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Optimal Internal Stabilization of the Linear System of Elasticity

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Abstract

We address the non-linear optimal design problem which consists in finding the best position and shape of a feedback damping mechanism for the stabilization of the linear system of elasticity. Non-existence of classical designs are related to the over-damping phenomenon. Therefore, by means of Young measures, a relaxation of the original problem is proposed. Due to the vector character of the elasticity system, the relaxation is carried out through div-curl Young measures which let the analysis be direct and the dimension independent. Finally, the relaxed problem is solved numerically, and a penalization technique to recover quasi-optimal classical designs from the relaxed ones is implemented in several numerical experiments.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. For a given function $\boldsymbol{u} = (u_1, u_2, \dots, u_N) : [0, T[\times \Omega \to \mathbb{R}^N$, depending on time t and position $\boldsymbol{x} = (x_1, \dots, x_n)$, partial derivatives with respect to t will be denoted by $'$ and derivatives with respect to x_j by $,_j$, that is,

$$u_i' = \frac{\partial u_i}{\partial t}, \quad u_i'' = \frac{\partial^2 u_i}{\partial t^2} \quad \text{and} \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}.$$

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We will also use the vector notation

$$\boldsymbol{u}' = \left(u_1', \ldots, u_N'\right), \quad \boldsymbol{u}'' = \left(u_1'', \ldots, u_N''\right) \quad \text{and} \quad \nabla_x \boldsymbol{u} = \left(u_{i,j}\right), \ 1 \leqq i, j \leqq N.$$

We introduce the classical symmetric tensors of linear elasticity, namely, the linearized strain tensor

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T\right) \tag{1}$$

and the stress tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \left(\sigma_{ij} = a_{ijkl}\varepsilon_{kl}\right) \tag{2}$$

where the coefficient of elasticity $a_{ijkl} \in W^{1,\infty}(\Omega)$, $i, j, k, l = 1, \ldots, N$, are such that

$$a_{ijkl} = a_{klij} = a_{jikl} \quad \text{and} \quad a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqq \alpha \varepsilon_{ij}\varepsilon_{ij} \ \text{in } \Omega \tag{3}$$

for some fixed $\alpha > 0$.

Then, we consider the following damped system:

$$\begin{cases} \boldsymbol{u}'' - \nabla_x \cdot \boldsymbol{\sigma} + a(\boldsymbol{x}) \mathcal{X}_\omega(\boldsymbol{x}) \boldsymbol{u}' = \boldsymbol{0} & \text{in } (0, T) \times \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } (0, T) \times \Gamma_0, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } (0, T) \times \Gamma_1, \\ \boldsymbol{u}(0, \cdot) = \boldsymbol{u_0}, \quad \boldsymbol{u}'(0, \cdot) = \boldsymbol{u_1} & \text{in } \Omega, \end{cases} \tag{4}$$

where $\omega \subset \Omega$ is a subset of positive Lebesgue measure, $\mathcal{X}_\omega$ is the characteristic function of ω, $\nabla_x \cdot$ is the divergence operator considered with respect to the spatial variable $\boldsymbol{x}$, $\boldsymbol{n} = (n_1, \ldots, n_N)$ is the outward unit normal vector to Γ_1, $0 < T \leqq \infty$, and $a = a(\boldsymbol{x}) \in L^\infty(\Omega; \mathbb{R}_+)$ is a damping potential satisfying

$$a(\boldsymbol{x}) \geqq a_0 > 0 \quad \text{almost everywhere } \boldsymbol{x} \in \omega.$$

It is known (see [1,8,14]) that system (Equation 4) is well posed in the following sense: if we introduce the space

$$V_0 = \left\{\boldsymbol{u} \in \left(H^1(\Omega)\right)^N : \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_0\right\}$$

and take $(\boldsymbol{u_0}, \boldsymbol{u_1}) \in V_0 \times \left(L^2(\Omega)\right)^N$, then there exists a unique *weak* solution $\boldsymbol{u}$ of Equation (4) in the class

$$\boldsymbol{u} \in C([0, T[\,; V_0) \cap C^1\left([0, T[\,; L^2(\Omega)^N\right).$$

The energy at time t of this solution is given by

$$E(t) = \frac{1}{2}\int_\Omega \left(\left|\boldsymbol{u}'\right|^2 + \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u})\right) \mathrm{d}x.$$

where $\boldsymbol{\sigma} : \boldsymbol{\varepsilon}$ designates the trace $\sum_{i,j=1,N} \sigma_{ij}\varepsilon_{ij}$. Multiplying the first vector equation in Equation (4) by $\boldsymbol{u}'$ and integrating by parts, one easily deduces that

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_{\Omega} a(\boldsymbol{x})\, \mathcal{X}_{\omega}(\boldsymbol{x}) \left|\boldsymbol{u}'\right|^2 \mathrm{d}x, \quad \forall t > 0.$$

Therefore, the energy is a non-increasing function of time.

Regarding the physical meaning of system (Equation 4), the dissipative term $a(\boldsymbol{x})\, \mathcal{X}_{\omega}(\boldsymbol{x})\, \boldsymbol{u}'$ may be seen as a feedback *damping* mechanism which measures the velocity of vibrations through the use of sensors, and acts on the system according to these measurements by means of actuators. In this sense, $\mathcal{X}_{\omega}$ indicates the place and shape of actuators.

It is then natural, and very important in practice, to analyze the question of determining the best position and shape of sensors and actuators that minimize the energy of the system over a time interval (see [7,11] and the references therein). This is the main problem we address in this work. In mathematical terms, we consider the following non-linear optimal design problem:

$$(P) \qquad \inf_{\omega\in\Omega_L} J(\mathcal{X}_{\omega}) = \int_0^T E(t)\, \mathrm{d}t = \frac{1}{2}\int_0^T\int_{\Omega}\left(\left|\boldsymbol{u}'\right|^2 + \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u})\right) \mathrm{d}x\, \mathrm{d}t, \quad (5)$$

where $\boldsymbol{u}$ is the solution of system (Equation 4), and for a fixed $0 < L < 1$,

$$\Omega_L = \{\omega \subset \Omega : |\omega| = L\,|\Omega|\},$$

$|\omega|$ and $|\Omega|$ being the Lebesgue measure of ω and Ω, respectively.

The same optimization problem for the damped wave equation has been recently considered by the authors in [17,18] where the possible non-well-posedeness character of (P) was observed, that is, the non-existence of a minimizer in the class of characteristic functions. Then, a full relaxation of the original problem was carried out by means of Young measures (which are a powerful tool to understand the limit behavior of minimizing sequences in non-linear functionals) (see [3,19] and the references there in), and a suitable representation of divergence-free vector fields which enables one to transform the original problem into a non-convex, vector variational one.

The aim of this work is to extend the results in [17,18] to the case of the system of linear elasticity. To this end, we consider the relaxed problem

$$(\mathrm{RP}) \inf_{s\in L^{\infty}(\Omega)} J(s) = \int_0^T E(t)\, \mathrm{d}t \quad (6)$$

where as above $E(t)$ is the energy associated with the new system

$$\begin{cases} \boldsymbol{u}'' - \nabla_x \cdot \boldsymbol{\sigma} + a(\boldsymbol{x})\, s(\boldsymbol{x})\, \boldsymbol{u}' = \boldsymbol{0} & \text{in } (0,T)\times\Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } (0,T)\times\Gamma_0, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0} & \text{on } (0,T)\times\Gamma_1, \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u_0}, \quad \boldsymbol{u}'(0,\cdot) = \boldsymbol{u_1} & \text{in } \Omega, \end{cases} \quad (7)$$

and now the competing functions s satisfy the pointwise and volume constraints

$$0 \leqq s(\boldsymbol{x}) \leqq 1 \quad \text{and} \quad \int_{\Omega} s(\boldsymbol{x})\,\mathrm{d}x = L\,|\Omega|. \tag{8}$$

Our main theoretical result reads as follows.

Theorem 1.1. *Assume that the initial data of system* (*Equation* 4) *have the regularity*

$$(\boldsymbol{u_0}, \boldsymbol{u_1}) \in \left(\left(H^2(\Omega)\right)^N \cap V_0\right) \times V_0. \tag{9}$$

Then (RP) *is a full relaxation of* (P) *in the sense:*

(i) *There are optimal solutions for* (RP).
(ii) *The minimum of* (RP) *equals the infimum of* (P).
(iii) *Minimizing sequences for* (P) *correspond to sequences converging weakly to optimal solutions of* (RP). *In particular, first-order laminates with arbitrary normals and weights given by optimal solutions of* (RP) *are minimizing for* (P).

The first part of this work is devoted to the proof of this relaxation result. This is done in Section 2. As we will see in Lemma 2.3, the assumption (Equation 9) on the initial data is a sufficient condition in order to avoid concentration of energy phenomena and therefore this will enable us to use the Young measures theory to compute the cost limit of a minimizing sequence for problem (P).

