A FUNCTIONAL CALCULUS FOR ALMOST SECTORIAL OPERATORS AND APPLICATIONS TO ABSTRACT EVOLUTION EQUATIONS

F. PERIAGO AND B. STRAUB

ABSTRACT. In this paper, we construct a functional calculus for linear operators A whose spectrum lies in a sector of the complex plane and whose resolvent satisfies $\|(z-A)^{-1}\| \leq M|z|^{\gamma}$ for some $-1<\gamma<0$ and all z outside the sector. By means of this functional calculus, we define complex powers and semigroups associated with A and some of its powers. Finally, these abstract results are applied to prove existence and uniqueness of classical solution for three different types of abstract differential equations.

1. INTRODUCTION

Sectorial operators, that is, linear operators A defined in Banach spaces, whose spectrum lies in a sector $S_\omega = \{z \in \mathbb{C}\setminus\{0\} \mid |\arg z| \leq \omega\} \cup \{0\}$ for some $0 \leq \omega < \pi$, and whose resolvent satisfies an estimate

$$(1) \qquad \|(z-A)^{-1}\| \leq M|z|^{-1} \qquad \text{for all } z \in \mathbb{C}\setminus S_\omega,$$

have been studied extensively during the last 40 years, both in abstract settings and for their applications to partial differential equations. From an abstract point of view, it is of particular interest to construct functional calculi for this class of operators and wide classes of holomorphic functions. Then complex powers of an operator A and semigroups associated with these powers are defined through the functional calculi. Finally, these abstract results can be applied to linear and non-linear partial differential equations, mainly in the parabolic case, that is, when $0 \leq \omega < \pi/2$ (see [1, 12, 20] for material regarding functional calculi, and [8, 10, 15, 17] for the applications).

Many important elliptic differential operators belong to the class of sectorial operators, especially when they are considered in the Lebesgue spaces or in spaces of continuous functions (see [3] and [10, Chapter 3]). However, if we look at spaces

2000 *Mathematics Subject Classification.* 47A60, 47D06, 34K30, 35K90.

Key words and phrases. Almost sectorial operators, functional calculus, fractional powers, semigroups, abstract evolution equations.

The first author was partially supported by a grant of the Technical University of Cartagena (Spain) under the 2000 mobility program (PFMP-UPCT-2000). The second author gratefully acknowledges the support of the Australian Research Council.

of more regular functions such as the spaces of Hölder continuous functions, we find that these elliptic operators do no longer satisfy the estimate (1) and therefore are not sectorial, as was pointed out by W. von Wahl in the case of the Laplacian (see [21] or [10, Example 3.1.33]).

Nevertheless, for these operators estimates such as

$$\|(z-A)^{-1}\| \leq M\,|z|^{\gamma} \qquad \text{for all } z \in \mathbb{C} \setminus S_\omega \tag{2}$$

and some $-1 < \gamma < 0$, can be obtained.

Our first aim in this paper is to construct a functional calculus for operators A defined in a Banach space, whose spectrum lies in a sector S_ω and whose resolvent satisfies (2). This is done in Section 2, where some examples of elliptic differential operators which satisfy these conditions are also included. Our approach is based on the ideas underlying the McIntosh functional calculus for sectorial operators (see [12] for the Hilbert space case and [1, 7] for the construction in a Banach space). We emphasise that no assumption on the density of the domain of the base operator A is made in our construction. An alternative functional calculus, based on C-semigroups, can be found in [4].

The class of holomorphic functions that we consider is large enough to include complex powers and exponentials. Then, by means of the functional calculus, in Section 3 we define complex powers of A for all exponents $\alpha \in \mathbb{C}$, as well as the semigroups associated with these powers. In particular, we prove that if A satisfies (2) for some $0 < \omega < \pi/2$, then the semigroup associated with the fractional power A^α for $0 < \alpha < \frac{\pi}{2\omega}$, is analytic and of growth order $\frac{\gamma+1}{\alpha}$. The latter essentially means that the semigroup has a singularity of type $\mathrm{O}(t^{-\frac{\gamma+1}{\alpha}})$ as t goes to zero.

Section 4 is devoted to applications to abstract differential equations. We consider the inhomogeneous linear and the semilinear first order abstract Cauchy problem associated with an operator A satisfying (2), as well as an incomplete second order abstract Cauchy problem. Under suitable regularity conditions for the initial data, existence and uniqueness of classical solutions is proved.

Acknowledgements. The authors thank G. Dore for providing his unpublished paper [7]. The first author gratefully acknowledges the hospitality of the School of Mathematics at UNSW, Sydney, where this work was carried out during the winter of the Olympic year 2000.

2. An Extension of the McIntosh Functional Calculus

Throughout this paper, $(X, \|\cdot\|)$ denotes a complex Banach space. By an operator in X, we mean a linear mapping $A : D(A) \subseteq X \to X$ whose domain $D(A)$ is a linear subspace of X. As usual, $R(A)$ stands for the range of A, $\sigma(A)$ is the spectrum of A and $\rho(A)$ the resolvent set of A. Finally, we denote by $\mathcal{L}(X)$ the space of all bounded linear operators on X.

For $0 < \mu < \pi$, let S_μ^0 be the open sector

$$\{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \mu\}$$

and S_μ be its closure, that is,

$$S_\mu = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| \le \mu\} \cup \{0\}.$$

Note that we use the argument function arg with values in $]-\pi, \pi]$.

Definition 2.1. Let $-1 < \gamma < 0$ and $0 \le \omega < \pi$. By $\Theta_\omega^\gamma(X)$ we denote the set of all closed linear operators $A : D(A) \subseteq X \to X$ which satisfy

(i) $\sigma(A) \subseteq S_\omega$, and
(ii) for every $\omega < \mu < \pi$ there exists a constant $C_\mu > 0$ such that

$$\|(z - A)^{-1}\| \le C_\mu |z|^\gamma \qquad \text{for all } z \notin S_\mu. \tag{3}$$

For simplicity of notation, we write Θ_ω^γ instead of $\Theta_\omega^\gamma(X)$ when there is no possibility for confusion.

Remark 2.2. Observe that every operator $A \in \Theta_\omega^\gamma$ satisfies $0 \in \rho(A)$. In particular, A is injective. To see this, let $\omega < \mu < \pi$. Then $]-\infty, 0[\not\subseteq S_\mu$, and by (3), the power series of the resolvent $(z - A)^{-1}$ about $z_0 < 0$,

$$(z - A)^{-1} = \sum_{k=0}^{\infty} (z_0 - z)^k \, (z_0 - A)^{-(k+1)},$$

converges in the open disk $D_{z_0} = \{z \in \mathbb{C} \mid |z - z_0| < C_\mu^{-1} |z_0|^{-\gamma}\}$. Taking $z_0 = -2^{-1} C_\mu^{-\frac{1}{1+\gamma}}$, it follows that $|z_0| < C_\mu^{-1} |z_0|^{-\gamma}$. Hence $0 \in D_{z_0} \subseteq \rho(A)$.

Next, we give some examples of operators which belong to Θ_ω^γ for some γ and ω, and are not sectorial.

Example 2.3. Let $0 < \alpha < 1$, $m \in \mathbb{N}$, and Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^{4m}. In the Banach space $X = (C^\alpha(\overline{\Omega}), \|\cdot\|_\alpha)$ of Hölder continuous functions, consider the differential operator A given by

$$Au(x) = \sum_{|\beta| \le 2m} a_\beta(x) D^\beta u(x) \qquad \text{for all } x \in \overline{\Omega},$$

with domain $D(A) = \{u \in C^{2m+\alpha}(\overline{\Omega}) \mid D^\beta u|_{\partial\Omega} = 0 \text{ for all } |\beta| \le m - 1\}$. Here, β is a multiindex in $(\mathbb{N} \cup \{0\})^n$, $|\beta| = \sum_{j=1}^n \beta_j$ and $D^\beta = \Pi_{j=1}^n (\frac{1}{i} \frac{\partial}{\partial x_j})^{\beta_j}$. If the coefficients $a_\beta : \overline{\Omega} \to \mathbb{C}$ of A satisfy the conditions

(i) $a_\beta \in C^\alpha(\bar{\Omega})$ for all $|\beta| \le 2m$,
(ii) $a_\beta(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2m$, and
(iii) there exists a constant $M > 0$ such that

$$M^{-1} |\xi|^{2m} \le \sum_{|\beta| = 2m} a_\beta(x) \xi^\beta \le M |\xi|^{2m} \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and } x \in \overline{\Omega},$$

then there exist $\lambda, \varepsilon > 0$ such that the operator $-A + \lambda$ belongs to the class Θ_ω^γ with $\omega = \frac{\pi}{2} - \varepsilon$ and $\gamma = \frac{\alpha}{2m} - 1$. For the details, we refer to [21, Satz 1 and Satz 2]. Moreover, as shown in [21, Bemerkung 2], the exponent $\gamma = \frac{\alpha}{2m} - 1$ is sharp. In particular, the operator $-A+\lambda$ is not sectorial. Note that A is not densely defined since $D(A) \subseteq \{u \in C^\alpha(\overline{\Omega}) \mid u|_{\partial\Omega} = 0\}$.

