriales, Universidad Politécnica de Cartagena, Campus Muralla del Mar, 30202 Cartagena, Spain. Email: f.periago@upct.es, Phone: +34-968 338 909, Fax: +34-968 326 493.

system

$$\begin{cases} y_{tt} - y_{xx} = h_\omega, & (t,x) \in \left]0,T\right[\times \Omega \\ y|_{\partial\Omega} = 0, & t \in [0,T] \\ y(0,x) = y^0(x), \quad y_t(0,x) = y^1(x), & x \in \Omega \end{cases} \tag{1}$$

satisfies the exact controllability condition

$$y(T,x) = y_t(T,x) = 0, \quad x \in \Omega. \tag{2}$$

This problem has been solved by J. Lagnese [8] and independently by A. Haraux [3, 4]. This last author used the Hilbert Uniqueness Method (HUM) introduced by J. L. Lions [9] which reduces the exact controllability problem (1)-(2) to proving an observability inequality for the solutions of the adjoint system

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & (t,x) \in \left]0,T\right[\times \Omega \\ \phi|_{\partial\Omega}(t) = 0, & t \in [0,T] \\ \phi(0,x) = \phi^0(x), \quad \phi_t(0,x) = \phi^1(x), & x \in \Omega, \end{cases} \tag{3}$$

with $\left(\phi^0, \phi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$. To be more precise, consider the problem

$$\inf_{\left(\widetilde{\phi}^0, \widetilde{\phi}^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)} \mathcal{J}_{\mathcal{X}_\omega}\left(\widetilde{\phi}^0, \widetilde{\phi}^1\right) \tag{4}$$

where the functional $\mathcal{J}_{\mathcal{X}_\omega}$ is given by

$$\mathcal{J}_{\mathcal{X}_\omega}\left(\widetilde{\phi}^0, \widetilde{\phi}^1\right) = \frac{1}{2}\int_0^T \int_\Omega \mathcal{X}_\omega \widetilde{\phi}^2 dx dt + \ < \widetilde{\phi}^1, y^0 >_{H^{-1}, H_0^1} - \int_\Omega \widetilde{\phi}^0 y^1 dx \tag{5}$$

Here $< \cdot, \cdot >_{H^{-1}, H_0^1}$ stands for the duality product in $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and $\widetilde{\phi}$ is the solution of (3) associated with the initial conditions $\left(\widetilde{\phi}^0, \widetilde{\phi}^1\right)$. Then, it is proved that there exists $T^\star = T^\star(\Omega \setminus \omega) > 0$ such that for $T \geq T^\star$ the problem (4) has a unique minimizer $\left(\phi^0, \phi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$. $T^\star$ is the so-called a uniqueness time for the exact controllability problem (1)-(2) relative to ω (see [3, Definition 1.1.7]). In particular, if $\Omega = \left]0, \pi\right[$, then for any $T^\star \geq 2\pi$, $T^\star$ is a uniqueness time for (1)-(2) relative to any ω open subset of Ω ([3, Proposition 1.3.1]). For simplicity, from now on in this paper we assume that $\Omega = \left]0, \pi\right[$ and $T \geq T^\star = 2\pi$.

It is important to emphasize that the coercitivity of the functional $\mathcal{J}_{\mathcal{X}_\omega}$ follows from an observability inequality for the solutions of system (3) which holds for $T \geq T^\star$. Finally, the function $h_\omega = -\mathcal{X}_\omega \phi$, where $\mathcal{X}_\omega$ is the characteristic function of ω and ϕ is the solution of (3) associated with the solution $\left(\phi^0, \phi^1\right)$ of (4), is the exact control of minimal $L^2\left(0, T; L^2(\omega)\right)$ $-$norm for (1)-(2). The control $h_\omega = -\mathcal{X}_\omega \phi$ obtained in this way is the so-called HUM control. That is, ω *being fixed*, HUM provides the optimal (in the L^2-norm) control.

It is then natural to optimize the L^2-norm of the HUM controls with respect to ω in the class of the ω's having the same size. Identifying each subset ω with its characteristic function $\mathcal{X}_\omega$, we consider the nonlinear optimal design problem

$$(P) \qquad \inf_{\mathcal{X}\in\mathcal{U}_L} J(\mathcal{X}) = \int_0^T \int_\Omega \mathcal{X}\phi^2 dxdt$$

where for some fixed $0 < L < 1$,

$$\mathcal{U}_L = \left\{ \mathcal{X}_\omega \in L^\infty(\Omega;\{0,1\}) : \omega \subset \Omega \text{ is open} \quad \text{and} \quad \int_\Omega \mathcal{X}_\omega(x)\,dx = L\,|\Omega| \right\}.$$

Here $\mathcal{X}_\omega$ is the characteristic function of ω, $|\Omega|$ is the Lebesgue measure of Ω and ϕ is the solution of the adjoint system (3) associated with the minimizer (ϕ^0, ϕ^1) of (5).