A few more comments on Theorem 1.1 are in order. We will show in the second part of this work that for some values of the damping potential a there is a numerical evidence that problem (P) is ill posed, that is, there is no minimizer in the class of characteristic functions. This justifies the relaxation stated in points (i) and (ii) above. In what concerns point (iii), it tells us what the microstructure of the optimal damping designs looks like. As we will see in the proof of Theorem 1.1, this information is codified by the optimal Young measure associated with the relaxed problem (RP). Moreover, because of the particular form of this optimal measure (see Equation 21), we will deduce that if $\mathcal{X}_{\omega_j}$ is a minimizing sequence for (P), then the associated displacements $\boldsymbol{u}_j$ converge to the optimal displacement $\boldsymbol{u}$ for the relaxed problem (RP) in a strong sense. This is our Theorem 2.1.

Apart from considering a more complicated system than the scalar wave equation, the main novelty of this work concerns the method of proving Theorem 1.1. It is different from the one the authors used in [17,18] for the case of the wave equation. Our approach here does not require the introduction of auxiliary potentials associated with divergence-free vector fields. Instead of those, we use div-curl Young measures as given by [20]. See also [10]. This makes the treatment much more direct and dimension-independent.

It is also important to mention that the optimal design problem we have considered in this work could be analyzed by some other methods. In particular, by the homogenization method (see [2] for the case of optimal design problems with steady-state equations) and by using the Trotter–Kato theorem (see [9] for the case of the wave equation). The approach in this paper, although less standard, seems to be more general in scope. In particular, it may be applied to more complicated stabilization problems where, for instance, the damping mechanism acts not only in the lower-order term but also in the principal part of the operator. In this situation we talk about an optimal design problem or a bi-design situation. We refer to [15] for a specific example under the 1D wave equation. Our approach can also accommodate other more general types of cost functionals depending on derivatives of the state, and thus it is not limited to dealing with the energy as an optimization criterium.

In a second part, we address the numerical resolution of the relaxed problem (RP) by using a first-order gradient descent method. We present several experiments which highlight the influence of the over-damping phenomenon on the nature of (P). For small values of the damping potential a, we find that the problem (P) is well-posed. On the contrary, when a is large enough, the optimal density is no longer in $L^\infty(\Omega, \{0, 1\})$ but in $L^\infty(\Omega, [0, 1])$. The influence of the Lamé coefficients (in the isotropic case) on the optimal shape and position of the optimal damping set is also briefly analyzed. Finally, we illustrate point (iii) of Theorem 1.1 by extracting for the optimal density a minimizing sequence of characteristic functions.

2. Proof of the relaxation theorem

As we mentioned in the Introduction, our approach to prove Theorem 1.1 is based on the use of a class of Young measures associated with a pair of vector fields, the first one being divergence-free and the second one curl-free. For this reason, these measures are called div-curl Young measures. In order to make the paper easier to read, we first collect the main properties of these measures that we will need later on. For a proof of this we refer the reader to [10,20].

2.1. Preliminary on div-curl Young measures

Given a bounded $\mathcal{C}^2$ domain $\Omega \subset \mathbb{R}^N$ and a sequence of pairs of vector fields $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$ such that

$$\boldsymbol{F}_j : \Omega \to \mathcal{M}^{m\times N}, \quad \boldsymbol{G}_j : \Omega \to \mathcal{M}^{m\times N}$$

are uniformly bounded in $L^2\left(\Omega; \mathcal{M}^{m\times N}\right)$, it is well-known [19, Th. 6.2, p. 97] that we may associate with (a subsequence of) such a pair a family of probability measures $\nu = \{\nu_x\}_{\boldsymbol{x}\in\Omega}$ with the main property that if the sequence of functions $\left\{\phi\left(\boldsymbol{x}, \boldsymbol{F}_j(\boldsymbol{x}), \boldsymbol{G}_j(\boldsymbol{x})\right)\right\}$ weakly converges in $L^1(\Omega)$ for some Carathéodory integrand ϕ, then the weak limit is given by

$$\overline{\phi}(\boldsymbol{x}) = \int_{\mathcal{M}^{m\times N}\times\mathcal{M}^{m\times N}} \phi(\boldsymbol{x}, A, B)\, \mathrm{d}\nu_x(A, B).$$

If, in addition, the pair $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$ satisfies

$$\nabla_x \cdot \boldsymbol{F}_j = 0 \quad \text{and} \quad \operatorname{curl} \boldsymbol{G}_j = 0, \tag{10}$$

in a distributional sense, then the measure $\nu = \{\nu_x\}_{\boldsymbol{x}\in\Omega}$ is called the div-curl Young measure associated with $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$.

Besides the general properties of Young measures, div-curl Young measures enjoy the following fundamental property which is an immediate consequence of the well-known div-curl lemma.

Lemma 2.1 *If $\nu = \{\nu_x\}_{\boldsymbol{x}\in\Omega}$ is a div-curl Young measure, then for almost every where $\boldsymbol{x} \in \Omega$,*

$$\int_{\mathcal{M}^{m\times N}\times\mathcal{M}^{m\times N}} AB^T \mathrm{d}\nu_x\,(A, B) = \int_{\mathcal{M}^{m\times N}} A\mathrm{d}\nu_x^{(1)}\,(A) \int_{\mathcal{M}^{m\times N}} B^T \mathrm{d}\nu_x^{(2)}\,(B) \tag{11}$$

where $\nu_x^{(i)}$, $i = 1, 2$, are the marginals of ν_x on the two Cartesian factors, respectively.

An important subclass of this family of measures is the class of the so-called div-curl laminates. These are to div-curl Young measures what laminates are to gradient Young measures, and can be constructed as follows.

Lemma 2.2 *Suppose that A_i, B_i, $i = 1, 2$, are four $m \times N$ matrices such that*

$$(A_2 - A_1)\left(B_2^T - B_1^T\right) = 0. \tag{12}$$

Then the measure

$$\nu = s\delta_{(A_1,B_1)} + (1-s)\,\delta_{(A_2,B_2)} \tag{13}$$

is a div-curl Young measure for all $0 \leqq s \leqq 1$.

Notice that hypothesis (Equation 12) above is a sufficient condition in order for the Young measure (Equation 13) to satisfy the div-curl condition (Equation 11). For a concrete (very important in the context of optimal design in conductivity) example of div-curl laminate and its associated sequences $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$, we refer to [20].

2.2. Two preliminary key results

In the sequel we will have to deal with the class of homogeneous div-curl Young measures ν of the form

$$\nu = s\delta_{((M_1+C,\, -\boldsymbol{\sigma}(\overline{M})),M)} + (1-s)\,\delta_{((M_1,\, -\boldsymbol{\sigma}(\overline{M})),M)}, \tag{14}$$

with $0 \leqq s \leqq 1$, $C \in \mathbb{R}^N$ and $M = \left(M_1, \overline{M}\right) \in \mathcal{M}^{N\times(N+1)}$. The term M_1 stands for the first column of M and $\overline{M} \in \mathcal{M}^{N\times N}$ is the rest of the matrix. Moreover, $\boldsymbol{\sigma}\left(\overline{M}\right)$ stands for the $N \times N$ matrix with components $\left(\boldsymbol{\sigma}\left(\overline{M}\right)\right)_{ij} = a_{ijkl}\overline{M}_{kl,}$ with a_{ijkl} the coefficients given in Equation (3).

The following result is essential to avoid undesirable phenomena of concentration of energy.

Lemma 2.3 *Suppose that the initial data of system (Equation 4) have the regularity (Equation 9) and that for almost each $(t, \boldsymbol{x}) \in (0, T) \times \Omega$ we have a div-curl Young measure ν of the form (Equation 14) with $s = s(\boldsymbol{x})$ satisfying (Equation 8), $C = a\boldsymbol{u}$ and $M = \left(M_1, \overline{M}\right) = (\boldsymbol{u}', \nabla_x \boldsymbol{u})$, where $\boldsymbol{u}$ is the solution of Equation (7) associated with s. Assume also that there exists a divergence-free vector field*

$$\boldsymbol{F} \in L^2\left((0, T) \times \Omega; \mathcal{M}^{N\times(N+1)}\right)$$

such that

$$\boldsymbol{F}(t, \boldsymbol{x}) = \int_{\mathcal{M}^{N\times(N+1)}} A d\nu^{(1)}_{(t,x)}(A).$$

Then there exists a sequence $\mathcal{X}_{\omega_j}$, which is admissible for (P), and such that if $\boldsymbol{u}_j$ is the corresponding solution of Equation (4), then

$$\boldsymbol{G}_j = \left(\boldsymbol{u}'_j, \nabla_x \boldsymbol{u}_j\right) \to (\boldsymbol{u}', \nabla_x \boldsymbol{u}) \quad \textit{strong in} \quad \left(L^2((0, T) \times \Omega)\right)^N.$$

Proof. Due to the particular form of the measure ν (specifically, ν is div-curl Young measure), it is known (see for instance [10,12,20]) that there exist two vector fields $\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right)$ such that

$$\nabla_{(t,x)} \cdot \boldsymbol{H}_j \to 0 \quad \text{in } H^{-1}, \quad \boldsymbol{R}_j = \nabla_{(t,x)} \boldsymbol{v}_j,$$

with $\boldsymbol{v}_j$ satisfying the initial and boundary conditions of system (Equation 4), and whose associated Young measure is ν. Note that now $\nabla_{(t,x)}\cdot$ is the divergence operator with respect to t and $\boldsymbol{x}$. Moreover, both

$$\left\|\boldsymbol{H}_j\right\|^2 \quad \text{and} \quad \left\|\boldsymbol{R}_j\right\|^2,$$

where $\|\cdot\|$ stands for the usual norm in the space of matrices $\mathcal{M}^{N\times(N+1)}$, are equi-integrable. What is at stake here is the fact that by modifying the pair $(\boldsymbol{H}_j, \boldsymbol{R}_j)$ a bit, we can get a new pair denoted in the sequel by $(\boldsymbol{F}_j, \boldsymbol{G}_j)$ admissible for (P) so that the underlying Young measure as well as the equi-integrability are preserved.