Example 2.4. In the Banach space $X = (C_\infty(\mathbb{R}; \mathbb{C}^2), \|\cdot\|)$ of continuous functions which vanish at infinity, endowed with the norm

$$\|f\| \;=\; \|f_1\|_\infty + \|f_2\|_\infty \qquad \text{for all } f = (f_1, f_2) \in X,$$

consider the matrix multiplication operator

$$A_q : D(A_q) \subseteq X \;\to\; X, \quad f \;\mapsto\; A_q f \;=\; qf$$

with maximal domain $D(A_q) = \{f \in X \mid qf \in X\}$, where q is the matrix-valued function given by

$$q(x) \;=\; \begin{bmatrix} 1 + x^2 + i(1+x^2) & x^{4+2\gamma} \\ 0 & 1 + x^2 - i(1+x^2) \end{bmatrix},$$

with $-1 < \gamma < 0$. It is not difficult to see that A_q is closed and densely defined, and that $\sigma(A_q) = \{x \pm ix \mid x \geq 1\}$. Moreover, for every $\frac{\pi}{4} < \mu < \pi$, there is a constant $C = C(\mu, \gamma) > 0$ such that

$$\|(z - A_q)^{-1}\| \;\leq\; C|z|^\gamma \qquad \text{for all } z \notin S_\mu,$$

that is, $A_q \in \Theta_{\pi/4}^\gamma$.

Other examples of operators in the class Θ_ω^γ can be found, for example, in [9, Chapter 1].

We now introduce some special functions and classes of functions. Given $0 < \mu < \pi$, we set

$$H(S_\mu^0) \;=\; \{f : S_\mu^0 \to \mathbb{C} \mid f \text{ is holomorphic}\}$$

and

$$H^\infty(S_\mu^0) \;=\; \{f \in H(S_\mu^0) \mid f \text{ is bounded}\}.$$

On $\mathbb{C} \setminus \{-1\}$ we define the functions φ_0 and ψ_k for $k \in \mathbb{N}$, by

$$\varphi_0(z) \;=\; \frac{1}{1+z} \qquad \text{and} \qquad \psi_k(z) \;=\; \frac{z}{(1+z)^k}$$

and put

$$\Psi_0(S_\mu^0) \;=\; \{f \in H(S_\mu^0) \mid \sup_{z \in S_\mu^0} |\varphi_0^{-1}(z)\, f(z)| < \infty\}.$$

Note that for all $z \in S_\mu^0$, the estimate

$$|1+z| \;=\; \left|e^{-i\frac{\arg z}{2}}(1+z)\right| \;\geq\; \left|\cos\frac{\arg z}{2} + |z|\cos\frac{\arg z}{2}\right| \;\geq\; \cos\frac{\mu}{2}\,(1+|z|)$$

holds. Hence the space $\Psi_0(S^0_\mu)$ consists of all functions $f \in H(S^0_\mu)$ for which there exists a constant $K_\mu > 0$ such that

$$|f(z)| \;\le\; K_\mu\,(1+|z|)^{-1} \qquad \text{for all } z \in S^0_\mu.$$

For every $s < 0$, we put

$$\Psi^\gamma_s(S^0_\mu) \;=\; \bigl\{ f \in H(S^0_\mu) \;\big|\; \sup_{z\in S^0_\mu} |\psi^s_m(z)\, f(z)| < \infty \bigr\},$$

where m is the smallest integer such that $m \ge 2$ and $\gamma + 1 < -(m-1)s$. A function $f \in H(S^0_\mu)$ belongs to $\Psi^\gamma_s(S^0_\mu)$ if and only if there exists a constant $K_\mu > 0$ such that

$$|f(z)| \;\le\; K_\mu |z|^{-s}\,(1+|z|)^{ms} \qquad \text{for all } z \in S^0_\mu,$$

that is, f is $\mathrm{O}(|z|^{-s})$ as $|z| \to 0$ and $\mathrm{O}(|z|^{(m-1)s})$ as $|z| \to \infty$.

Finally, we set

$$\mathcal{F}^\gamma_0(S^0_\mu) \;=\; \bigcup_{s<0} \Psi^\gamma_s(S^0_\mu) \;\cup\; \Psi_0(S^0_\mu)$$

and

$$\mathcal{F}(S^0_\mu) \;=\; \bigl\{ f \in H(S^0_\mu) \;\big|\; \text{there exist } k, n \in \mathbb{N} \text{ such that } f\psi^k_n \in \mathcal{F}^\gamma_0(S^0_\mu) \bigr\}.$$

It is not difficult to show that if f belongs to $\mathcal{F}(S^0_\mu)$, then there are constants $r, K_\mu > 0$ such that

$$|f(z)| \;\le\; K_\mu\,(|z|^{-r} + |z|^r) \qquad \text{for all } z \in S^0_\mu. \tag{4}$$

Conversely, if $f \in H(S^0_\mu)$ satisfies the estimate (4), then, taking $k \in \mathbb{N}$ with $k > r$, it follows that $f\psi^k_m \in \Psi^\gamma_s(S^0_\mu)$ for all $r - k \le s < 0$ and $m \in \mathbb{N}$ such that $m \ge 2$ and $\gamma + 1 < -(m-1)s$.

The classes of functions introduced above satisfy the inclusions

$$\mathcal{F}^\gamma_0(S^0_\mu) \;\subseteq\; H^\infty(S^0_\mu) \;\subseteq\; \mathcal{F}(S^0_\mu) \;\subseteq\; H(S^0_\mu).$$

The aim of this section is to construct a functional calculus for operators in the class Θ^γ_ω and functions in $\mathcal{F}(S^0_\mu)$. We first develop a functional calculus for the class $\mathcal{F}^\gamma_0(S^0_\mu)$ and then extend it to $\mathcal{F}(S^0_\mu)$.

Throughout the paper, Γ_θ with $0 < \theta < \pi$, will denote the path

$$\bigl\{ re^{-i\theta} \;\big|\; r > 0 \bigr\} \;\cup\; \bigl\{ re^{i\theta} \;\big|\; r > 0 \bigr\},$$

oriented such that S^0_θ lies to the left of Γ_θ.

Theorem 2.5. *Let A be an operator in the class Θ^γ_ω and $\omega < \theta < \mu < \pi$. Then for every $f \in \mathcal{F}^\gamma_0(S^0_\mu)$, the integral*

$$f(A) \;=\; \frac{1}{2\pi i} \int_{\Gamma_\theta} f(z)\,(z-A)^{-1}\,dz \tag{5}$$

is absolutely convergent and defines a bounded linear operator on X. Moreover, its value does not depend on the choice of θ for $\omega < \theta < \mu$.

Proof. Let $\omega < \tilde{\mu} < \theta$. Then for all $z \in \Gamma_\theta$ we have

$$\|f(z)\,(z-A)^{-1}\| \;\le\; \begin{cases} C_{\tilde{\mu}} K_\mu\, |z|^{\gamma-s}(1+|z|)^{ms} & \text{if } f \in \Psi_s^\gamma(S_\mu^0) \text{ for some } s<0,\\ C_{\tilde{\mu}} K_\mu\, |z|^{\gamma}(1+|z|)^{-1} & \text{if } f \in \Psi_0(S_\mu^0). \end{cases}$$

Hence (5) is absolutely convergent and defines a bounded linear operator on X.

Now let $\omega < \tilde{\mu} < \theta_1 < \theta_2 < \mu$. For $0<\delta<1$ consider the path

$$\Delta_\delta \;=\; \{re^{i\theta_1} \mid \delta \le r \le \delta^{-1}\} \,\cup\, \{\delta^{-1}e^{i\phi} \mid \theta_1 \le \phi \le \theta_2\}$$
$$\cup\; \{re^{i\theta_2} \mid \delta \le r \le \delta^{-1}\} \,\cup\, \{\delta e^{i\phi} \mid \theta_1 \le \phi \le \theta_2\}$$

oriented counter-clockwise. Since the function $f(z)\,(z-A)^{-1}$ is analytic in $\mathbb{C}\setminus S_\omega$, Cauchy's Theorem yields

$$\int_{\Delta_\delta} f(z)\,(z-A)^{-1}\,dz \;=\; 0. \tag{6}$$

Taking limits in (6) as $\delta \to 0$, we conclude that

$$\int_0^\infty f(re^{i\theta_1})\,(re^{i\theta_1}-A)^{-1}e^{i\theta_1}\,dr \;=\; \int_0^\infty f(re^{i\theta_2})\,(re^{i\theta_2}-A)^{-1}e^{i\theta_2}\,dr$$

since the integrals along the paths $\{\delta e^{i\phi} \mid \theta_1 \le \phi \le \theta_2\}$ and $\{\delta^{-1}e^{i\phi} \mid \theta_1 \le \phi \le \theta_2\}$ vanish as $\delta \to 0$. An analogous reasoning shows that

$$\int_0^\infty f(re^{-i\theta_1})(re^{-i\theta_1}-A)^{-1}e^{-i\theta_1}dr \;=\; \int_0^\infty f(re^{-i\theta_2})(re^{-i\theta_2}-A)^{-1}e^{-i\theta_2}dr. \qquad \square$$

Remark 2.6. Note that as A and A^{-1} are closed, $f(A)$ commutes with these operators on their respective domains.