Notice that both the solution ϕ of the adjoint system (3) and (ϕ^0, ϕ^1) depend on $\mathcal{X}$.

Up to the knowledge of the author, this problem has been considered for the first time by A. Münch in [10] where two numerical schemes are implemented for solving numerically (P). The first one of these two methods is developed in the level set framework and requires the computation of the shape derivative of the cost function J. It is assumed in this approach that (P) is well-posed, that is, the optimal solution is a characteristic function. The second scheme, which is easier to implement and less sensitive to numerical approximations, is based on the conjecture that a relaxation for (P) consists in replacing the set $\mathcal{U}_L$ by its convex envelope. This permits to solve the relaxed problem by using a gradient descent method.

The first aim of this paper is to give a rigorous proof of this conjecture. This is our Theorem 2.1. The proof of this result uses standard arguments in controllability theory and optimal control and is based on a uniform (with respect to $\mathcal{X} \in \mathcal{U}_L$) observability inequality for the solutions of the adjoint system (3). See Proposition 2.1. It is important to emphasize that since the design variable does not appear in the principal part of the wave operator and the cost function is very well-adapted to the weak form of the state equation, the relaxation procedure does not require the use of more sophisticated techniques like the homogenization method or the classical tools of non-convex, vector, variational problems. We refer for instance to [14] for an application of these two methods to an optimal design problem for the heat equation.

We also notice that the fact that it is not known if problem (P) is well-posed, is one of the main reasons for studying a relaxed problem. Roughly speaking, the unboundedness of the number of connected components of ω may be the reason for (P) to be ill-posed. In a similar context, for an example of a shape design problem which is ill-posed we refer to [5]. The same phenomenon was observed numerically in [12] for the stabilization of the wave equation and in connection with the over-damping phenomenon.

In a second part, we study the first-order optimality conditions for the relaxed problem and provide a characterization of its minimizers. This is done in Section 3. Finally, we list some interesting related open problems.

2 Relaxation

As indicated in the Introduction, throughout this section we assume that $\Omega =]0,\pi[$ and $T \geq 2\pi$. The following technical result will be a key point in the sequel.

Proposition 2.1 (Uniform observability inequality) *There exists a positive constant C, which only depends on T, such that*

$$\left\|\phi^0\right\|_{L^2}^2+\left\|\phi^1\right\|_{H^{-1}}^2 \leq C\int_0^T\int_\Omega \mathcal{X}\phi^2 dxdt \qquad \textit{for all } \mathcal{X}\in\mathcal{U}_L \textit{ and } \left(\phi^0,\phi^1\right)\in L^2\times H^{-1}. \tag{6}$$

Proof. We begin by writing the solution $\phi \in C\left(0,T;L^2\left(\Omega\right)\right) \cap C^1\left(0,T;H^{-1}\left(\Omega\right)\right)$ of (3) in the form

$$\phi\left(t,x\right)=\sum_{n=1}^{\infty} b_n\cos\left(nt+\alpha_n\right)\sin\left(nx\right).$$

Note that this formula for $\phi\left(t,x\right)$ is obtained from the more commonly used formula

$$\sum_{n=1}^{\infty}\left[A_n\cos\left(nt\right)+B_n\sin\left(nt\right)\right]\sin\left(nx\right)$$

just by taking

$$b_n=\sqrt{A_n^2+B_n^2} \quad \text{and} \quad \alpha_n=-\tan^{-1}\frac{B_n}{A_n}.$$

By Parseval's identity,

$$\left\|\phi\left(t\right)\right\|_{L^2(\Omega)}^2+\left\|\phi_t\left(t\right)\right\|_{H^{-1}(\Omega)}^2=\frac{\pi}{2}\sum_{n=1}^{\infty}b_n^2, \quad t\geq 0,$$

and

$$\int_0^{2\pi}\phi^2\left(t,x\right)dt=\frac{\pi}{2}\sum_{n=1}^{\infty}b_n^2\sin^2\left(nx\right).$$

Now let ω be the subset associated with $\mathcal{X}\in\mathcal{U}_L$. We must prove that there exists $C>0$, independent of ω, such that

$$\int_\omega \sin^2\left(nx\right)dx \geq C \quad \text{for all } n\in\mathbb{N}.$$