Since ν is of the form (Equation 14) and $\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right)$ is its associated sequence, for j large, the pair $\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right)$ is closer to $\Lambda_{1,C}$ than Λ_0 in proportion $s(\boldsymbol{x})$, where

$$\Lambda_{1,C} = \left\{(A, B) \in \mathcal{M}^{N\times(N+1)} \times \mathcal{M}^{N\times(N+1)} : A_1 = B_1 + C, \quad \overline{A} = -\boldsymbol{\sigma}\left(\overline{B}\right)\right\}$$

and

$$\Lambda_0 = \left\{(A, B) \in \mathcal{M}^{N\times(N+1)} \times \mathcal{M}^{N\times(N+1)} : A_1 = B_1, \quad \overline{A} = -\boldsymbol{\sigma}\left(\overline{B}\right)\right\}.$$

Let us denote by $\mathcal{X}_{\omega_j}$ the characteristic function indicating this property, that is,

$$\mathcal{X}_{\omega_j}(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \operatorname{dist}\left(\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right), \Lambda_{1,C}\right) < \operatorname{dist}\left(\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right), \Lambda_0\right) \\ 0 & \text{else.} \end{cases}$$

We claim that this sequence of characteristic functions do not depend on time (as it has been indicated in its definition) even though the fields $\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right)$ do. This is a direct consequence of the explicit form assumed on the underlying Young measure in Equation (14). Indeed, it corresponds to a (div-curl) laminate and the direction of lamination should be perpendicular to all rows of the differences for the first Cartesian factor (the one corresponding to the div-free constraint)

$$\left(M_1 + C, \;\; -\boldsymbol{\sigma}\left(\overline{M}\right)\right) - \left(M_1, \;\; -\boldsymbol{\sigma}\left(\overline{M}\right)\right) = (C, \mathbf{0}).$$

Notice that the second Cartesian factor does not provide any information on the direction of lamination since this component is the same on the two Dirac masses. The component corresponding to C above is the time direction. So perpendicular directions should have a vanishing time component and are not restricted otherwise. This, in turn, implies that laminates are "vertical" (time axis) with any normal in space. In this way, the essential dependence on time of the sequence $\mathcal{X}_{\omega_j}$ is incompatible with this direction of lamination.

Obviously,

$$\lim_{j \to \infty} \int_{\Omega} \mathcal{X}_{\omega_j} (\boldsymbol{x})\, \mathrm{d}x = L\, |\Omega|\,.$$

We aim to modify the pair $\left(\boldsymbol{H}_j, \boldsymbol{R}_j\right)$ in a suitable way to obtain a new pair $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$, *admissible for (P),* whose associated measure is the same ν. To this end, we modify the sequence $\mathcal{X}_{\omega_j}$ in a set whose measure converges to zero as $j \to \infty$ and satisfying

$$\int_{\Omega} \mathcal{X}_{\omega_j} (\boldsymbol{x})\, \mathrm{d}x = L\, |\Omega|\,.$$

This new sequence of characteristic functions (not relabelled) is admissible for the original optimal design problem and satisfies

$$\lim_{j \to \infty} \int_0^T \int_{\Omega} \left[\left| \boldsymbol{H}_j^1 - \left(\boldsymbol{v}_j' + a\,(\boldsymbol{x})\, \mathcal{X}_{\omega_j} (\boldsymbol{x})\, \boldsymbol{v}_j \right) \right|^2 + \left\| \overline{\boldsymbol{H}}_j + \boldsymbol{\sigma}\left(\boldsymbol{v}_j\right) \right\|^2 \right] \mathrm{d}x\, \mathrm{d}t = 0, \tag{15}$$

where as before $\boldsymbol{H}_j^1$ denotes the first column of $\boldsymbol{H}_j$ and $\overline{\boldsymbol{H}}_j$ is the $N \times N$ matrix composed of the remaining columns.

Let $\boldsymbol{u}_j$ be the solution of Equation (4) associated with the admissible sequence $\mathcal{X}_{\omega_j}$ and consider the pair

$$\boldsymbol{F}_j = \left(\boldsymbol{u}_j' + a\,(\boldsymbol{x})\, \mathcal{X}_{\omega_j} (\boldsymbol{x})\, \boldsymbol{u}_j, -\boldsymbol{\sigma}_j \right) \quad \text{and} \quad \boldsymbol{G}_j = \left(\boldsymbol{u}_j', \nabla_x \boldsymbol{u}_j \right).$$

We claim that the measure associated with this new pair is also ν. To prove this, consider the sequence $\boldsymbol{w}_j = \boldsymbol{u}_j - \boldsymbol{v}_j$. It is easy to see that $\boldsymbol{w}_j$ solves the non-homogeneous system

$$\begin{cases} \boldsymbol{w}_j'' - \nabla_x \cdot \boldsymbol{\sigma}\left(\boldsymbol{w}_j\right) + a(\boldsymbol{x})\mathcal{X}_{\omega_j}(\boldsymbol{x})\boldsymbol{w}_j' = \\ \qquad \nabla_{(t,x)} \cdot \left(\boldsymbol{H}_j^1 - \left(\boldsymbol{v}_j' + a(\boldsymbol{x})\mathcal{X}_{\omega_j}(\boldsymbol{x})\boldsymbol{v}_j\right),\ \overline{\boldsymbol{H}}_j + \boldsymbol{\sigma}\left(\boldsymbol{v}_j\right)\right) & \text{in} \quad (0,T)\times\Omega, \\ \boldsymbol{w}_j = \boldsymbol{0} & \text{on} \quad (0,T)\times\Gamma_0, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0} & \text{on} \quad (0,T)\times\Gamma_1, \\ \boldsymbol{w}_j(0,\cdot) = \boldsymbol{0}, \quad \boldsymbol{w}_j'(0,\cdot) = \boldsymbol{0} & \text{in} \quad \Omega. \end{cases}$$

Moreover, from Equation (15), it follows that

$$\boldsymbol{w}_j \rightharpoonup 0 \quad \text{weakly in } \left(H^1\left((0,T)\times\Omega\right)\right)^N. \tag{16}$$

On the other hand, since $\boldsymbol{u}_j$ is a solution of Equation (4),

$$-\nabla_x \cdot \boldsymbol{\sigma}\left(\boldsymbol{w}_j\right) = -\boldsymbol{u}_j'' - a\left(\boldsymbol{x}\right)\mathcal{X}_{\omega_j}\left(\boldsymbol{x}\right)\boldsymbol{u}_j' + \nabla_x \cdot \boldsymbol{\sigma}\left(\boldsymbol{v}_j\right).$$

Multiplying this equation by $\boldsymbol{w}_j$ and integrating by parts, as $\boldsymbol{w}_j$ satisfies zero initial and boundary conditions, we obtain the following:

$$\begin{aligned} &\int_0^T\int_\Omega a_{klmn}\varepsilon_{mn}\left(\boldsymbol{w}_j\right)\varepsilon_{kl}\left(\boldsymbol{w}_j\right)\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T\int_\Omega \boldsymbol{u}_j''\boldsymbol{w}_j\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega a\left(\boldsymbol{x}\right)\mathcal{X}_{\omega_j}\left(\boldsymbol{x}\right)\boldsymbol{u}_j'\boldsymbol{w}_j\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T\int_\Omega a_{klmn}\varepsilon_{mn}\left(\boldsymbol{v}_j\right)\varepsilon_{kl}\left(\boldsymbol{w}_j\right)\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

From the assumptions on the initial data, it can be proved (see [1,14]) that

$$\boldsymbol{u}_j' \in L^\infty\left((0,T);V_0\right), \quad \boldsymbol{u}_j'' \in L^\infty\left((0,T);(L^2(\Omega))^N\right)$$

and both sequences are uniformly bounded with respect to j. This, together with the weak convergence (Equation 16), implies that the two first terms in the right-hand side above converge to zero. Moreover, since the measure associated with $\boldsymbol{R}_j = \nabla_{(t,x)}\boldsymbol{v}_j$ is a delta, $\varepsilon_{mn}\left(\boldsymbol{v}_j\right)$ is strongly convergent in L^2. Therefore, the third term also converges to zero. Finally, by using the coercivity condition (Equation 3) and the second Korn inequality [5, p. 192], we get the convergence

$$\int_0^T\int_\Omega \left\|\nabla_x\boldsymbol{w}_j\right\|^2\mathrm{d}x\,\mathrm{d}t \to 0. \tag{17}$$

As for derivatives with respect to time, from the identity

$$\left|\boldsymbol{w}_j'\right|^2 = \left(\boldsymbol{w}_j'\cdot\boldsymbol{w}_j\right)' - \boldsymbol{w}_j''\cdot\boldsymbol{w}_j$$

and by using similar arguments,

$$\int_0^T\int_\Omega \left|\boldsymbol{w}_j'\right|^2\mathrm{d}x\,\mathrm{d}t \to 0. \tag{18}$$

From Equations (17) and (18) it follows (see [19, p. 101]) that the sequences $\boldsymbol{G}_j$ and $\boldsymbol{R}_j$ share the same associated measure. Moreover, since $\|\boldsymbol{R}_j\|^2$ is equi-integrable, so is $\|\boldsymbol{G}_j\|^2$.