Theorem 2.7 (Product Formula)**.** *Let $A \in \Theta_\omega^\gamma$ and $\omega < \mu < \pi$. Then for all $f, g \in \mathcal{F}_0^\gamma(S_\mu^0)$, we have $fg \in \mathcal{F}_0^\gamma(S_\mu^0)$ and*

$$(fg)(A) \;=\; f(A)\,g(A).$$

Proof. Since $|(fg)(z)| \le \|f\|_\infty\,|g(z)|$ for all $z \in S_\mu^0$, it is clear that $fg \in \mathcal{F}_0^\gamma(S_\mu^0)$. Take $\omega < \theta_1 < \theta_2 < \mu$. With the resolvent equation $(z-A)^{-1} - (w-A)^{-1} = (w-z)(z-A)^{-1}(w-A)^{-1}$, we obtain

$$\begin{aligned} -4\pi^2 f(A)\,g(A) \;&=\; \int_{\Gamma_{\theta_1}}\int_{\Gamma_{\theta_2}} f(z)\,g(w)\,(z-A)^{-1}(w-A)^{-1}\,dw\,dz \\ &=\; \int_{\Gamma_{\theta_1}} f(z)\int_{\Gamma_{\theta_2}} \frac{g(w)}{w-z}\,dw\;(z-A)^{-1}\,dz \\ &\quad+\; \int_{\Gamma_{\theta_2}} g(w)\int_{\Gamma_{\theta_1}} \frac{f(z)}{z-w}\,dz\;(w-A)^{-1}\,dw. \end{aligned}$$

By Cauchy's Theorem, $\int_{\Gamma_{\theta_1}} \frac{f(z)}{z-w}\,dz = 0$ and $\int_{\Gamma_{\theta_2}} \frac{g(w)}{w-z}\,dw = 2\pi i\, g(z)$, which completes the proof. $\square$

Proposition 2.8. *Let $A \in \Theta_\omega^\gamma$, and $k, n \in \mathbb{N} \cup \{0\}$ with $k > n$. Then for every $\omega < \mu < \pi$, the function $\varphi_{k,n}$ given by*

$$\varphi_{k,n}(z) = \frac{z^n}{(1+z)^k} \qquad \text{for all } z \in \mathbb{C} \setminus \{-1\},$$

belongs to $\mathcal{F}_0^\gamma(S_\mu^0)$ and

$$\varphi_{k,n}(A) = A^n (1+A)^{-k}.$$

In particular, the operator $\varphi_{k,n}(A)$ is injective.

Proof. The fact that $\varphi_{k,n} \in \mathcal{F}_0^\gamma(S_\mu^0)$ is evident. For $0 < \delta < 1$, consider the path Δ_δ composed of

$$\begin{aligned} &\{re^{-i\theta} \mid \delta \le r \le \delta^{-1}\} \cup \{\delta^{-1} e^{i\phi} \mid \theta \le \phi \le 2\pi - \theta\} \\ &\quad \cup \{re^{i\theta} \mid \delta \le r \le \delta^{-1}\} \cup \{\delta e^{i\phi} \mid \theta \le \phi \le 2\pi - \theta\} \end{aligned}$$

oriented clockwise. By the Residues Theorem, $\frac{1}{2\pi i} \int_{\Delta_\delta} \varphi_{k,n}(z)\,(z-A)^{-1}\,dz$ equals the negative of the residue of $\varphi_{k,n}(z)\,(z-A)^{-1}$ in $z = -1$. Moreover,

$$\lim_{\delta \to 0} \int_{\Delta_\delta} \varphi_{k,n}(z)\,(z-A)^{-1}\,dz = \int_{\Gamma_\theta} \varphi_{k,n}(z)\,(z-A)^{-1}\,dz$$

since the integrals along $\{\delta^{-1} e^{i\phi} \mid \theta \le \phi \le 2\pi - \theta\}$ and $\{\delta e^{i\phi} \mid \theta \le \phi \le 2\pi - \theta\}$ vanish as $\delta \to 0$. As $\varphi_{k,n}(z)\,(z-A)^{-1}$ has a pole of order k at $z = -1$,

$$\operatorname{Res}\big(\varphi_{k,n}(z)\,(z-A)^{-1}, z = -1\big) = \frac{f_n^{(k-1)}(-1)}{(k-1)!},$$

where $f_n(z) = z^n\,(z-A)^{-1}$. A simple calculation shows that

$$\frac{f_n^{(k-1)}(z)}{(k-1)!} = (-1)^{k-1} \left(\sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{n-j}\,(z-A)^{-(n-j)} \right) (z-A)^{-(k-n)}$$

and hence

$$\operatorname{Res}\big(\varphi_{k,n}(z)\,(z-A)^{-1}, z = -1\big) = -A^n (1+A)^{-k}.$$

The injectivity of $\varphi_{k,n}(A)$ follows from the fact that A is injective. □

Our goal is to construct a functional calculus for functions in the class $\mathcal{F}(S_\mu^0)$. Recall that if $f \in \mathcal{F}(S_\mu^0)$, then there exist $k, m \in \mathbb{N}$ such that $f\psi_m^k \in \mathcal{F}_0^\gamma(S_\mu^0)$. This together with Proposition 2.8 enables us to give the following definition.

Definition 2.9. Let $A \in \Theta_\omega^\gamma$, $\omega < \mu < \pi$, and $f \in \mathcal{F}(S_\mu^0)$. Take $k, m \in \mathbb{N}$ such that $f\psi_m^k \in \mathcal{F}_0^\gamma(S_\mu^0)$. We define the linear operator $f(A)$ on the domain

$$D(f(A)) = \{x \in X \mid (f\psi_m^k)(A)x \in D(A^{(m-1)k})\}$$

by

$$f(A) = (\psi_m^k(A))^{-1}(f\psi_m^k)(A),$$

where the operator $(f\psi_m^k)(A)$ is given by (5).

Note that the operator $f(A)$ is closed since it is the product of the closed operator $(\psi_m^k(A))^{-1}$ with the bounded operator $(f\psi_m^k)(A)$. We prove next that the above definition of $f(A)$ does not depend on the choice of k and m.

Proposition 2.10. *Let A be an operator in the class Θ_ω^γ and $\omega < \mu < \pi$. Let $f \in \mathcal{F}(S_\mu^0)$ and $j, k, m, n \in \mathbb{N}$ be such that $f\psi_m^j$, $f\psi_n^k \in \mathcal{F}_0^\gamma(S_\mu^0)$. Then for any $x \in X$,*

$$(f\psi_m^j)(A)x \in D[(\psi_m^j(A))^{-1}] \quad \text{if and only if} \quad (f\psi_n^k)(A)x \in D[(\psi_n^k(A))^{-1}],$$

and in this case

$$(\psi_m^j(A))^{-1}(f\psi_m^j)(A)x = (\psi_n^k(A))^{-1}(f\psi_n^k)(A)x.$$

Consequently, Definition 2.9 does not depend on k and m.

Proof. If $m \neq n$, say $m > n$, then the function $\psi_m^k\psi_n^{-k}$ belongs to $\mathcal{F}_0^\gamma(S_\mu^0)$ and consequently so does $f\psi_m^k = (\psi_m^k\psi_n^{-k})\,(f\psi_n^k)$. By Theorem 2.7,

$$(f\psi_m^k)(A) = (\psi_m^k\psi_n^{-k})(A)\,(f\psi_n^k)(A).$$

Let $x \in X$. With Proposition 2.8 we obtain that

$$(f\psi_n^k)(A)x \in D[(\psi_n^k(A))^{-1}] = D[A^{(n-1)k}]$$

if and only if

$$(f\psi_m^k)(A)x \in (1+A)^{-(m-n)k}D[A^{(n-1)k}] = D[(\psi_m^k(A))^{-1}],$$

and then

$$\begin{aligned}(\psi_m^k(A))^{-1}(f\psi_m^k)(A)x &= (1+A)^m A^{-k}\,(1+A)^{-(m-n)k}(f\psi_n^k)(A)x \\ &= (\psi_n^k(A))^{-1}(f\psi_n^k)(A)x.\end{aligned}$$

Now suppose $j > k$. As $f\psi_m^j$, $f\psi_m^k$ and ψ_m^{j-k} belong to $\mathcal{F}_0^\gamma(S_\mu^0)$, Theorem 2.7 and Proposition 2.8 yield

$$(f\psi_m^j)(A) = \psi_m^{j-k}(A)\,(f\psi_m^k)(A) = A^{j-k}(1+A)^{-m(j-k)}\,(f\psi_m^k)(A).$$

Hence for $x \in X$, $(f\psi_m^j)(A)x \in D[(\psi_m^j(A))^{-1}]$ if and only if $(f\psi_m^k)(A)x$ belongs to $D[(\psi_m^k(A))^{-1}]$, and in this case

$$\begin{aligned}(\psi_m^j(A))^{-1}(f\psi_m^j)(A)x &= (\psi_m^j(A))^{-1}\psi_m^{j-k}(A)(f\psi_m^k)(A)x \\ &= (\psi_m^k(A))^{-1}(f\psi_m^k)(A)x.\end{aligned}$$

□

Theorem 2.11 (Product Formula). *Let $A \in \Theta_\omega^\gamma$ and $\omega < \mu < \pi$. Then for all $f, g \in \mathcal{F}(S_\mu^0)$ the following hold.*

(i) $f(A)\, g(A) \subseteq (fg)(A)$.
(ii) *If $D[(fg)(A)] \subseteq D[g(A)]$, then* $f(A)\, g(A) = (fg)(A)$.
(iii) *If $g(A)$ is bounded, then* $f(A)\, g(A) = (fg)(A)$.