It is clear that

$$\inf_\omega\int_\omega \sin^2\left(nx\right)dx,$$

where $\omega \subset \Omega$ is open and $|\omega| = L|\Omega|$, is attained at

$$\omega_n = \left]0, \frac{|\omega|}{2n}\right[\left[\bigcup_{k=1}^{n-1}\right] \frac{k\pi}{n} - \frac{|\omega|}{2n}, \frac{k\pi}{n} + \frac{|\omega|}{2n} \left[\bigcup\right] \pi - \frac{|\omega|}{2n}, \pi\left[.\right.$$

Note that ω_n is obtained by placing two intervals of size $\frac{|\omega|}{2n}$ at the extremes of $]0,\pi[$ and $n-1$ intervals of length $\frac{|\omega|}{n}$ centered at each of the zeros of $\sin^2(nx)$ in the interval $]0,\pi[$. Using the periodicity properties of $\sin^2(nx)$, a simple computation shows that

$$\begin{aligned} \int_{\omega_n} \sin^2(nx)\,dx &= 2n \int_0^{\frac{|\omega|}{2n}} \sin^2(nx)\,dx \\ &= 2\int_0^{\frac{|\omega|}{2}} \sin^2(y)\,dy \\ &= 2\left[\frac{y}{2} - \frac{\sin(2y)}{4}\right]_0^{\frac{|\omega|}{2}} \\ &= \frac{|\omega|}{2} - \frac{\sin|\omega|}{2}. \end{aligned}$$

■

Let $\overline{\mathcal{U}}_L$ be the space

$$\overline{\mathcal{U}}_L = \left\{\theta \in L^\infty(\Omega;[0,1]),\quad \int_\Omega \theta(x)\,dx = L|\Omega|\right\}$$

endowed with the weak-$\star$ topology of $L^\infty(\Omega)$. As is well-known (see [7, Prop. 7.2.14, p. 289] and [2, Th. 2.3 and Lemma 2.1 p. 94-96]), $\overline{\mathcal{U}}_L$ is the weak-$\star$ closure of $\mathcal{U}_L$ in $L^\infty(\Omega)$.

Consider now the relaxed problem

$$(RP) \qquad \inf_{\theta\in\overline{\mathcal{U}}_L} \overline{J}(\theta) = \int_0^T \int_\Omega \theta\phi^2 dxdt \tag{7}$$

where ϕ is the solution of (3) associated with the initial data (ϕ^0,ϕ^1) that minimize the new functional $\overline{\mathcal{J}}_\theta : L^2(\Omega)\times H^{-1}(\Omega) \to \mathbb{R}$ defined by

$$\overline{\mathcal{J}}_\theta\left(\widetilde{\phi}^0,\widetilde{\phi}^1\right) = \frac{1}{2}\int_0^T \int_\Omega \theta\widetilde{\phi}^2 dxdt + \ < \widetilde{\phi}^1, y^0 >_{H^{-1},H_0^1} - \int_\Omega \widetilde{\phi}^0 y^1 dx. \tag{8}$$

Here again, $\widetilde{\phi}$ is the solution of the adjoint system (3) corresponding to the initial conditions $\left(\widetilde{\phi}^0,\widetilde{\phi}^1\right)$. The coercitivity and therefore the existence of a unique minimizer for $\overline{\mathcal{J}}_\theta$ follows from Proposition 2.1 and the density of $\mathcal{U}_L$ in $\overline{\mathcal{U}}_L$. From this and by applying the HUM, it follows that for any $T \geq 2\pi$, $\theta \in \overline{\mathcal{U}}_L$ and

$\left(y^0, y^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists $h_\theta \in L^2\left(0, T; L^2(\Omega)\right)$ such that the solution y of the system

$$\begin{cases} y_{tt} - y_{xx} = \theta h_\theta, & (t,x) \in]0,T[\times \Omega \\ y(t,0) = y(t,\pi) = 0, & t \in [0,T] \\ y(0,x) = y^0(x), \quad y_t(0,x) = y^1(x), & x \in \Omega \end{cases} \tag{9}$$

satisfies $y(T,x) = y_t(T,x) = 0$ for all $x \in \Omega$. Indeed, $h_\theta = -\phi$, where ϕ is the solution of (3) associated with the minimizer $\left(\phi^0, \phi^1\right)$ of (8).

We are now in position to prove the main result of this section.