The conclusion is then a consequence of the fact that the projection onto the second Cartesian factor of the measure in Equation (14) is a delta since both Dirac masses have the same second Cartesian factor. It is well known (see [19, Prop. 6.12, p. 11]) that this fact, together with the equiintegrability, implies the strong convergence claimed. Notice that this second Cartesian factor corresponds to the gradient variable (including time). □

Let us now prove that weak-⋆ convergence in $L^{\infty}(\Omega)$ of a sequence of minimizing sequences for problem (P) implies strong convergence of the associated displacement fields.

Theorem 2.1. *Suppose that the initial condition $(\boldsymbol{u_0}, \boldsymbol{u_1})$ have the regularity (Equation 9). Suppose that $\mathcal{X}_{\omega_j}$ is a minimizing sequence for the optimization problem (P) and let $\boldsymbol{u}_j$ be the associated sequence of displacement fields. If*

$$\mathcal{X}_{\omega_j} \rightharpoonup s \quad \textit{weak-}\star \ \textit{in } L^{\infty}(\Omega),$$

then

$$\boldsymbol{u}_j \to \boldsymbol{u} \quad \textit{strong in } \left(H^1((0,T)\times\Omega)\right)^N,$$

where $\boldsymbol{u}$ is the corresponding solution of system (Equation 7).

Proof. Let $\mathcal{X}_{\omega_j}$ be a minimizing sequence for problem (P). We start by rewriting the system of PDE's in Equation (4) as

$$\nabla_{(t,x)} \cdot \left(\boldsymbol{u}_j' + a(\boldsymbol{x})\,\mathcal{X}_{\omega_j}(\boldsymbol{x})\,\boldsymbol{u}_j, -\boldsymbol{\sigma}\right) = 0,$$

where $\boldsymbol{u}_j$ is the sequence of associated displacements, $\left(\boldsymbol{u}_j' + a(\boldsymbol{x})\,\mathcal{X}_{\omega_j}(\boldsymbol{x})\,\boldsymbol{u}_j, -\boldsymbol{\sigma}\right)$ is a sequence of matrices of order $N \times (N+1)$ which have the term $\boldsymbol{u}_j' + a(\boldsymbol{x})\,\mathcal{X}_{\omega_j}(\boldsymbol{x})\,\boldsymbol{u}_j$ in the first column, and the divergence operator $\nabla_{(t,x)}\cdot$ now includes the time variable (as the first variable) too.

We introduce the two sequences of vector fields

$$\boldsymbol{F}_j = \left(\boldsymbol{u}_j' + a(\boldsymbol{x})\,\mathcal{X}_{\omega_j}(\boldsymbol{x})\,\boldsymbol{u}_j, -\boldsymbol{\sigma}_j\right) \quad \text{and} \quad \boldsymbol{G}_j = \left(\boldsymbol{u}_j', \nabla_x \boldsymbol{u}_j\right),$$

so that the pair $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$ is div-curl-free ($\nabla_{(t,x)} \cdot \boldsymbol{F}_j = 0$ and curl $\boldsymbol{G}_j = 0$). As in the static case (see [2,5]), it can be proved that the vector fields $\boldsymbol{F}_j$ and $\boldsymbol{G}_j$ are uniformly bounded in L^2, and therefore we may associate with this pair (rather with a subsequence of the pair) a div-curl Young measure $\nu = \left\{\nu_{(t,x)}\right\}_{(t,\boldsymbol{x})\in(0,T)\times\Omega}$.

Since $\mathcal{X}_{\omega_j}(\boldsymbol{x})$ only takes on two values, ν is supported in the union of the two manifolds Λ_0 and $\Lambda_{1,C}$ introduced in Lemma 2.3. In this case, the vector $C \in \mathbb{R}^N$ plays the role of $a(\boldsymbol{x})\boldsymbol{u}$ and as far as derivatives are concerned is like a constant.

Let $\nu^{(2)}$ designate the projection of ν onto the second copy of $\mathcal{M}^{N\times(N+1)}$ and let $S = S(t, \boldsymbol{x})$ be the matrix in $\mathcal{M}^{N\times N}$ given by

$$S(t, \boldsymbol{x}) = \int_{\mathcal{M}^{N\times(N+1)}} \left[\operatorname{diag}(B_1)\operatorname{diag}(B_1) + \boldsymbol{\sigma}\left(\overline{B}\right)\overline{B}^T\right] d\nu^{(2)}_{(t,x)}(B)$$

where diag(B_1) stands for a $N \times N$ diagonal matrix with the vector B_1 in the principal diagonal. Note also that B_1 plays the role of $\boldsymbol{u}'$ and $\overline{B} \equiv \nabla_x \boldsymbol{u}$.

Due to the symmetry properties (Equation 3) and the basic property of Young measures (the weak lower semicontinuity [19], Th. 6.11, p. 110), it is elementary to check that

$$\lim_{j\to\infty} J\left(\mathcal{X}_{\omega_j}\right) \geqq \frac{1}{2}\int_0^T\int_\Omega \operatorname{tr} S(t, \boldsymbol{x})\, dx\, dt.$$

Since the trace is a linear operator,

$$\operatorname{tr}(S) = \int_{\mathcal{M}^{N\times(N+1)}} \left[|B_1|^2 + \boldsymbol{\sigma}\left(\overline{B}\right) : \boldsymbol{\varepsilon}\left(\overline{B}\right)\right] d\nu^{(2)}(B),$$

where $\boldsymbol{\varepsilon}\left(\overline{B}\right) = \frac{1}{2}\left(\overline{B} + \overline{B}^T\right)$. By Jensen's inequality,

$$\operatorname{tr}(S) \geqq |M_1|^2 + \boldsymbol{\sigma}\left(\overline{M}\right) : \boldsymbol{\varepsilon}\left(\overline{M}\right) \tag{19}$$

with M the first moment of $\nu^{(2)}$.

If we can show that the right-hand side of (Equation 19) can be obtained by a div-curl laminate, then that value will be optimal and the underlying state law will be encoded in such a laminate. In addition, we could apply Lemma 2.3. But, this is elementary to check. It suffices to take into account that due to the strict convexity this minimum value can only be achieved when

$$\nu^{(2)} = \delta_M \tag{20}$$

and hence

$$\nu = s\delta_{((M_1+C,\, -\boldsymbol{\sigma}(\overline{M})),M)} + (1-s)\,\delta_{((M_1,\, -\boldsymbol{\sigma}(\overline{M})),M)}, \tag{21}$$

with $0 \leqq s \leqq 1$, is our desired div-curl laminate. Note that the difference of the two pair of matrices is $((C, 0), 0)$ which satisfies (Equation 12).

Once we know that the minimum value $|M_1|^2 + \boldsymbol{\sigma}\left(\overline{M}\right) : \boldsymbol{\varepsilon}\left(\overline{M}\right)$ is eligible, it becomes the cost for the relaxed problem (RP), and the optimal ν is given as the convex combination of these two deltas, one in each manifold. Conclude now by Lemma 2.3.

We have thus shown that given an arbitrary subsequence of the full sequence $\mathcal{X}_{\omega_j}$, we can always find a further subsequence so that we have the claimed strong convergence in $\left(H^1((0, T) \times \Omega)\right)^N$ for the corresponding solutions. Because of uniqueness of the limit, the full sequence will converge strongly. This completes the proof. □

2.3. Proof of Theorem 1.1

We now have all the necessary ingredients to prove Theorem 1.1. The proof is standard in non-convex optimal control problems (see for instance [19,20]) and is essentially contained in the proof of Theorem 2.1, but is included here for completeness to stress how this approach can be used to deal with more general cost functionals.