Proof. Let $j, k, m \in \mathbb{N}$ such that $g\psi_m^j$, $f\psi_m^k \in \mathcal{F}_0^\gamma(S_\mu^0)$.

To see that (i) holds, let $x \in D[f(A)\, g(A)]$. By Remark 2.6, Theorem 2.7 and Proposition 2.8, we have

$$\begin{aligned} f(A)\, g(A)x &= \psi_m^k(A)^{-1}(f\psi_m^k)(A)\, \psi_m^j(A)^{-1}(g\psi_m^j)(A)x \\ &= \psi_m^{k+j}(A)^{-1}(fg\psi_m^{k+j})(A)x \\ &= (fg)(A)x. \end{aligned}$$

Now assume that $D[(fg)(A)] \subseteq D[g(A)]$ and take $x \in D[(fg)(A)]$. The same reasoning as above yields

$$\begin{aligned} (\psi_m^{k+j} fg)(A)x &= (f\psi_m^k)(A)\, (g\psi_m^j)(A)x \\ &= (f\psi_m^k)(A)\psi_m^j(A)\psi_m^j(A)^{-1}(g\psi_m^j)(A)x \\ &= \psi_m^j(A)(f\psi_m^k)(A)\, g(A)x. \end{aligned}$$

Hence $g(A)x \in D[f(A)]$ and $(fg)(A)x = f(A)\, g(A)x$. This proves (ii).

Part (iii) is an immediate consequence of (ii). □

Our next result is a first step towards spectral theory for operators $f(A)$ with $f \in \mathcal{F}(S_\mu^0)$.

Proposition 2.12. *Let $A \in \Theta_\omega^\gamma$ and $\omega < \mu < \pi$. Suppose $f \in \mathcal{F}(S_\mu^0)$ satisfies that $f(z) \neq 0$ for all $z \in S_\mu^0$. If $f^{-1} \in \mathcal{F}(S_\mu^0)$, then $f(A)$ is injective and $(f(A))^{-1} = f^{-1}(A)$.*

Proof. Let $j, k, m \in \mathbb{N}$ such that $f\psi_m^k$ and $f^{-1}\psi_m^j$ belong to $\mathcal{F}_0^\gamma(S_\mu^0)$. Take $x \in D[f(A)]$. By Proposition 2.8, Remark 2.6 and Theorem 2.7, it follows that

$$\begin{aligned} (f^{-1}\psi_m^j)(A)\, f(A)x &= (f^{-1}\psi_m^j)(A)\, (\psi_m^k(A))^{-1}(f\psi_m^k)(A)x \\ &= (\psi_m^k(A))^{-1}(f^{-1}f\psi_m^{j+k})(A)x \\ &= (\psi_m^k(A))^{-1}\psi_m^{k+j}(A)x \\ &= \psi_m^j(A)x. \end{aligned}$$

Hence $f(A)x \in D[f^{-1}(A)]$ and $f^{-1}(A)f(A)x = x$. □

The functional calculus on the class $\mathcal{F}(S_\mu^0)$ does not transform, in general, bounded functions into bounded operators. However, the following partial result will be very useful in the next section.

Lemma 2.13. *Let $A \in \Theta_\omega^\gamma$ and $\omega < \theta < \mu < \pi$. Suppose $f \in H^\infty(S_\mu^0)$ satisfies the following two conditions.*

(a) The function $z \mapsto f(z)\,(z-A)^{-1}$ is absolutely integrable on Γ_θ.

(b) $\sup\limits_{-\mu<\phi<\mu} |f(re^{i\phi})| \to 0$ as $r \to \infty$.

Then $f(A)$ is the bounded linear operator given by

$$f(A) \;=\; \frac{1}{2\pi i}\int_{\Gamma_\theta} f(z)\,(z-A)^{-1}\,dz.$$

Proof. Since $f\psi_2 \in \mathcal{F}_0^\gamma(S_\mu^0)$, we need to show that

$$\int_{\Gamma_\theta} f(z)\,A(1+A)^{-2}\,(z-A)^{-1}\,dz \;=\; \int_{\Gamma_\theta} \frac{z}{(1+z)^2}\,f(z)\,(z-A)^{-1}\,dz.$$

The resolvent equation $(z-A)^{-1}-(w-A)^{-1} = (w-z)(z-A)^{-1}(w-A)^{-1}$ yields

$$\begin{aligned}\int_{\Gamma_\theta} & f(z)\,A(1+A)^{-2}\,(z-A)^{-1}\,dz \\ &= \int_{\Gamma_\theta} \frac{f(z)}{1+z}\,A(1+A)^{-2}\,dz \;+\; \int_{\Gamma_\theta} \frac{f(z)}{(1+z)^2}\,A(1+A)^{-1}\,dz \\ &\quad - \int_{\Gamma_\theta} \frac{f(z)}{(1+z)^2}\,dz \;+\; \int_{\Gamma_\theta} \frac{zf(z)}{(1+z)^2}\,(z-A)^{-1}\,dz.\end{aligned}$$

By Cauchy's Theorem, the first three integrals on the right hand side vanish. This gives the claim. ☐

3. Complex Powers and Semigroups

In this section, we study the operators which, by means of the functional calculus developed in the preceding section, are associated with the functions $\phi_\alpha(z) = z^\alpha$ and $g_{\alpha,t}(z) = e^{-tz^\alpha}$. Throughout this section we assume that A is an operator in the class Θ_ω^γ. In particular, A is injective. However, we make no assumptions on the domain of A, that is, A may have non-dense domain.

It is worth pointing out that a satisfactory theory of complex powers for one-to-one sectorial operators with possibly non-dense domain and/or range, which is valid for all complex exponents, has been obtained only very recently (see [7, 11]).

Further, we note that if $A \in \Theta_\omega^\gamma$, then it is not hard to see that for every $\omega < \mu < \pi$, there is a constant $M_\mu > 0$ such that

$$\|(z-A)^{-1}\| \;\le\; M_\mu \qquad \text{for all } z \notin S_\mu.$$

Hence A belongs to the class of operators considered in [5, 19]. However, the theory of complex powers developed there requires that A is densely defined.

For every $\alpha \in \mathbb{C}$, the function ϕ_α given by

$$\phi_\alpha(z) \;=\; z^\alpha \qquad \text{for all } z \in \mathbb{C}\setminus\,]-\infty,0]\,,$$

belongs to $\mathcal{F}(S^0_\mu)$ for $0 < \mu < \pi$, since ϕ_α satisfies an estimate of the form (4). This enables us to give the following definition.

Definition 3.1. Let $\alpha \in \mathbb{C}$. We define the complex power A^α to be the operator

$$A^\alpha \;=\; \phi_\alpha(A).$$

As a straightforward consequence of the functional calculus developed in Section 2, we obtain the following properties of the operators A^α.

Theorem 3.2. *Let $A \in \Theta^\gamma_\omega$. Then for all $\alpha, \beta \in \mathbb{C}$ the following hold.*

(i) The operator A^α is closed.
(ii) $A^\alpha A^\beta \subseteq A^{\alpha+\beta}$. Moreover, if $D(A^{\alpha+\beta}) \subseteq D(A^\beta)$, then $A^\alpha A^\beta = A^{\alpha+\beta}$.
(iii) A^α is injective and $(A^\alpha)^{-1} = A^{-\alpha}$.
(iv) $A^n = \underbrace{A \cdots A}_{n\text{-times}}$ for all $n \in \mathbb{N}$ and $A^0 = I$.