THEOREM 2.1 *The functional $\overline{J}$ as given by (7) is convex and continuous for the weak-$\star$ topology of $L^\infty(\Omega)$. In particular, there exists $\theta^* \in \overline{\mathcal{U}}_L$ such that*

$$\inf_{\mathcal{X} \in \mathcal{U}_L} J(\mathcal{X}) = \min_{\theta \in \overline{\mathcal{U}}_L} \overline{J}(\theta) = \overline{J}(\theta^*).$$

Proof. Let us first prove the continuity of $\overline{J}$. Due to the density of $\mathcal{U}_L$ in $\overline{\mathcal{U}}_L$ it suffices to show that if $\mathcal{X}_n \in \mathcal{U}_L$ is such that

$$\mathcal{X}_n \rightharpoonup \theta \quad \text{weak-}\star \text{ in } L^\infty(\Omega),$$

then

$$\int_0^T \int_\Omega \mathcal{X}_n \phi_n^2 dxdt \to \int_0^T \int_\Omega \theta \phi^2 dxdt$$

where ϕ_n and ϕ are the solutions of (3) corresponding to the minimizers of $\mathcal{J}_{\mathcal{X}_n}$ and $\overline{\mathcal{J}}_\theta$, respectively. We proceed in three steps:

Step 1: Mise en scène of HUM. By Proposition 2.1, the operator $\Lambda_n : L^2(\Omega) \times H^{-1}(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ defined by

$$\Lambda_n\left(\phi^0, \phi^1\right) = \left((\psi_n)_t(0), -\psi_n(0)\right), \tag{10}$$

where $\psi_n \in C\left(0, T; H_0^1(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right)$ is the solution of the backward system

$$\begin{cases} (\psi_n)_{tt} - (\psi_n)_{xx} = -\mathcal{X}_n \phi, & (t,x) \in]0,T[\times \Omega \\ \psi_n(t,0) = \psi_n(t,\pi) = 0, & t \in [0,T] \\ \psi_n(T,x) = (\psi_n)_t(T,x) = 0, & x \in \Omega \end{cases} \tag{11}$$

and $\phi \in C\left(0, T; L^2(\Omega)\right) \cap C^1\left(0, T; H^{-1}(\Omega)\right)$ is the solution of the adjoint system (3) with initial data $\left(\phi^0, \phi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ is an isomorphism. Hence, the equation

$$\Lambda_n\left(\phi_n^0, \phi_n^1\right) = \left(y^1, -y^0\right) \tag{12}$$

has a unique solution $\left(\phi_n^0, \phi_n^1\right)$ which satisfies

$$< \Lambda_n\left(\phi_n^0, \phi_n^1\right), \left(\phi_n^0, \phi_n^1\right) >_{H_0^1, H^{-1}} \quad = \quad \int_0^T \int_\Omega \mathcal{X}_n \phi_n^2 dxdt. \tag{13}$$

By (10) and (12),

$$\left((\psi_n)_t(0), -\psi_n(0)\right) = \left(y^1, -y^0\right) \tag{14}$$

and so the function $-\mathcal{X}_n\phi_n$ is an internal exact control for (11) corresponding to the initial data (14). Moreover, since by construction $-\mathcal{X}_n\phi_n$ is the HUM control, it is also the one of minimal L^2-norm.

Step 2: Uniform a priori estimates. From (6), (12) and (13) it follows that

$$\int_\Omega y^1\phi_n^0 dx \;-\; < y^0, \phi_n^1 >_{H_0^1, H^{-1}} = \int_0^T \int_\Omega \mathcal{X}_n \phi_n^2 dx dt \geq c_1 \left(\left\|\phi_n^0\right\|_{L^2}^2 + \left\|\phi_n^1\right\|_{H^{-1}}^2 \right).$$

Since $\left(y^0, y^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ are fixed,

$$\left| \int_\Omega y^1\phi_n^0 dx \;-\; < y^0, \phi_n^1 >_{H_0^1, H^{-1}} \right| \leq c_2 \left(\left\|\phi_n^0\right\|_{L^2} + \left\|\phi_n^1\right\|_{H^{-1}} \right).$$

Therefore,

$$\left\|\phi_n^0\right\|_{L^2} + \left\|\phi_n^1\right\|_{H^{-1}} \leq c_3 \tag{15}$$

and

$$\int_0^T \int_\Omega (\mathcal{X}_n\phi_n)^2 dx dt = \int_\Omega y^1\phi_n^0 dx - \; < y^0, \phi_n^1 >_{H_0^1, H^{-1}} \leq c_4. \tag{16}$$