Proof of Theorem 1.1. To begin with, we put the original problem into the setting of calculus of variations. So, for $(\boldsymbol{u}, A, B) \in \mathbb{R}^N \times \mathcal{M}^{N\times(N+1)} \times \mathcal{M}^{N\times(N+1)}$, we introduce the functions

$$W(\boldsymbol{u}, A, B) = \begin{cases} |B_1|^2 + \boldsymbol{\sigma}\left(\overline{B}\right) : \boldsymbol{\varepsilon}\left(\overline{B}\right) & \text{if } (A, B) \in \Lambda_0 \cup \Lambda_{1,\boldsymbol{au}} \\ +\infty & \text{else} \end{cases}$$

and

$$V(\boldsymbol{u}, A, B) = \begin{cases} 1 & \text{if } (A, B) \in \Lambda_{1,\boldsymbol{au}} \\ 0 & \text{if } (A, B) \in \Lambda_0 \backslash \Lambda_{1,\boldsymbol{au}} \\ +\infty & \text{else.} \end{cases}$$

Then it is not hard to check that the original problem is equivalent to

$$\text{Minimize in } (\boldsymbol{u}, \boldsymbol{F}) : \frac{1}{2}\int_0^T\int_\Omega W\left(\boldsymbol{u}(t,\boldsymbol{x}), \boldsymbol{F}(t,\boldsymbol{x}), \nabla_{(t,x)}\boldsymbol{u}(t,\boldsymbol{x})\right) \mathrm{d}x\,\mathrm{d}t$$

subject to

$$\boldsymbol{F} \in L^2\left((0,T)\times\Omega; \mathcal{M}^{N\times(N+1)}\right), \quad \nabla_{(t,x)} \cdot \boldsymbol{F} = 0,$$

$\boldsymbol{u} \in H^1\left((0,T)\times\Omega; \mathbb{R}^N\right)$ satisfies the same initial and boundary conditions as in system (Equation 4), and

$$\int_\Omega V\left(\boldsymbol{u}(t,\boldsymbol{x}), \boldsymbol{F}(t,\boldsymbol{x}), \nabla_{(t,\boldsymbol{x})}\boldsymbol{u}(t,\boldsymbol{x})\right) \mathrm{d}x\,\mathrm{d}t = L|\Omega| \quad \text{for all } t \geqq 0.$$

Relaxation for non-convex functionals like our case is based on the computation of the constrained quasi-convexification of W. For fixed $(\boldsymbol{u}, A, B, s)$, it is defined as

$$CQW(\boldsymbol{u}, A, B, s) = \min_\nu \int_{\mathcal{M}^{N\times(N+1)}\times\mathcal{M}^{N\times(N+1)}} W(\boldsymbol{u}, R, S)\,\mathrm{d}\nu(R,S)$$

where ν is a div-curl Young measure of the form

$$\nu = s\nu_1 + (1-s)\nu_0,$$

with $0 \leqq s \leqq 1$ and

$$\operatorname{supp}\nu_0 \subset \Lambda_0, \ \operatorname{supp}\nu_1 \subset \Lambda_{1,\boldsymbol{au}} \quad \text{and} \quad (A, B) \text{ the first moment of } \nu.$$

The relaxation of the original problem is then given by

$$\text{Minimize in } (s, \boldsymbol{u}, \boldsymbol{F}) : \frac{1}{2}\int_0^T\int_\Omega CQW\left(\boldsymbol{u}(t,\boldsymbol{x}), \boldsymbol{F}(t,\boldsymbol{x}), \nabla_{(t,x)}\boldsymbol{u}(t,\boldsymbol{x})\right)\mathrm{d}x\,\mathrm{d}t$$

subject to

$$\begin{cases} 0 \leqq s(\boldsymbol{x}) \leqq 1, & \int_\Omega s(\boldsymbol{x})\,\mathrm{d}x = L\,|\Omega| \\ (\boldsymbol{F}, \boldsymbol{u}) \text{ are as before.} & \end{cases}$$

These computations have been carried out in the proof of Theorem 2.1 where we have found that the optimal measure furnishing the value of $CQW(\boldsymbol{u}, A, B, s)$ is a first-order laminate of the form (Equation 14). From this, we have also deduced that the relaxed problem in terms of measures

$$\text{Minimize in } \nu \text{ the cost function } J(\nu)$$

where

$$J(\nu) = \frac{1}{2}\int_0^T\int_\Omega\int_{\mathcal{M}^{N\times(N+1)}}\left[|B_1|^2 + \boldsymbol{\sigma}\left(\overline{B}\right) : \boldsymbol{\varepsilon}\left(\overline{B}\right)\right]\mathrm{d}\nu_{(t,x)}^{(2)}(B)\,\mathrm{d}x\,\mathrm{d}t$$

ν being of the form (Equation 14), is equivalent to (RP). In particular,

$$\inf_\nu J(\nu) = \inf_s J(s).$$

Now let $\mathcal{X}_{\omega_j}$ be a minimizing sequence for (P). Then, as is well-known [19, Th. 6.11, p. 110],

$$\lim_{j\to\infty} J\left(\mathcal{X}_{\omega_j}\right) \geqq J(\nu), \tag{22}$$

where ν is the measure associated with $\mathcal{X}_{\omega_j}$ as in Theorem 2.1. This proves that

$$\inf_\nu J(\nu) \leqq \inf_{\mathcal{X}_\omega} J(\mathcal{X}_\omega).$$

Conversely, if ν is an admissible measure of the form (Equation 14), then by Lemma 2.3 there exists an admissible pair $\left(\boldsymbol{F}_j, \boldsymbol{G}_j\right)$ associated with some sequence of admissible characteristic functions $\mathcal{X}_{\omega_j}$ such that $\left\|\boldsymbol{G}_j\right\|^2$ is equi-integrable. Thanks to this equi-integrability, we have

$$\lim_{j\to\infty} J\left(\mathcal{X}_{\omega_j}\right) = J(\nu),$$

and this implies that

$$\inf_\nu J(\nu) \geqq \inf_{\mathcal{X}_\omega} J(\mathcal{X}_\omega).$$

This proves (ii). For the proof of (i), take a minimizing sequence $\mathcal{X}_{\omega_j}$ for (P) and let ν be its associated Young measure. Again by Lemma 2.3, we may find another sequence, say $\mathcal{X}_{\overline{\omega}_j}$, such that its associated $\left\| \nabla_{(t,x)} \boldsymbol{u}_j \right\|^2$ is equi-integrable. Hence,

$$\lim_{j\to\infty} J\left(\mathcal{X}_{\overline{\omega}_j}\right) = J(\nu),$$

but since $\mathcal{X}_{\omega_j}$ is minimizing,

$$\lim_{j\to\infty} J\left(\mathcal{X}_{\omega_j}\right) \leqq \lim_{j\to\infty} J\left(\mathcal{X}_{\overline{\omega}_j}\right) = J(\nu).$$

From Equation (22) it follows that

$$\lim_{j\to\infty} J\left(\mathcal{X}_{\omega_j}\right) = J(\nu).$$

This proves that ν is a minimizer for the problem in measures; therefore its associated function s is a minimizer for (RP).

Finally, note that the optimal measure associated with an optimal s of the relaxed problem is a first-order laminate whose projection on the second Cartesian factor is a delta in $\nabla_{(t,x)}\boldsymbol{u}$. This implies that the normal to this optimal laminates is independent of time and can take any direction in space.

Note that when we talk about normals, because our div-curl Young measures also incorporate time as a variable, we mean normals in space-time. It turns out that, as a result of optimization according to our computations above, such normals are indeed "plain" normals in space (having a vanishing time component) whose direction (in space) is not restricted in any way because the projection on space of the two-mass points for optimal measures is the same, and hence the difference vanishes. This implies no restriction on normals. This has also been emphasized before. □

Remark 1. It is also important to note that given an optimal relaxed design s, if $\mathcal{X}_{\omega_j}$ is an admissible sequence of characteristics functions such that:

(i) $\mathcal{X}_{\omega_j} \rightharpoonup s$ weak-$\star$ in $L^\infty(\Omega)$, and,
(ii) the associated $\left\| \nabla_{(t,x)} \boldsymbol{u}_j \right\|^2$ are equi-integrable,

then $\mathcal{X}_{\omega_j}$ is a minimizing sequence for (P). This is again a consequence of the fact that the optimal measure projects a delta located at $\nabla_{(t,x)}\boldsymbol{u}$ on its second Cartesian factor. The condition on the equi-integrability of $||\nabla_{(t,x)}\boldsymbol{u_j}||^2$ may be obtained by assuming enough regularity on the initial data.

Remark 2. We also point out that the result we have obtained may be extended to a non-homogeneous system of the type:

$$\begin{cases} \boldsymbol{u}'' - \nabla_x \cdot \boldsymbol{\sigma} + a(\boldsymbol{x})\,\mathcal{X}_\omega(\boldsymbol{x})\,\boldsymbol{u}' = \boldsymbol{f} & \text{in } (0,T)\times\Omega, \\ \boldsymbol{u} = 0 & \text{on } (0,T)\times\Gamma_0, \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = 0 & \text{on } (0,T)\times\Gamma_1, \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u_0}, \quad \boldsymbol{u}'(0,\cdot) = \boldsymbol{u_1} & \text{in } \Omega, \end{cases} \tag{23}$$

for any $\boldsymbol{f} \in L^\infty((0,T); L^2(\Omega)^N)$.

3. Numerical analysis of problems (P) and (RP)

In this second part, we address the problem of computing numerically the optimal density for (RP). Based on this optimal relaxed density, we propose a penalization technique to recover quasi-optimal classical designs for (P). We first describe an algorithm of minimization and then present some numerical experiments.

3.1. Algorithm of minimization

We briefly discuss the resolution of the relaxed problem (RP) using a gradient descent method. In this respect, we compute the first variation of the cost function J with respect to s. For any $\eta \in \mathbb{R}^+$, $\eta << 1$, and any $s_1 \in L^\infty(\Omega)$, we associate to the perturbation $s^\eta = s + \eta s_1$ of s the derivative of J with respect to s in the direction s_1 as follows:

$$\frac{\partial J(s)}{\partial s} \cdot s_1 = \lim_{\eta \to 0} \frac{J(s + \eta s_1) - J(s)}{\eta}. \tag{24}$$

Following the proof of [18] in the similar context of the wave equation, we obtain the following result.