Proposition 3.3. *Let $A \in \Theta^\gamma_\omega$ and $\omega < \theta < \pi$. Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, and $n \in \mathbb{N}$ such that $n > \operatorname{Re}\alpha$. Then*

$$A^\alpha \;=\; (1+A)^{n+1}\, \frac{1}{2\pi i} \int_{\Gamma_\theta} \frac{z^\alpha}{(1+z)^{n+1}}\, (z-A)^{-1}\, dz. \tag{7}$$

Consequently, $D(A^{n+1}) \subseteq D(A^\alpha)$.

Proof. By Theorem 2.11, part (iii), and Theorem 2.5,

$$A^\alpha (1+A)^{-(n+1)} \;=\; \frac{1}{2\pi i} \int_{\Gamma_\theta} \frac{z^\alpha}{(1+z)^{n+1}}\, (z-A)^{-1}\, dz.$$

Let $x \in X$. By Theorem 2.11, part (iii), it follows that $x \in D(A^\alpha)$ if and only if

$$\int_{\Gamma_\theta} \frac{z^\alpha}{(1+z)^{n+1}}\, (z-A)^{-1} x\, dz \;\in\; D(A^{n+1}),$$

and in this case the equality (7) holds.

If $x \in D(A^{n+1})$, then the integral

$$\int_{\Gamma_\theta} \frac{z^\alpha}{(1+z)^{n+1}}\, (z-A)^{-1} (1+A)^{n+1} x\, dz$$

is absolutely convergent and, as $(1+A)^{n+1}$ is closed, $x \in D(A^\alpha)$ by the first part of the proof. □

In difference to the case of sectorial operators, having $0 \in \rho(A)$ does not imply that the complex powers $A^{-\alpha}$ with $\operatorname{Re}\alpha > 0$, are bounded. The operator $A^{-\alpha}$ belongs to $\mathcal{L}(X)$ whenever $\operatorname{Re}\alpha > 1 + \gamma$.

Proposition 3.4. *Let $A \in \Theta^\gamma_\omega$. Then for all $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 1+\gamma$, the complex power $A^{-\alpha}$ is bounded.*

Proof. Let $\omega < \theta < \mu < \pi$. Take $k, m \in \mathbb{N}$ such that $k > \operatorname{Re}\alpha$, $m \geq 2$ and $z^{-\alpha}\psi_m^k \in \mathcal{F}_0^\gamma(S_\mu^0)$. We have to show that for every $x \in X$,

$$\int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk}} (z-A)^{-1} x \, dz \ \in \ D(A^{(m-1)k}).$$

As $\int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk}} \, dz = 0$, we can write

$$\begin{aligned} \int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk}} (z-A)^{-1} x \, dz &= \int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk-1}} \frac{(z-A)^{-1} + (1+A)^{-1}}{1+z} x \, dz \\ &= \int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk-1}} (z-A)^{-1}(1+A)^{-1} x \, dz \\ &= (1+A)^{-1} \int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk-1}} (z-A)^{-1} x \, dz. \end{aligned}$$

Note that for $\operatorname{Re}\alpha > 1+\gamma$ the integral

$$\int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk-1}} (z-A)^{-1} x \, dz$$

is absolutely convergent. Repeating this argument $(mk-k-1)$-times yields

$$\int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{mk}} (z-A)^{-1} x \, dz \ = \ (1+A)^{-(m-1)k} \int_{\Gamma_\theta} \frac{z^{k-\alpha}}{(1+z)^{k}} (z-A)^{-1} x \, dz,$$

which completes the proof. □

Remark 3.5. The result of Proposition 3.4 is best possible, that is, the complex powers A^α with $\operatorname{Re}\alpha > -(1+\gamma)$, are in general not bounded. Consider, for example, the operator A_q as given by Example 2.4.

For $\alpha \in \mathbb{C}$, the complex power A_q^α of A_q coincides with the multiplication operator associated with the matrix-valued function

$$q_\alpha(x) \ = \ \begin{bmatrix} (1+x^2+i(1+x^2))^\alpha & \frac{x^{4+2\gamma}[(1+x^2+i(1+x^2))^\alpha - (1+x^2-i(1+x^2))^\alpha]}{2i(1+x^2)} \\ 0 & (1+x^2-i(1+x^2))^\alpha \end{bmatrix}.$$

The term in the upper right corner, and consequently the operator A_q^α, is unbounded if $\operatorname{Re}\alpha > -(1+\gamma)$.

It is natural to ask whether the fractional power A^α of an operator $A \in \Theta_\omega^\gamma$ belongs to a similar class, that is, if $A^\alpha \in \Theta_{\tilde{\omega}}^{\tilde{\gamma}}$ for some $-1 < \tilde{\gamma} < 0$ and $0 \leq \tilde{\omega} < \pi$. The answer is given next.

Proposition 3.6. *Let $A \in \Theta_\omega^\gamma$ and $1+\gamma < \alpha < \frac{\pi}{\omega}$. Then $A^\alpha \in \Theta_{\alpha\omega}^{-1+\frac{\gamma+1}{\alpha}}$.*

Proof. By Theorem 3.2, the linear operator A^α is closed and injective. It remains to show that $\sigma(A^\alpha) \subseteq S_{\alpha\omega}$ and that the resolvent of A^α satisfies the estimate (3).

Clearly, $\phi_\alpha(S_\nu) = S_{\alpha\nu}$ for all $0 < \nu < \min\{\pi, \frac{\pi}{\alpha}\}$. In particular, $\phi_\alpha(S_\omega) = S_{\alpha\omega}$. Let $\alpha\omega < \mu < \pi$ and $\xi \notin S_\mu$. Take $\omega < \nu < \min\{\pi, \frac{\mu}{\alpha}\}$. Consider the function $\phi(z) = \xi - z^\alpha$ on S_ν^0. We have $\phi \in \mathcal{F}(S_\nu^0)$ and $\phi(z) \neq 0$ for all $z \in S_\nu^0$. Moreover,

$$\begin{aligned} |\phi(z)| &= \left| e^{i\frac{\arg(z^\alpha)+\arg(-\xi)}{2}} (z^\alpha - \xi) \right| \\ &\geq \left| \cos \frac{\arg(z^\alpha) - \arg(-\xi)}{2} \right| \left(|z^\alpha| + |\xi|\right) \\ &= \left| \sin \frac{\alpha \arg z - \arg \xi}{2} \right| \left(|z|^\alpha + |\xi|\right) \\ &\geq \min\left\{ \sin \frac{\mu - \alpha\nu}{2}, \sin \frac{\mu + \alpha\nu}{2} \right\} \left(|z|^\alpha + |\xi|\right) \end{aligned}$$

for all $z \in S_\nu^0$. It follows that $\phi^{-1} \in \mathcal{F}(S_\nu^0)$. In fact, setting $K_{\mu,\nu}$ equal to the reciprocal of $\min\{\sin \frac{\mu-\alpha\nu}{2}, \sin \frac{\mu+\alpha\nu}{2}\}$, we have $K_{\mu,\nu} > 0$ and

$$|\phi^{-1}(z)| \leq K_{\mu,\nu} \left(|z|^\alpha + |\xi|\right)^{-1} \qquad \text{for all } z \in S_\nu^0. \tag{8}$$

By Proposition 2.12, $\phi(A) = \xi - A^\alpha$ is injective and

$$(\xi - A^\alpha)^{-1} = \phi^{-1}(A).$$

From the assumption $\alpha > \gamma + 1$ and the estimate (8), it is clear that ϕ^{-1} satisfies the hypotheses of Lemma 2.13. Hence $(\xi - A^\alpha)^{-1}$ is bounded and

$$(\xi - A^\alpha)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_\theta} \phi^{-1}(z) \, (z - A)^{-1} \, dz$$

for some $\omega < \theta < \nu$. In particular,

$$\|(\xi - A^\alpha)^{-1}\| \leq \frac{K_{\mu,\nu} \, C_\nu}{\pi} \int_0^\infty \frac{r^\gamma}{|\xi| + r^\alpha} \, dr = \frac{C \, K_{\mu,\nu} \, C_\nu}{\pi} |\xi|^{-1+\frac{\gamma+1}{\alpha}},$$

where $C = C(\alpha, \gamma) = \int_0^\infty s^\gamma (1 + s^\alpha)^{-1} \, ds$. □

Remark 3.7. For $0 < \alpha < 1+\gamma$, in general, a result similar to Proposition 3.6 can not be expected. To see this, consider again the operator A_q given in Example 2.4. If, for some $z \in \mathbb{C}$, the inverse of $z - A_q^\alpha$ exists, then it must coincide with the multiplication operator associated with the matrix-valued function

$$\begin{bmatrix} \frac{1}{z-(1+x^2+i(1+x^2))^\alpha} & \frac{x^{4+2\gamma}}{2i(1+x^2)} \left[\frac{1}{z-(1+x^2+i(1+x^2))^\alpha} - \frac{1}{z-(1+x^2-i(1+x^2))^\alpha} \right] \\ 0 & \frac{1}{z-(1+x^2-i(1+x^2))^\alpha} \end{bmatrix}.$$

The upper right hand term is unbounded for $\alpha < 1 + \gamma$. Therefore $\sigma(A_q^\alpha) = \mathbb{C}$.