Step 3: Pass to the limit. By (15), (16) and taking into account the continuous dependence of the solutions of the homogeneous wave equation with respect to the initial data, up to subsequences still labelled by n, we have the convergence

$$\begin{cases} \left(\phi_n^0, \phi_n^1\right) \rightharpoonup \left(\phi^0, \phi^1\right) & \text{weak in } L^2(\Omega) \times H^{-1}(\Omega), \\ \left(\phi_n, (\phi_n)_t\right) \rightharpoonup (\phi, \phi_t) & \text{weak-}\star \text{ in } L^\infty\left(0, T; L^2(\Omega) \times H^{-1}(\Omega)\right), \\ \mathcal{X}_n\phi_n \rightharpoonup \phi^* & \text{weak in } L^2\left(0, T; L^2(\Omega)\right) \end{cases}$$

where ϕ is a solution of

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0, & (t, x) \in]0, T[\times \Omega \\ \phi(t, 0) = \phi(t, \pi) = 0, & t \in [0, T] \\ \phi(0, x) = \phi^0(x), \quad \phi_t(0, x) = \phi^1(x), & x \in \Omega. \end{cases}$$

Aubin's lemma implies that

$$\phi_n \to \phi \quad \text{strong in } L^\infty\left(0, T; H^{-1}(\Omega)\right)$$

and since

$$\mathcal{X}_n \rightharpoonup \theta \quad \text{weak-}\star \text{ in } L^\infty\left(0, T; H^{-1}(\Omega)\right)$$

we get the weak convergence

$$\mathcal{X}_n\phi_n \rightharpoonup \theta\phi \quad \text{in } \mathcal{D}'((0, T) \times \Omega).$$

This clearly forces $\phi^* = \theta\phi$. On the other hand,

$$\left(\psi_n, (\psi_n)_t\right) \rightharpoonup (\psi, \psi_t) \quad \text{weak-}\star \text{ in } L^\infty\left(0, T; H_0^1(\Omega) \times L^2(\Omega)\right),$$

where ψ solves

$$\begin{cases} \psi_{tt} - \psi_{xx} = -\theta\phi, & (t,x) \in]0,T[\times \Omega \\ \psi(t,0) = \psi(t,\pi) = 0, & t \in [0,T] \\ \psi(T,x) = \psi(T,x) = 0, & x \in \Omega. \end{cases}$$

Passing to the limit in (12) and (14),

$$\Lambda\left(\phi^0, \phi^1\right) = \left(\psi_t(0), -\psi(0)\right) = \left(y^1, -y^0\right), \tag{17}$$

where Λ is the HUM isomorphism associated with the controlled system (9). Therefore,

$$\lim_{n\to\infty} \int_0^T \int_\Omega \mathcal{X}_n \phi_n^2 dxdt = \lim_{n\to\infty} \left[\int_\Omega y^1 \phi_n^0 dx - \ < y^0, \phi_n^1 >_{H_0^1, H^{-1}}\right] = \int_0^T \int_\Omega \theta\phi^2 dxdt. \tag{18}$$

Note also that the weak limit $\left(\phi^0, \phi^1\right)$ of any subsequence of $\left(\phi_n^0, \phi_n^1\right)$ is uniquely defined by (17). This implies that the limit (18) holds for the whole sequences $\left(\phi_n^0, \phi_n^1\right)$ and $\left(\mathcal{X}_n \phi_n^2\right)$.

Finally, let us prove that $\overline{J}$ is convex. For $\theta \in \overline{\mathcal{U}}_L$ and using once again the HUM, an easy computation shows that

$$\begin{aligned} -\frac{1}{2}\overline{J}(\theta) &= -\frac{1}{2}\int_0^T \int_\Omega \theta\phi^2 dxdt \\ &= -\frac{1}{2}\left(\int_\Omega y^1 \phi_n^0 dx - \ < y^0, \phi_n^1 >_{H_0^1, H^{-1}}\right) \\ &= \frac{1}{2}\int_0^T \int_\Omega \theta\phi^2 dxdt + \ < y^0, \phi^1 >_{H_0^1, H^{-1}} - \int_\Omega y^1 \phi^0 dx \\ &= \min_{(\phi^0,\phi^1)\in L^2\times H^{-1}} \overline{\mathcal{J}}_\theta\left(\phi^0, \phi^1\right). \end{aligned}$$

This proves that $-\frac{1}{2}\overline{J}(\theta)$ is concave since it is the minimum of affine functions. Hence, $\overline{J}(\theta)$ is convex. ■