Theorem 3.1. *If* $(\boldsymbol{u_0}, \boldsymbol{u_1}) \in \left(\left(H^2(\Omega)\right)^N \cap V_0 \right) \times V_0$*, then the derivative of* J *with respect to* s *in any direction* s_1 *exists and takes the form*

$$\frac{\partial J(s)}{\partial s} \cdot s_1 = \int_\Omega a(\boldsymbol{x}) s_1(\boldsymbol{x}) \int_0^T \boldsymbol{u}'(t, \boldsymbol{x}) \cdot \boldsymbol{p}(t, x) \, \mathrm{d}t \, \mathrm{d}x \tag{25}$$

where $\boldsymbol{u}$ *is the solution of Equation* (7) *and* $\boldsymbol{p}$ *is the solution in* $C([0, T]; (H_0^1(\Omega))^N) \cap C^1([0, T]; (L^2(\Omega))^N)$ *of the adjoint problem*

$$\begin{cases} \boldsymbol{p}'' - \nabla_x \cdot \boldsymbol{\sigma}(\boldsymbol{p}) - a(\boldsymbol{x}) s(\boldsymbol{x}) \boldsymbol{p}' = \boldsymbol{u}'' + \nabla_x \cdot \boldsymbol{\sigma}(\boldsymbol{u}), & \textit{in} \ \ (0, T) \times \Omega, \\ \boldsymbol{p} = 0, & \textit{on} \ \ (0, T) \times \Gamma_0, \\ \boldsymbol{p} \cdot \boldsymbol{n} = 0, & \textit{on} \ \ (0, T) \times \Gamma_1, \\ \boldsymbol{p}(T, \cdot) = 0, \quad \boldsymbol{p}'(T, \cdot) = \boldsymbol{u}'(T, \cdot) & \textit{in} \ \ \Omega. \end{cases} \tag{26}$$

Notice that the integral (Equation 25) is well defined, that is $\boldsymbol{u}' \cdot \boldsymbol{p} \in C([0, T], L^1(\Omega))$ since from the regularity assumed on $(\boldsymbol{u_0}, \boldsymbol{u_1})$, we have $\boldsymbol{u}'' + \nabla_x \cdot \boldsymbol{\sigma}(\boldsymbol{u}) \in C([0, T]; (L^2(\Omega))^N)$ and hence $\boldsymbol{p} \in C([0, T]; (L^2(\Omega))^N)$.

In order to take into account the volume constraint on s, we introduce the Lagrange multiplier $\gamma \in \mathbb{R}$ and the functional

$$J_\gamma(s) = J(s) + \gamma ||s||_{L^1(\Omega)}. \tag{27}$$

By using Theorem 3.1, we obtain that the derivative of J_γ is

$$\frac{\partial J_\gamma(s)}{\partial s} \cdot s_1 = \int_\Omega s_1(\boldsymbol{x}) \left(a(\boldsymbol{x}) \int_0^T \boldsymbol{u}'(t, \boldsymbol{x}) \cdot \boldsymbol{p}(t, \boldsymbol{x}) \mathrm{d}t + \gamma \right) \mathrm{d}x \tag{28}$$

which permits to define the following descent direction:

$$s_1(\boldsymbol{x}) = -\left(a(\boldsymbol{x})\int_0^T \boldsymbol{u}'(t,\boldsymbol{x})\cdot\boldsymbol{p}(t,\boldsymbol{x})\mathrm{d}t + \gamma\right), \quad \forall \boldsymbol{x}\in\Omega. \tag{29}$$

Consequently, for any function $\eta \in L^\infty(\Omega, \mathbb{R}^+)$ with $||\eta||_{L^\infty(\Omega)}$ small enough, we have $J_\gamma(s+\eta s_1) \leqq J_\gamma(s)$. The multiplier γ is then determined so that, for any function $\eta \in L^\infty(\Omega,\mathbb{R}^+)$ and $\eta \neq 0$, $||s+\eta s_1||_{L^1(\Omega)} = L|\Omega|$ leading to

$$\gamma = \frac{(\int_\Omega s(\boldsymbol{x})\mathrm{d}x - L|\Omega|) - \int_\Omega \eta(\boldsymbol{x})a(\boldsymbol{x})\int_0^T \boldsymbol{u}'(t,\boldsymbol{x})\cdot\boldsymbol{p}(t,\boldsymbol{x})\,\mathrm{d}t\mathrm{d}x}{\int_\Omega \eta(\boldsymbol{x})\mathrm{d}x}. \tag{30}$$

At last, the function η is chosen so that $s(\boldsymbol{x}) + \eta(\boldsymbol{x})s_1(\boldsymbol{x}) \in [0,1]$, for all $\boldsymbol{x}\in\Omega$. A simple and efficient choice consists in taking $\eta(\boldsymbol{x}) = \epsilon s(\boldsymbol{x})(1-s(\boldsymbol{x}))$ for all $\boldsymbol{x}\in\Omega$ with ϵ a small real positive.

Consequently, the descent algorithm to solve numerically the relaxed problem (RP) may be structured as follows: let $\Omega\subset\mathbb{R}^N$, $(\boldsymbol{u_0},\boldsymbol{u_1}) \in ((H^2(\Omega))^N\cap V_0)\times V_0$, $L\in(0,1)$, $T>0$, and $\epsilon<1$, $\epsilon_1 << 1$ be given:

- Initialization of the density function $s^0 \in L^\infty(\Omega;]0,1[)$;
- For $k \geqq 0$, iteration until convergence (that is, $|J(s^{k+1}) - J(s^k)| \leqq \epsilon_1|J(s^0)|$) as follows:
  - Computation of the solution $\boldsymbol{u}_{s^k}$ of Equation (7) and then the solution $\boldsymbol{p}_{s^k}$ of Equation (26), both corresponding to $s = s^k$.
  - Computation of the descent direction s_1^k defined by Equation (29) where the multiplier γ^k is defined by Equation (30).
  - Update the density function in Ω:

$$s^{k+1} = s^k + \epsilon s^k(1-s^k)s_1^k \tag{31}$$

    with $\varepsilon\in\mathbb{R}^+$ small enough in order to ensure the decrease of the cost function and $s^{k+1}\in L^\infty(\Omega,[0,1])$.

3.2. Numerical experiments

In this section, we present some numerical simulations for $N=2$ and the unit square $\Omega = (0,1)^2$. Moreover, for simplicity, we consider the case $\Gamma_0 = \partial\Omega$ and assume that Ω is composed of an isotropic homogeneous material for which

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).$$

$\lambda>0$ and $\mu>0$ are the Lamé coefficients and δ designates the Kronecker symbol. The stress tensor becomes simply: $\boldsymbol{\sigma}(\boldsymbol{u}) = \lambda tr(\nabla_x\cdot\boldsymbol{u})\boldsymbol{I_{N\times N}} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$.

Systems (Equation 7) and (Equation 26) are solved in space using a C^0-finite element method with mass lumping (we refer to [6,16]). Precisely, introducing a triangulation $\mathcal{T}_h$ of Ω ($h = \max_{T\in\mathcal{T}_h}|T|$), we approximate $L^2(\Omega)$ and $H^1(\Omega)$ by the following finite-dimensional spaces $V_h = \{v_h | v_h \in C^0(\overline{\Omega}), v_{h|T}\in\mathbb{P}_1\ \forall T\in\mathcal{T}_h\}$ where $\mathbb{P}_1$ designates the space of the polynomials of degree $\leqq 1$. The time discretization is performed in a standard way using centered finite differences of order two. At last, without loss of generality, we consider a constant damping function $a(\boldsymbol{x}) = a\mathcal{X}_\Omega(\boldsymbol{x})$ in Ω since the dependence in $\boldsymbol{x}$ is contained in the density s.

In the sequel, we treat the following simple conditions in $((H^2(\Omega))^N \cap V_0) \times V_0$, sufficient to illustrate the complexity of the problem:

$$\boldsymbol{u_0} = (\sin(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\sin(\pi x_2)), \quad \boldsymbol{u_1} = (0, 0). \tag{32}$$

Results are obtained with $h = 10^{-2}$, $\epsilon_1 = 10^{-5}$, $L = 10^{-1}$, $T = 1$, $s^0(\boldsymbol{x}) = L$ on Ω and $\epsilon = 10^{-2}$ (see the algorithm).

We highlight that the gradient algorithm may lead to local minima of J. For this reason, we consider constant initial density s^0 as indicated above which permit to privilege no location for ω.

3.2.1. Influence of the damping constant value a Similarly to the wave equation case considered in [17,18], numerical simulations exhibit a bifurcation phenomenon with respect to the value of the damping constant a. When this value is small enough, say $a < a^\star(\Omega, L, \lambda, \mu, \boldsymbol{u_0}, \boldsymbol{u_1})$, depending on the data, the optimal density is always a characteristic function, which suggests that the original problem (P) is well-posed. On the other hand, when the critical value $a^\star$ is reached, it appears that the optimal density takes values strictly in (0, 1). This suggests that (P) is no more well-posed and fully justifies the introduction of the relaxed problem (RP). For $(\lambda, \mu) = (1/2, 1)$, Fig. 1 depicts the iso-values of the optimal density s_{opt}—obtained at the convergence of the algorithm—for several values of a: for $a = 5$, $s_{\text{opt}} \in \{0, 1\}$, while for example for $a = 10$, $s_{\text{opt}} \in [0, 1]$.