Next, we turn our attention to generator properties of the powers A^α. Throughout the remainder of this section, we assume that A belongs to the class Θ_ω^γ for some ω with $0 < \omega < \frac{\pi}{2}$.

Given $0 < \alpha < \frac{\pi}{2\omega}$ and $t \in S^0_{\frac{\pi}{2}-\alpha\omega}$, take μ with $\omega < \mu < \min\{\pi, \frac{\pi - 2|\arg t|}{2\alpha}\}$. Consider the function $g_{\alpha,t}$ on S^0_μ, given by

$$g_{\alpha,t}(z) \;=\; e^{-tz^\alpha} \qquad \text{for all } z \in S^0_\mu.$$

Since $|g_{\alpha,t}(z)| \leq e^{-|t||z|^\alpha \cos(|\arg t| + \alpha\mu)}$ for all $z \in S^0_\mu$, it follows by Lemma 2.13 that

$$\mathcal{T}_\alpha(t) \;=\; \frac{1}{2\pi i} \int_{\Gamma_\theta} g_{\alpha,t}(z)\, (z - A)^{-1}\, dz$$

defines a bounded linear operator on X. Here $\omega < \theta < \mu$.

Note that the above holds in particular for $0 < \alpha \leq 1$.

In the following, we investigate the structure of the operator families $\{\mathcal{T}_\alpha(t) \mid t \in S^0_{\frac{\pi}{2}-\alpha\omega}\}$ for $0 < \alpha < \frac{\pi}{2\omega}$. We show that they form analytic semigroups of growth order $\frac{\gamma+1}{\alpha}$ as defined below.

Definition 3.8. Let $0 < \delta < \frac{\pi}{2}$ and $\beta > 0$. A family $\{\mathcal{T}(t) \mid t \in S^0_\delta\}$ of bounded linear operators on X, is said to be an analytic semigroup of growth order β if the following conditions hold.

(i) $\mathcal{T}(t+s) = \mathcal{T}(t)\,\mathcal{T}(s)$ for all $t, s \in S^0_\delta$.
(ii) The mapping $t \mapsto \mathcal{T}(t)$ is analytic in S^0_δ.
(iii) There exists a constant $C > 0$ such that

$$\|\mathcal{T}(t)\| \;\leq\; C\, t^{-\beta} \qquad \text{for all } t > 0.$$

(iv) If $\mathcal{T}(t)x = 0$ for some $t \in S^0_\delta$, then $x = 0$.

Semigroups $(\mathcal{T}(t))_{t>0}$ of growth order β were introduced by G. Da Prato [2] (see also [13, 14, 18, 22]). We note that our definition differs from the classical definition of such semigroups in two main points. First, we do not require the set

$$X_0 \;=\; \bigcup_{t>0} \mathcal{T}(t)X$$

to be dense in X. Second, we replace the strong continuity of the mapping $t \mapsto \mathcal{T}(t)$ for $t > 0$ by condition (ii).

Suppose $\mathcal{T}(\cdot)$ is an analytic semigroup of growth order β. By the continuity set of $\mathcal{T}(\cdot)$, we mean the set

$$\Omega_{\mathcal{T}} \;=\; \{x \in X \mid \lim_{\substack{t \to 0 \\ t>0}} \mathcal{T}(t)x \;=\; x\}.$$

The operator G defined on the domain

$$D(G) \;=\; \{x \in X \mid \lim_{\substack{t \to 0 \\ t>0}} \frac{\mathcal{T}(t)x - x}{t} \text{ exists}\}$$

by $Gx = \lim_{t\to 0} \frac{1}{t}(\mathcal{T}(t)x - x)$, is called the infinitesimal generator of $\mathcal{T}(\cdot)$. As in [2, Teorema [1.I]], one shows that G is closable. We call the closure $\overline{G}$ of G the complete generator of the semigroup $\mathcal{T}(\cdot)$.

We now return to the operator families $\{\mathcal{T}_\alpha(t) \mid t \in S^0_{\frac{\pi}{2}-\alpha\omega}\}$ introduced above.

Theorem 3.9. *Let $A \in \Theta^\gamma_\omega$ for some $0 < \omega < \frac{\pi}{2}$, and $0 < \alpha < \frac{\pi}{2\omega}$. Then the family $\{\mathcal{T}_\alpha(t) \mid t \in S^0_{\frac{\pi}{2}-\alpha\omega}\}$ of bounded linear operators on X, forms an analytic semigroup of growth order $\frac{\gamma+1}{\alpha}$. More precisely, the following properties hold.*

(i) $\mathcal{T}_\alpha(t+s) = \mathcal{T}_\alpha(t)\,\mathcal{T}_\alpha(s)$ for all $t, s \in S^0_{\frac{\pi}{2}-\alpha\omega}$.

(ii) There exists a constant $C = C(\gamma, \alpha) > 0$ such that

$$\|\mathcal{T}_\alpha(t)\| \;\le\; C\, t^{-\frac{\gamma+1}{\alpha}} \qquad \text{for all } t > 0. \tag{9}$$

(iii) The range $R(\mathcal{T}_\alpha(t))$ of $\mathcal{T}_\alpha(t)$ with $t \in S^0_{\frac{\pi}{2}-\alpha\omega}$, is contained in $D(A^\infty)$. In particular, $R(\mathcal{T}_\alpha(t)) \subseteq D(A^\beta)$ for all $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$, and

$$A^\beta \mathcal{T}_\alpha(t)x \;=\; \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz^\alpha}(z-A)^{-1}x\,dz \qquad \text{for all } x \in X.$$

Consequently, there is $C = C(\gamma, \alpha, \beta) > 0$ such that

$$\|A^\beta \mathcal{T}_\alpha(t)\| \;\le\; C\, t^{-\frac{\gamma+\operatorname{Re}\beta+1}{\alpha}} \qquad \text{for all } t > 0. \tag{10}$$

(iv) The function $t \mapsto \mathcal{T}_\alpha(t)$ is analytic in $S^0_{\frac{\pi}{2}-\alpha\omega}$ and

$$\frac{d^k}{dt^k}\big(\mathcal{T}_\alpha(t)\big) \;=\; (-1)^k A^{k\alpha}\mathcal{T}_\alpha(t) \qquad \text{for all } t \in S^0_{\frac{\pi}{2}-\alpha\omega}.$$

(v) The operators $\mathcal{T}_\alpha(t)$ with $t \in S^0_{\frac{\pi}{2}-\alpha\omega}$, are injective.

(vi) Let $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$. Then $\mathcal{T}_\alpha(t)A^\beta \subseteq A^\beta\mathcal{T}_\alpha(t)$ for all $t \in S^0_{\frac{\pi}{2}-\alpha\omega}$.

(vii) Let Ω_α denote the continuity set of $\mathcal{T}_\alpha(\cdot)$. If $\beta > 1+\gamma$, then $D(A^\beta) \subseteq \Omega_\alpha$.

Proof. Property (i) is an immediate consequence of Theorem 2.11, part (iii).

To see (ii), choose $\omega < \theta < \mu < \min\{\pi, \frac{\pi}{2\alpha}\}$. For every $t > 0$, we have by Lemma 2.13,

$$\mathcal{T}_\alpha(t) \;=\; \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-tz^\alpha}(z-A)^{-1}\,dz.$$

Hence

$$\|\mathcal{T}_\alpha(t)\| \;\le\; \frac{C_\mu}{\pi}\int_0^\infty e^{-tr^\alpha\cos\alpha\theta} r^\gamma\,dr \;=\; C_\mu\,\Gamma(\frac{\gamma+1}{\alpha})\,\frac{(\cos\alpha\theta)^{-\frac{\gamma+1}{\alpha}}}{\pi}\,t^{-\frac{\gamma+1}{\alpha}}.$$

For (iii), let $t \in S^0_{\frac{\pi}{2}-\alpha\omega}$. Choose μ accordingly. Since $\mathcal{T}_\alpha(t)$ is bounded, Theorem 2.11, part (iii), yields

$$A^\beta\mathcal{T}_\alpha(t) \;=\; z^\beta(A)\,g_{\alpha,t}(A) \;=\; (z^\beta g_{\alpha,t})(A)$$

for every $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$. But the function $z^\beta g_{\alpha,t}$ belongs to $\mathcal{F}_0^\gamma(S_\mu^0)$. Thus, $A^\beta \mathcal{T}_\alpha(t) \in \mathcal{L}(X)$ which implies $R(\mathcal{T}_\alpha(t)) \subseteq D(A^\beta)$. The estimate (10) follows from the representation

$$A^\beta \mathcal{T}_\alpha(t) \;=\; \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz^\alpha}(z-A)^{-1}\,dz$$

for all $t > 0$, with $\omega < \theta < \mu < \min\{\pi, \frac{\pi}{2\alpha}\}$, as in the proof of (ii).