3 First-order optimality conditions

Next, we aim to characterize the minimizers of the relaxed problem (RP). We then should study the variations of $\overline{J}$ in the space $\overline{\mathcal{U}}_L$ of admissible designs. To this end, we recall that the tangent cone $T'_{\overline{\mathcal{U}}_L}(\theta^*)$ to the set $\overline{\mathcal{U}}_L$ at θ^* in $L^\infty(\Omega)$ is defined as the set of functions $\overline{\theta} \in L^\infty(\Omega)$ such that for any sequence of positive real numbers $(\varepsilon_n) \searrow 0$ there exists a sequence $\overline{\theta}_n \in L^\infty(\Omega)$ which satisfies: (a) $\overline{\theta}_n \to \overline{\theta}$ uniformly, and (b) $\theta^* + \varepsilon_n \overline{\theta}_n \in \overline{\mathcal{U}}_L$ for all $n \in \mathbb{N}$. The following characterization of $T'_{\overline{\mathcal{U}}_L}(\theta^*)$ is known (see the proof of [1, Prop. 2.1]).

Lemma 3.1 $T'_{\overline{\mathcal{U}}_L}(\theta^*)$ *is composed of the functions* $\overline{\theta} \in L^\infty(\Omega)$ *such that:*

(i) $\int_{\Omega} \overline{\theta}(x)\,dx = 0,$

(ii) *As* $n \to \infty$, $\left\| \mathcal{X}_{Q_n^0} \overline{\theta}_- \right\|_\infty \to 0$ *and* $\left\| \mathcal{X}_{Q_n^1} \overline{\theta}_+ \right\|_\infty \to 0$, *where* $\overline{\theta}_-$ *(resp.* $\overline{\theta}_+$*) are the negative (resp. positive) parts of* $\overline{\theta}$ *and*

$$Q_n^0 = \{x \in \Omega : \theta^*(x) \leq 1/n\}, \quad Q_n^1 = \{x \in \Omega : \theta^*(x) \geq 1 - 1/n\}.$$

Theorem 3.1 *The functional* $\overline{J}$ *is Gâteaux differentiable on the set* $\overline{\mathcal{U}}_L$ *and its derivative at* $\theta \in \overline{\mathcal{U}}_L$ *in the admissible direction* $\overline{\theta}$ *is given by*

$$< \overline{J}'(\theta), \overline{\theta} > \;=\; -\int_0^T \int_\Omega \overline{\theta} \phi_\theta^2 dx dt,$$

ϕ_θ *being the solution of the adjoint system (3) with initial data* $\left(\phi_\theta^0, \phi_\theta^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ *which are associated through the HUM isomorphism*

$$\Lambda\left(\phi_\theta^0, \phi_\theta^1\right) = \left(y^1, -y^0\right)$$

with the initial conditions of the controlled system

$$\begin{cases} y_{tt} - y_{xx} = -\theta \phi_\theta, & (t,x) \in]0,T[\times \Omega \\ y|_{\partial\Omega} = 0, & t \in [0,T] \\ y(0,x) = y^0(x), \quad y_t(0,x) = y^1(x), & x \in \Omega \\ y(T,x) = y_t(T,x) = 0, & x \in \Omega. \end{cases}$$

In particular, $\theta^* \in \overline{\mathcal{U}}_L$ *is a minimizer for (RP) if and only if*

$$\int_0^T \int_\Omega \overline{\theta} \phi_{\theta^*}^2 dx dt \leq 0 \quad \forall\, \overline{\theta} \in T'_{\overline{\mathcal{U}}_L}(\theta^*). \tag{19}$$

Proof. For a fixed $\theta \in \overline{\mathcal{U}}_L$, by writing down the optimality condition for the continuous, convex and coercive functional $\overline{\mathcal{J}}_\theta$, it is not hard to show that if $\left(\phi_\theta^0, \phi_\theta^1\right)$ is the unique minimizer of $\overline{\mathcal{J}}_\theta$, then the associated solution ϕ_θ satisfies the conditions

$$\begin{cases} (1_\theta) & \int_0^T \int_\Omega \theta \phi_\theta \widetilde{\phi}_{\widetilde{\phi}_0} dx dt = \int_\Omega \widetilde{\phi}_0 y^1 dx & \forall\, \widetilde{\phi}_0 \in L^2(\Omega) \\ (2_\theta) & \int_0^T \int_\Omega \theta \phi_\theta \widetilde{\phi}_{\widetilde{\phi}_1} dx dt = - < \widetilde{\phi}_1, y^0 >_{H^{-1}, H_0^1} & \forall\, \widetilde{\phi}_1 \in H^{-1}(\Omega), \end{cases}$$

where $\widetilde{\phi}_{\widetilde{\phi}_0}$ (resp. $\widetilde{\phi}_{\widetilde{\phi}_1}$) is the solution of the system