The well-posedness when a is small may be explained as follows: from (Equation 25), one may write that (we introduce the notation $\overline{J}(s, a) = J(s(a))$)

$$\overline{J}(s + \eta s_1, a) = \overline{J}(s, a) + \eta \int_\Omega a(\boldsymbol{x}) s_1(\boldsymbol{x}) \int_0^T \boldsymbol{u_{(s)}}' \cdot \boldsymbol{p_{(s)}}\, \mathrm{d}t\, \mathrm{d}x + O(\eta^2 a) \tag{33}$$

such that for the conservative case $s = 0$ and $a(\boldsymbol{x}) = a\mathcal{X}_\Omega(\boldsymbol{x})$,

$$\begin{aligned}\overline{J}(\eta s_1, a) &= \overline{J}(0, a) + \eta a \int_\Omega s_1(\boldsymbol{x}) \int_0^T \boldsymbol{u_{t(0)}} \cdot \boldsymbol{p_{(0)}}\, \mathrm{d}t\, \mathrm{d}x + O(\eta^2 a)\\ &= \overline{J}(s_1, 0) + \eta a \int_\Omega s_1(\boldsymbol{x}) \int_0^T \boldsymbol{u_{(0)}}' \cdot \boldsymbol{p_{(0)}}\, \mathrm{d}t\, \mathrm{d}x + O(\eta^2 a)\end{aligned} \tag{34}$$

where $\boldsymbol{u_{(0)}}$, $\boldsymbol{p_{(0)}}$ are the solutions of Equations (7) and (26) in the conservative case. Then, writing that $J(\eta s_1, a) = J(s_1, \eta a)$, one obtains

$$\overline{J}(s_1, \eta a) = \overline{J}(s_1, 0) + \eta a \int_\Omega s_1(\boldsymbol{x}) \int_0^T \boldsymbol{u_{(0)}}' \cdot \boldsymbol{p_{(0)}}\, \mathrm{d}t\, \mathrm{d}x + O(\eta^2 a). \tag{35}$$

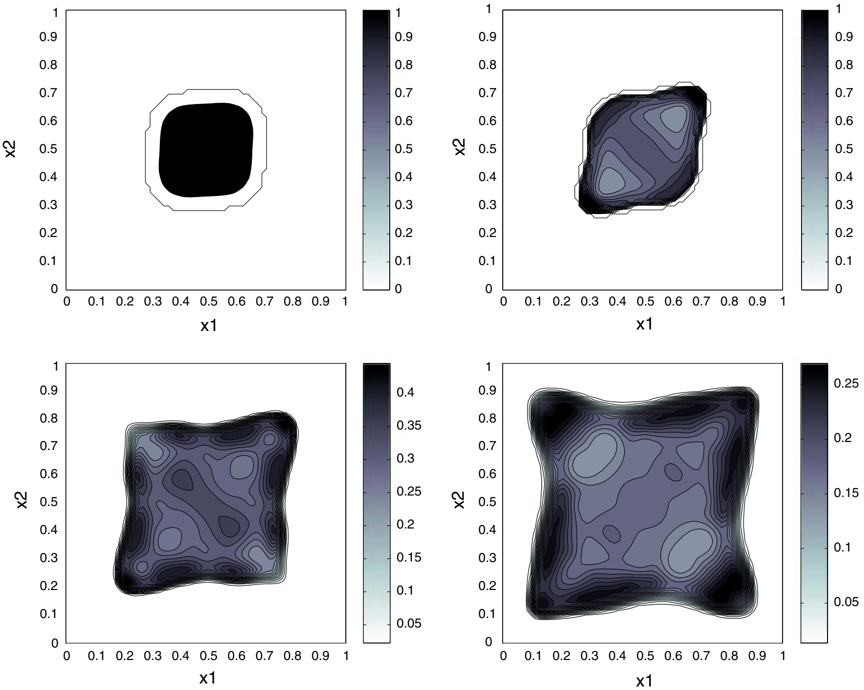

Fig. 1. $T = 1$, $(\lambda, \mu) = (1/2, 1)$, $a(\boldsymbol{x}) = a\mathcal{X}_\Omega(\boldsymbol{x})$—iso-value of the optimal density $s_{\rm opt}$ on Ω for $a = 5$ (*top left*), $a = 10$ (*top right*), $a = 25$ (*bottom left*) and $a = 50$ (*bottom right*)

For η small, the last term may be neglected. Therefore, the optimal density associated with the damping coefficient ηa (small) is related to the minima in Ω of the negative function $\boldsymbol{x} \to \int_0^T \boldsymbol{u}_{(\mathbf{0})}' \cdot \boldsymbol{p}_{(\mathbf{0})} \mathrm{d}t$. For the initial condition (Equation 32), this function is strictly convex (see Fig. 2) and the minimum is reaches at point $(1/2, 1/2)$. We conclude that the optimal distribution s_1 which minimizes the second term in (Equation 35) is a characteristic function centered on $(1/2, 1/2)$. The conclusion is the same if a remains small: for $a = 5$, Fig. 1 top left depicts the iso-value of the density $s_{\rm opt}$ for $a = 5$ (Fig. 3).

On the other hand, when a is large enough, the last term in (Equation 35) can not be neglected. In this case, the ill-posedness is related to the over-damping phenomenon: *when* $\min_\omega a(\boldsymbol{x})$ *goes to infinity, the damping term* $a(\boldsymbol{x})\mathcal{X}_\omega$ *acts as penalization term and enforces the solution* $\boldsymbol{u}$ *to be constant in time in* ω*: at the limit, there is no more dissipation in* ω *(and so in* Ω*) and the energy is constant*. In order to avoid this phenomenon (which thus appears if $as(\boldsymbol{x})$ is too large), the density s must take (at least locally) values lower than 1 in order to compensate a: consequently, (P) can not be well-posed in this case. This is illustrated on Fig. 1 for $a = 10$, $a = 25$ and $a = 50$. Table 1 gives the corresponding values of the energy. Remark that the functions $a \to J(s_{\rm opt}(a))$, $a \to E(T)/E(0)$ are decreasing since the subset

$$V_a = \left\{ a(\boldsymbol{x}) = as(\boldsymbol{x}), 0 \leqq s(\boldsymbol{x}) \leqq 1, \int_\Omega s(\boldsymbol{x}) \mathrm{d}x = L|\Omega| \right\} \tag{36}$$

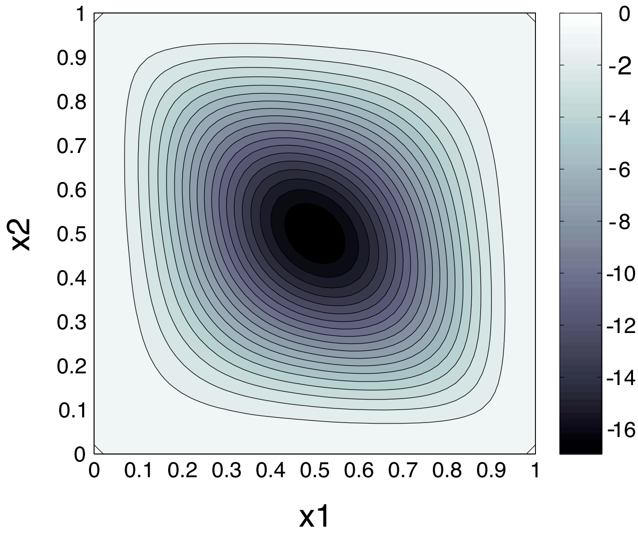

Fig. 2. $T = 1$, $(\lambda, \mu) = (1/2, 1)$, $a(\boldsymbol{x}) = 0\mathcal{X}_{\Omega}(\boldsymbol{x})$—iso-values of $\boldsymbol{x} \to \int_0^T \boldsymbol{u_{(0)}}' \cdot \boldsymbol{p_{(0)}} \mathrm{d}t$

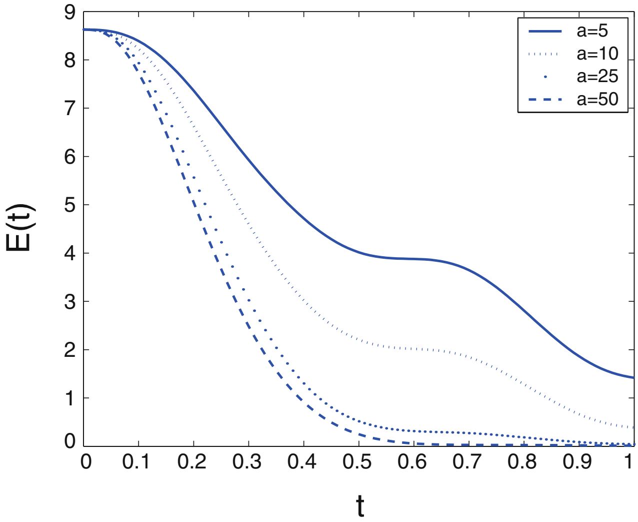

Fig. 3. $T = 1$, $(\lambda, \mu) = (1/2, 1)$, $a(\boldsymbol{x}) = a\mathcal{X}_{\Omega}(\boldsymbol{x})$—energy $E(t)$ versus $t \in [0, 1]$ for several values of a

of admissible damping functions is increasing with a. Without any upper bound on a, one may dissipate totally the system in finite time as proved in [4] for the 1-D wave equation with $a(x) = x^{-1}\mathcal{X}_{\Omega}(x)$ and $\Omega = (0, 1)$.