To see (iv), take $0 < \delta < \frac{\pi}{2} - \alpha\omega$. Choose $\omega < \theta < \mu < \min\{\pi, \frac{\pi - 2\delta}{2\alpha}\}$. Then for every $k \in \mathbb{N} \cup \{0\}$, the function

$$(z,t) \;\mapsto\; \frac{\partial^k}{\partial t^k}\big(e^{-tz^\alpha}(z-A)^{-1}\big) \;=\; (-1)^k z^{k\alpha} e^{-tz^\alpha}(z-A)^{-1}$$

is analytic in t and continuous on $\Gamma_\theta \times S_\delta^0$. Moreover, the function $z^{k\alpha}e^{-tz^\alpha}$ belongs to $\mathcal{F}_0^\gamma(S_\mu^0)$ and the integral

$$\int_{\Gamma_\theta} z^{k\alpha} e^{-tz^\alpha}(z-A)^{-1}\,dz$$

converges uniformly on compact subsets of S_δ^0. Hence $\mathcal{T}_\alpha(\cdot)$ is analytic on S_δ^0 and

$$\frac{d^k}{dt^k}\big(\mathcal{T}_\alpha(t)\big) \;=\; \frac{(-1)^k}{2\pi i}\int_{\Gamma_\theta} z^{k\alpha}e^{-tz^\alpha}(z-A)^{-1}\,dz \;=\; (-1)^k A^{k\alpha}\mathcal{T}_\alpha(t),$$

the last equality being a consequence of (iii).

The Dominated Convergence Theorem yields

$$\lim_{t\to 0}\int_{\Gamma_\theta}\frac{z}{(1+z)^2}e^{-tz^\alpha}(z-A)^{-1}x\,dz \;=\; \int_{\Gamma_\theta}\frac{z}{(1+z)^2}(z-A)^{-1}x\,dz$$

for all $x \in X$. By Theorem 2.11 and Proposition 2.8, this is equivalent to

$$\lim_{t\to 0}\mathcal{T}_\alpha(t)\,A(1+A)^{-2}x \;=\; A(1+A)^{-2}x \qquad \text{for all } x \in X.$$

If $\mathcal{T}_\alpha(t_0)x = 0$ for some $t_0 \in S^0_{\frac{\pi}{2}-\alpha\omega}$, then

$$\mathcal{T}_\alpha(t+t_0)x \;=\; \mathcal{T}_\alpha(t)\,\mathcal{T}_\alpha(t_0)x \;=\; 0 \qquad \text{for all } t \in S^0_{\frac{\pi}{2}-\alpha\omega}.$$

As $\mathcal{T}_\alpha(\cdot)$ is analytic, the uniqueness theorem for analytic functions gives that $\mathcal{T}_\alpha(t)x = 0$ for all $t \in S^0_{\frac{\pi}{2}-\alpha\omega}$. It follows that

$$0 \;=\; \lim_{t\to 0} A(1+A)^{-2}\,\mathcal{T}_\alpha(t)x \;=\; \lim_{t\to 0}\mathcal{T}_\alpha(t)\,A(1+A)^{-2}x \;=\; A(1+A)^{-2}x$$

and, as $A(1+A)^{-2}$ is injective, $x = 0$. This proves (v).

Property (vi) follows from part (iii) and Theorem 2.11, part (i).

Finally, for (vii), we take $x \in D(A^\beta)$. By Theorem 2.11 and Lemma 2.13,

$$\mathcal{T}_\alpha(t)x - x \;=\; \frac{1}{2\pi i}\int_{\Gamma_\theta}\frac{1}{1+z^\beta}(e^{-tz^\alpha}-1)(z-A)^{-1}(1+A^\beta)x\,dz,$$

and the Dominated Convergence Theorem yields $\lim_{t\to 0}\mathcal{T}_\alpha(t)x = x$. □

We denote the generator of the semigroup $\mathcal{T}_\alpha(\cdot)$ by A_α. In the last part of this section, we aim to identify the complete generator $\overline{A_\alpha}$ of $\mathcal{T}_\alpha(\cdot)$. To this end, consider the Banach space $X_D = (\overline{D(A)}, \|\cdot\|)$. Recall that the part C_D in X_D, of an operator C in X, is the operator defined on the domain

$$D(C_D) \;=\; \{x \in D(C) \cap X_D \mid Cx \in X_D\}$$

by $C_D x = Cx$ for $x \in D(C_D)$. It is not difficult to show that $A \in \Theta^\gamma_\omega(X)$ implies $A_D \in \Theta^\gamma_\omega(X_D)$. Moreover, the operator A_D is densely defined.

In order to establish the connection between the complete generator $\overline{A_\alpha}$ of $\mathcal{T}_\alpha(\cdot)$, and the fractional power $-(A_D)^\alpha$, we need the following lemmas.

Lemma 3.10. *Let $A \in \Theta^\gamma_\omega(X)$. Then for every $\alpha \in \mathbb{C}$,*

$$(A_D)^\alpha \;=\; [A^\alpha]_D,$$

that is, $(A_D)^\alpha$ is the part of A^α in X_D.

Proof. From the construction of $(A_D)^\alpha$, it is clear that $(A_D)^\alpha \subseteq [A^\alpha]_D$.

For the converse inclusion, let $x \in D([A^\alpha]_D)$. Take $\omega < \theta < \mu < \pi$ and $k, m \in \mathbb{N}$ such that $z^\alpha \psi^k_m \in \mathcal{F}^\gamma_0(S^0_\mu)$. By Definition 2.9 and Proposition 2.8,

$$[A^\alpha]_D x \;=\; (1+A)^{mk}\, A^{-k}\, \frac{1}{2\pi i} \int_{\Gamma_\theta} \frac{z^{\alpha+k}}{(1+z)^{mk}}\, (z-A)^{-1} x\, dz.$$

As $x \in X_D$, $(z-A)^{-1}x = (z-A_D)^{-1}x$ for all $z \in \rho(A) \subseteq \rho(A_D)$. This gives $(z^\alpha \psi^k_m)(A)x = (z^\alpha \psi^k_m)(A_D)x$ and $(z^\alpha \psi^k_m)(A)x \in D(A^{-k}) \cap X_D$.

Moreover, since $[A^\alpha]_D x \in X_D$ and $(1+A)^{-mk}$ leaves X_D invariant, we have that $A^{-k}\,(z^\alpha \psi^k_m)(A_D)x = (1+A)^{-mk}\,[A^\alpha]_D x$ belongs to the set

$$\{x \in D(A^{mk}) \mid A^{mk}x \in X_D\} = D((A_D)^{mk}).$$

This implies $[A^\alpha]_D x = (1+A_D)^{mk}\,[A^{-k}]_D\,(z^\alpha \psi^k_m)(A_D)x$.

It remains to show that $[A^{-k}]_D = (A_D)^{-k}$. But this follows from

$$\begin{aligned} D([A^{-k}]_D) \;&=\; \{A^k x \mid x \in D(A^k) \subseteq X_D \text{ and } A^k x \in X_D\} \\ &=\; \{(A_D)^k x \mid x \in D((A_D)^k)\} \\ &=\; D((A_D)^{-k}), \end{aligned}$$

and $[A^{-k}]_D x = (A_D)^{-k} x$ for all $x \in D((A_D)^{-k})$. □

Lemma 3.11. *Let $A \in \Theta^\gamma_\omega$ for some $0 < \omega < \frac{\pi}{2}$. Let $0 < \alpha < \frac{\pi}{2\omega}$ and $n \in \mathbb{N}$ such that $n > \alpha$. Then $D(A^{n+1}) \subseteq D(A_\alpha)$ and $A_\alpha x = -A^\alpha x$ for all $x \in D(A^{n+1})$.*

Proof. Let $x = (1+A)^{-(n+1)}y$ for some $y \in X$. By Theorem 2.11 and Lemma 2.13,

$$\frac{\mathcal{T}_\alpha(t)x - x}{t} \;=\; \frac{1}{2\pi i} \int_{\Gamma_\theta} \frac{1}{(1+z)^{n+1}}\, \frac{e^{-tz^\alpha} - 1}{t}\, (z-A)^{-1} y\, dz$$

for all $t > 0$. Letting $t \to 0$, the Dominated Convergence Theorem and Proposition 3.3 yield $x \in D(A_\alpha)$ and

$$A_\alpha x = -\frac{1}{2\pi i}\int_{\Gamma_\theta} \frac{z^\alpha}{(1+z)^{n+1}}(z-A)^{-1}y\,dz = -A^\alpha(1+A)^{-(n+1)}y = -A^\alpha x. \quad \square$$

Proposition 3.12. *Let $A \in \Theta_\omega^\gamma$ with $0 < \omega < \frac{\pi}{2}$. Then the following hold.*

(i) $\overline{A_\alpha} \subseteq -(A_D)^\alpha$ for all $0 < \alpha < \frac{\pi}{2\omega}$.

(ii) If $\gamma + 1 < \alpha < \frac{\pi}{2\omega}$, then $-(A_D)^\alpha \subseteq \overline{A_\alpha}$.