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & (t,x) \in]0,T[\times \Omega \\ \varphi|_{\partial\Omega}(t) = 0, & t \in [0,T] \\ \varphi(0,x) = \widetilde{\phi}_0(x) \text{ (resp. 0)}, \quad \varphi_t(0,x) = 0 \text{ (resp. } \widetilde{\phi}_1(x)\text{)}, & x \in \Omega. \end{cases}$$

Assume now that $\overline{\theta} \in L^\infty(\Omega)$ is an admissible direction, that is, for $\varepsilon > 0$ the function $\theta + \varepsilon\overline{\theta} \in \overline{\mathcal{U}}_L$. Denoting by $\left(1_{\theta+\varepsilon\overline{\theta}}\right)$ and $\left(2_{\theta+\varepsilon\overline{\theta}}\right)$ the optimality conditions for the minimizer $\left(\phi^0_{\theta+\varepsilon\overline{\theta}},\phi^1_{\theta+\varepsilon\overline{\theta}}\right)$ of $\overline{\mathcal{J}}_{\theta+\varepsilon\overline{\theta}}$, and subtracting $\left(1_{\theta+\varepsilon\overline{\theta}}\right)-(1_\theta)$ and $\left(2_{\theta+\varepsilon\overline{\theta}}\right)-(2_\theta)$ we get

$$\begin{cases} \int_0^T\int_\Omega \left[\left(\theta+\varepsilon\overline{\theta}\right)\phi_{\theta+\varepsilon\overline{\theta}}-\theta\phi_\theta\right]\widetilde{\phi}_{\widetilde{\phi}_0}dxdt=0 & \forall\ \widetilde{\phi}_0\in L^2(\Omega)\\ \int_0^T\int_\Omega \left[\left(\theta+\varepsilon\overline{\theta}\right)\phi_{\theta+\varepsilon\overline{\theta}}-\theta\phi_\theta\right]\widetilde{\phi}_{\widetilde{\phi}_1}dxdt=0 & \forall\ \widetilde{\phi}_1\in H^{-1}(\Omega).\end{cases}$$

Choosing as initial conditions $\left(\widetilde{\phi}_0=\phi^0_{\theta+\varepsilon\overline{\theta}},\ \widetilde{\phi}_1=\phi^1_{\theta+\varepsilon\overline{\theta}}\right)$ and $\left(\widetilde{\phi}_0=\phi^0_\theta,\ \widetilde{\phi}_1=\phi^1_\theta\right)$ in these expressions, by linearity,

$$\int_0^T\int_\Omega\left[\left(\theta+\varepsilon\overline{\theta}\right)\phi^2_{\theta+\varepsilon\overline{\theta}}-\theta\phi_\theta\phi_{\theta+\varepsilon\overline{\theta}}\right]dxdt=0$$

and

$$\int_0^T\int_\Omega\left[\left(\theta+\varepsilon\overline{\theta}\right)\phi_{\theta+\varepsilon\overline{\theta}}\phi_\theta-\theta\phi^2_\theta\right]dxdt=0.$$

From this, it is not hard to show that

$$\int_0^T\int_\Omega\left[\left(\theta+\varepsilon\overline{\theta}\right)\phi^2_{\theta+\varepsilon\overline{\theta}}-\theta\phi^2_\theta\right]dxdt=-\varepsilon\int_0^T\int_\Omega\overline{\theta}\phi_{\theta+\varepsilon\overline{\theta}}\phi_\theta dxdt.$$

Hence,

$$\lim_{\varepsilon\to0}\frac{\overline{J}\left(\theta+\varepsilon\overline{\theta}\right)-\overline{J}(\theta)}{\varepsilon}=-\lim_{\varepsilon\to0}\int_0^T\int_\Omega\overline{\theta}\phi_{\theta+\varepsilon\overline{\theta}}\phi_\theta dxdt=-\int_0^T\int_\Omega\overline{\theta}\phi^2_\theta dxdt,$$

the last equality being a consequence of the fact that the weak convergence $\left(\theta+\varepsilon\overline{\theta}\right)\rightharpoonup\theta$ weak-$\star$ in $L^\infty(\Omega)$ implies the convergence

$$\phi_{\theta+\varepsilon\overline{\theta}}\rightharpoonup\phi_\theta\quad\text{weak-}\star\ \text{ in } L^\infty\left(0,T;L^2(\Omega)\right).$$

This may be proved as in the proof of Theorem 2.1.

Finally, the first-order necessary optimality condition for (RP)

$$<\overline{J}'(\theta),\overline{\theta}>\geq0\quad\forall\ \overline{\theta}\in T'_{\overline{\mathcal{U}}_L}(\theta^*),$$

translate into (19). Since both $\overline{J}$ and $\overline{\mathcal{U}}_L$ are convex, this condition is also sufficient.