We have also observed that for $a < a^\star$, the optimal density is independent of the initialization s^0, which suggests a unique minimum. For $a \geqq a^\star$, we may obtain several minima although their corresponding cost are very similar.

Table 1. $(\lambda, \mu) = (1/2, 1)$, $T = 1$—value of the cost function and energy ratio with respect to a: $J(s_{opt}) \approx O(a^{-1/2})$ and $E(T)/E(0) \approx O(a^{-2})$

	$a = 5$	$a = 10$	$a = 25$	$a = 50$
$J(s_{opt})$	4.7640	3.5004	2.3534	2.0883
$E(T)/E(0)$	1.644×10^{-1}	4.488×10^{-2}	4.933×10^{-3}	1.186×10^{-3}

Table 2. $\mu = 1$, $T = 1$—value of the energy ratio with respect to λ

	$\lambda = 0.5$	$\lambda = 2.5$	$\lambda = 5$	$\lambda = 25$	$\lambda = 50$
$E(T)/E(0)$	0.1644	0.2603	0.3249	0.3979	0.3991

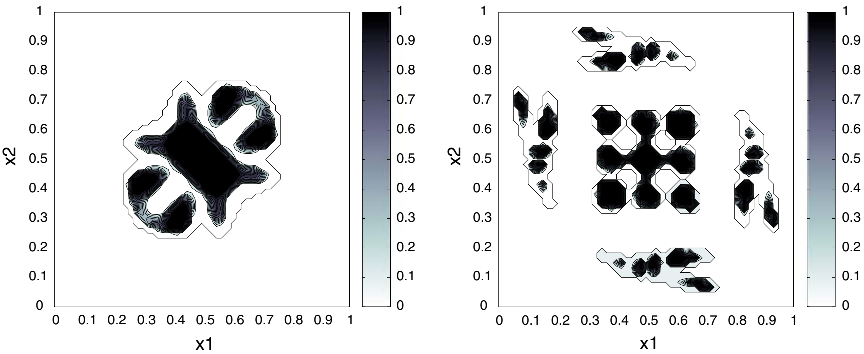

Fig. 4. $T = 1$, $\mu = 1$, $a(\boldsymbol{x}) = 5\mathcal{X}_\Omega(\boldsymbol{x})$—iso-value of the optimal density s_{opt} on Ω for $\lambda = 5$ (*left*) and $\lambda = 50$ (*right*)

3.2.2. Influence of the Lamé coefficients The nature of (P) depends strongly on the data of the problem: thus, for a fixed value of a, we obtain that (P) is well-posed as soon as L (or equivalently $|\Omega| - L$) is small enough (in other words, $a^\star$ is a decreasing function of L). We examine in this section the influence of the Lamé coefficient λ. We recall that when λ is arbitrarily large (or equivalently when the Poisson coefficient is near to $1/2$), we obtain the nearly incompressible situation. At the limit, the solution $\boldsymbol{u}$ fulfills the relation $div\ \boldsymbol{u}(t) = 0$ on Ω for all $t > 0$ (assuming $div \boldsymbol{u_0} = 0$) (see for instance [13, chapter 2]). The system (Equation 7) is then more constrained and the structure Ω more rigid.

For finite increasing values of λ, we observe on Table 2 that the ratio $E(T)/E(0)$ which quantifies the stabilization of the system increases. As an effect of this additional rigidity, large value of λ leads to a reduction of the stabilization. As shown on Fig. 4, the effect on the optimal position of the damping set is notable (to be compared with Fig. 1 top left). At last, if large values of λ change the dynamic of the system, it seems that the nature of (P) is unchanged: for the damping constant $a = 5$ considered here, optimal densities are characteristic functions.

The incompressible limit will be examined both theoretically and numerically in a future work.

3.2.3. Penalization of the optimal density In the case where the optimal density s_{opt} is not in $L^\infty((0, T) \times \Omega; \{0, 1\})$, one may associate to s_{opt} a characteristic

Table 3. $(\lambda, \mu) = (1/2, 1)$, $T = 1 - a = 50$—value of the cost function for the penalized characteristic density

N	1	2	3	4	5	6	7	16
$J(s^{\text{pen}}_{N,N})$	3.824	3.261	2.446	2.238	2.161	2.137	2.109	2.096

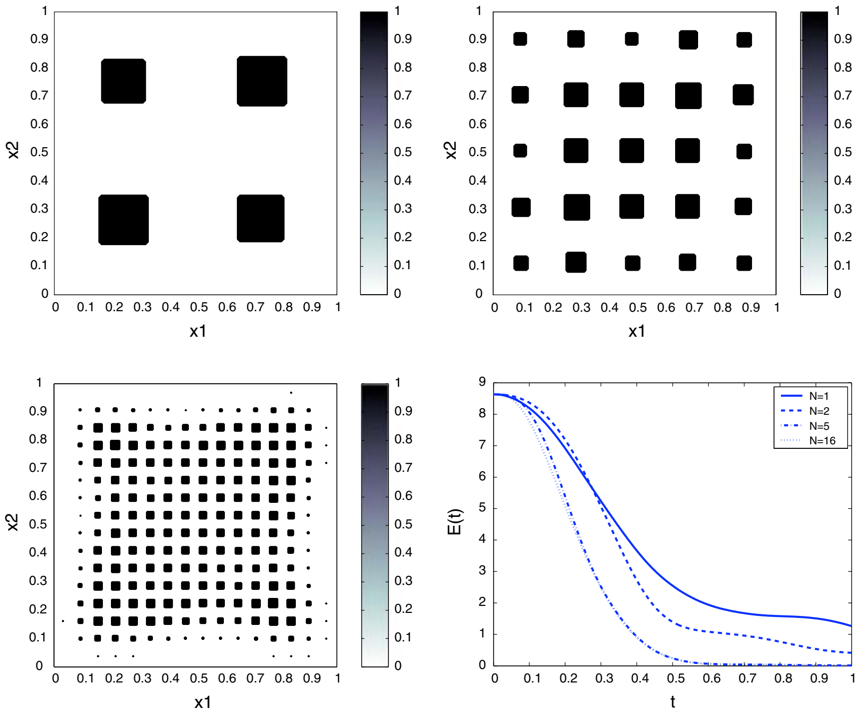

Fig. 5. $T = 1$, $(\lambda, \mu) = (1/2, 1)$, $a(\boldsymbol{x}) = 50\mathcal{X}_\Omega(\boldsymbol{x})$—iso-value of the penalized density for $N = 2$, $N = 5$ and $N = 16$—*bottom right*: $E(t)$ versus t associated with $s^{\text{pen}}_{N,N}$

function $s^{\text{pen}} \in L^\infty((0, T) \times \Omega; \{0, 1\})$ whose cost $J(s^{\text{pen}})$ is arbitrarily near to $J(s_{\text{opt}})$. Following [18], one may proceed as follows: we first decompose the domain $(0, 1) \times (0, 1)$ into $M \times N$ cells such that $\Omega = \cup_{i=1,M}[x_i, x_{i+1}] \times \cup_{j=1,N}[y_j, y_{j+1}]$ where $\{x_i\}_{(i=1,M+1)}$ and $\{y_j\}_{(j=1,N+1)}$ designate two uniform subdivisions of the interval $(0, 1)$. Then, we associate to each cell the mean value $m_{i,j} \in [0, 1]$ defined by

$$m_{i,j} = \frac{1}{(x_{i+1} - x_i)(y_{j+1} - y_j)} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} s_{\text{opt}}(x, y)\, dx\, dy \tag{37}$$

At last, we define the function $s^{\text{pen}}_{M,N}$ in $L^\infty(\Omega, \{0, 1\})$ by

$$s^{\text{pen}}_{M,N}(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathcal{X}_{[x_i, (1-\sqrt{m_{i,j}})x_i + \sqrt{m_{i,j}}x_{i+1}] \times [y_j, (1-\sqrt{m_{i,j}})y_j + \sqrt{m_{i,j}}y_{j+1}]}(x, y). \tag{38}$$

We easily check that $||s^{\text{pen}}_{M,N}||_{L^1(\Omega)} = ||s_{\text{opt}}||_{L^1(\Omega)}$, for all $M, N > 0$. Thus, the characteristic function $s^{\text{pen}}_{M,N}$ takes advantage of the information codified in the density s_{opt}, as discussed in Remark 1.

Let us illustrate this point with the optimal density obtained for $(\lambda, \mu) = (1/2, 1)$ and $a = 50$ (see Fig. 1 bottom right). The corresponding value of the cost function is $J(s_{\text{opt}}) \approx 2.0883$. Table 3 collects the value of $J(s^{\text{pen}}_{M,N})$ for several values of $M = N$ and suggests the convergence of $J(s^{\text{pen}}_{M,N})$ toward $J(s_{\text{opt}}) \approx 2.0883$ as N increases (Fig. 5).

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