■

4 Concluding remarks and open problems

Some numerical experiments [10] suggest that, at least if the initial data are regular enough, then (P) is well-posed. In this sense, it is important to mention [6] where an optimal design problem for an elliptic equation is studied and sufficient conditions on the data are given to insure the existence of classical solutions. In the case considered in this work, this is by now an open problem. We do believe that Theorem 3.1 may be of some help in this direction. Anyway, even if (P) is well-posed, Theorem 2.1 is not useless: at the numerical viewpoint, the relaxed formulation allows to implement a very efficient descent gradient method which, in some cases, may capture the solutions of the original problem (see [10]). It is also interesting to stand out that if (P) is well-posed, then this implies the uniqueness of solutions for (P). Indeed, this is a consequence of the fact that the relaxed functional $\overline{J}$ is convex and that of the extremal points of the convex set $\overline{\mathcal{U}}_L$ are the characteristic functions. For the simple initial conditions $y^0(x) = \sin \pi x$, $y^1(x) = 0$ and $\Omega =]0, 1[$, this was observed numerically in [10] where the optimal ω is an interval centered at 0.5. The dependance of the optimal ω with respect to the controllability time T was also numerically observed in [10].

As for possible extensions of the results of this work, it is natural to look at the $N-$dimensional wave equation. The first difficulty arises not on the optimal design problem but in the controllability one. Some additional conditions must be assumed on the support ω in order to the geometrical controllability condition be fulfilled. Otherwise, it is not easy to identify the space of controllable data [3, 4]. One of these conditions is that ω to be a neighbour of the boundary [15]. Apart from this, it seems that the arguments in this work extend to the $N-$ dimensional case. A numerical study in 2D has been recently developed in [11].

More general distributed parameter systems could also be considered. The present paper is just a first (and small) step in this program.

Acknowledgements. The author is very grateful to Arnaud Münch for proposing him the problem considered in this work and for some stimulating conversations on it. He also thanks the referees whose remarks and suggestions led to an improvement on the original draft.

References

[1] E. Bednarczuk, M. Pierre, E. Rouy, J. Sokolowski, Tangent sets in some functional spaces, J. Math. Anal. Appl. 42 (2000) 871-886.

[2] M. C. Delfour, J. P. Zolésio, Shapes and Geometries - Analysis, Differential Calculus and Optimization. Advances in Design and Control, SIAM, 2001.

[3] A. Haraux, On a completion problem in the theory of distributed control of wave equations, in Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, 1988 (H. Brèzis and J. L. Lions, Ed.), Pitman Research Notes in Mathematics.

[4] A. Haraux, A generalized internal control for the wave equation in a rectangle, J. Math. Anal. Appl. 153 (1990) 190-216.

[5] P. Hebrard, A. Henrot, Optimal shape and position of the actuators for the stabilization of a string, Syst. Control Lett. 48 (2003) 199-209.

[6] A. Henrot, H. Maillot, Optimization of the shape and location of the actuators in an internal control problem, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 8 4 (2001) n^o 3 737-757.

[7] A. Henrot, M. Pierre, Variation and optimisation des formes: une analyse géometrique, Mathématiques et Applications 48, Springer, 2005.

[8] J. Lagnese, Control of wave processes with distributed controls supported on a subregion, SIAM J. Control Optim. 21 N^o 1 (1983) 68-85.

[9] J. L. Lions, Contrôlabilité exacte, perturbation et stabilisation des systèmes distribués, Tome 1, Masson, 1988.

[10] A. Münch, Optimal location of the support of the control for the 1-D wave equation: numerical investigations, Comput. Optim. Appl. 39 (3) (2008) 262-286.

[11] A. Münch, Optimal design of the support of the control for the 2-D wave equation : numerical investigations, Int. J. Numer. Anal. Model. 5 (2) (2008) 331-351.

[12] A. Münch, P. Pedregal, F. Periago, Optimal design of the damping set for the stabilization of the wave equation, J. Diff. Equ. 231 (1) (2006) 331-358.

[13] A. Münch, P. Pedregal, F. Periago, Optimal internal stabilization of the linear system of elasticity, Arch. Rat. Mech. Anal., in press.

[14] A. Münch, P. Pedregal, F. Periago, Relaxation of an optimal design problem for the heat equation, J. Math. Pures Appl. 89 (3) (2008) 225-247.

[15] E. Zuazua, in Contrôlabilité exacte, perturbation et stabilisation des systèmes distribués, by J. L. Lions, Tome 1, Masson, 1988.