Proof. (i) Let $x \in D(A_\alpha)$. By Proposition 2.8 and Proposition 2.10,

$$\begin{aligned} A_\alpha x &= \lim_{t\to 0}\frac{\mathcal{T}_\alpha(t)x - x}{t} \\ &= \lim_{t\to 0}(1+A)^{n+1}\frac{1}{2\pi i}\int_{\Gamma_\theta}\frac{1}{(1+z)^{n+1}}\frac{e^{-tz^\alpha}-1}{t}(z-A)^{-1}x\,dz. \end{aligned}$$

Since $\lim_{t\to 0}\frac{1}{2\pi i}\int_{\Gamma_\theta}\frac{1}{(1+z)^{n+1}}\frac{e^{-tz^\alpha}-1}{t}(z-A)^{-1}x\,dz = -A^\alpha(1+A)^{-(n+1)}x$ and $(1+A)^{n+1}$ is closed, it follows that $x \in D(A^\alpha)$ and $A_\alpha x = -A^\alpha x$. Moreover, we have $\mathcal{T}_\alpha(t)x \in D(A)$ and, as $D(A_\alpha) \subseteq \Omega_\alpha$, $x \in \overline{D(A)}$. Therefore $A^\alpha x = -A_\alpha x \in \overline{D(A)}$. Lemma 3.10 yields $x \in D[(A_D)^\alpha]$ and $A_\alpha x = -(A_D)^\alpha x$. This means $A_\alpha \subseteq -(A_D)^\alpha$ and, since $(A_D)^\alpha$ is closed, $\overline{A_\alpha} \subseteq -(A_D)^\alpha$.

(ii) Suppose $\gamma + 1 < \alpha < \frac{\pi}{2\omega}$. Take $x \in D[((A_D)^\alpha)^2]$. Since $D[(A_D)^\alpha]$ is contained in Ω_α, we have $\lim_{t\to 0}\mathcal{T}_\alpha(t)x = x$ and

$$-(A_D)^\alpha x = \lim_{t\to 0}\mathcal{T}_\alpha(t)[-(A_D)^\alpha x] = \lim_{t\to 0} -(A_D)^\alpha\mathcal{T}_\alpha(t)x = \lim_{t\to 0}A_\alpha\mathcal{T}_\alpha(t)x,$$

the second equality being a consequence of the fact that $\mathcal{T}_\alpha(t)$ and $(A_D)^\alpha$ commute on $D[(A_D)^\alpha]$, and the third one a consequence of Lemma 3.11. As A_α is closable, this gives $x \in D(\overline{A_\alpha})$ and $\overline{A_\alpha}x = -(A_D)^\alpha x$.

The proof will be complete if we prove that $D[((A_D)^\alpha)^2]$ is a core for $(A_D)^\alpha$. Since A_D is densely defined, it is not hard to show that $D[(A_D)^{n+1}]$ for $n \in \mathbb{N}$, is dense in $\overline{D(A)}$. Let $n \in \mathbb{N}$ with $n > \alpha$. Now, given $x \in D[(A_D)^\alpha]$, there is a sequence $(y_m)_{m\in\mathbb{N}} \subseteq D((A_D)^{n+1}) \subseteq D[(A_D)^\alpha]$ such that

$$y_m \overset{m\to\infty}{\longrightarrow} x + (A_D)^\alpha x.$$

Put $x_m = (1+(A_D)^\alpha)^{-1}y_m \in D[((A_D)^\alpha)^2]$. Then

$$x_m \overset{m\to\infty}{\longrightarrow} (1+(A_D)^\alpha)^{-1}(1+(A_D)^\alpha)x = x$$

and

$$\begin{aligned} (A_D)^\alpha x_m &= (A_D)^\alpha(1+(A_D)^\alpha)^{-1}y_m \\ &\overset{m\to\infty}{\longrightarrow} (A_D)^\alpha(1+(A_D)^\alpha)^{-1}(1+(A_D)^\alpha)x \\ &= (A_D)^\alpha x. \end{aligned}$$

$\square$

Finally, we prove that, at least in the range $1+\gamma < \alpha < \frac{\pi}{2\omega}$, there is a one-to-one correspondence between A^α and the semigroup $\mathcal{T}_\alpha(\cdot)$.

Theorem 3.13. *Let $A \in \Theta_\omega^\gamma$ with $0 < \omega < \frac{\pi}{2}$, and $1+\gamma < \alpha < \frac{\pi}{2\omega}$. Then for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$,*

$$(\lambda + A^\alpha)^{-1} \;=\; \int_0^\infty e^{-\lambda t}\mathcal{T}_\alpha(t)\,dt. \tag{11}$$

Consequently, if $A, B \in \Theta_\omega^\gamma$ and $g_{\alpha,t}(A) = g_{\alpha,t}(B)$ for $1+\gamma < \alpha < \frac{\pi}{2\omega}$ and all $t > 0$, then $A^\alpha = B^\alpha$.

Proof. Note first that as a consequence of (9),

$$R_\alpha(\lambda) \;=\; \int_0^\infty e^{-\lambda t}\mathcal{T}_\alpha(t)\,dt$$

defines a bounded linear operator on X. Fix $x \in D(A^\alpha)$. Then by Theorem 3.9, part (iv) and part (vii),

$$\begin{aligned} R_\alpha(\lambda)A^\alpha x \;&=\; -\lim_{\varepsilon\to 0}\int_\varepsilon^\infty e^{-\lambda t}\frac{d}{dt}\big(\mathcal{T}_\alpha(t)x\big)\,dt \\ &=\; -\lim_{\varepsilon\to 0}\Big[e^{-\lambda t}\mathcal{T}_\alpha(t)x\big|_\varepsilon^\infty + \lambda\int_\varepsilon^\infty e^{-\lambda t}\mathcal{T}_\alpha(t)x\,dt\Big] \\ &=\; x \;-\; \lambda R_\alpha(\lambda)x. \end{aligned}$$

Substituting $x = (\lambda + A^\alpha)^{-1}y$ with $y \in X$, in the above expression we obtain

$$R_\alpha(\lambda)A^\alpha(\lambda + A^\alpha)^{-1}y \;=\; (\lambda + A^\alpha)^{-1}y \;-\; \lambda R_\alpha(\lambda)\,(\lambda + A^\alpha)^{-1}y$$

and hence

$$R_\alpha(\lambda)y \;=\; (\lambda + A^\alpha)^{-1}y.$$

If $A, B \in \Theta_\omega^\gamma$ and $g_{\alpha,t}(A) = g_{\alpha,t}(B)$ for $1+\gamma < \alpha < \frac{\pi}{2\omega}$ and all $t > 0$, then from (11), it follows that $(\lambda + A^\alpha)^{-1} = (\lambda + B^\alpha)^{-1}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. This gives $D(A^\alpha) = D(B^\alpha)$. Moreover, for all $x \in D(A^\alpha)$,

$$x \;=\; (\lambda + A^\alpha)^{-1}(\lambda + A^\alpha)x \;=\; (\lambda + B^\alpha)^{-1}(\lambda + A^\alpha)x.$$

Hence $(\lambda + B^\alpha)x = (\lambda + A^\alpha)x$ and consequently $A^\alpha x = B^\alpha x$. □

4. Applications to Abstract Evolution Equations

Throughout this section, we assume that $A \in \Theta_\omega^\gamma$ with $0 < \omega < \frac{\pi}{2}$ and $-1 < \gamma < 0$. We make no assumption on the density of the domain of A. For simplicity of notation, we write $\mathcal{T}(t)$ instead of $\mathcal{T}_1(t)$, that is, $\{\mathcal{T}(t) \mid t \in S^0_{\frac{\pi}{2}-\omega}\}$ is the analytic semigroup of growth order $\gamma + 1$ associated with A. Recall that this semigroup satisfies an estimate

$$\|\mathcal{T}(t)\| \;\le\; C\,t^{-(\gamma+1)} \qquad \text{for all } t > 0, \tag{12}$$

where $C > 0$ is a constant. Finally, Ω will denote the continuity set of $\mathcal{T}(\cdot)$.

Consider the inhomogeneous linear abstract Cauchy problem

$$\text{(LP)} \qquad \begin{cases} u'(t) + Au(t) &= f(t), \qquad 0 < t < T, \\ u(0) &= u_0, \end{cases}$$

where $f :]0,T[\to X$ and $u_0 \in X$ are given. By a classical solution of (LP), we understand a function $u \in C([0,T[;X) \cap C^1(]0,T[;X)$ which takes values in $D(A)$ for all $0 < t < T$ and satisfies (LP).

Analogously to the case where $-A$ generates a C_0-semigroup, one shows that if (LP) has a classical solution, then this solution is unique since it is given by

$$(13) \qquad u(t) \;=\; \mathcal{T}(t)u_0 \;+\; \int_0^t \mathcal{T}(t-s)f(s)\,ds \qquad \text{for all } 0 < t < T,$$

provided $f \in L^1(0,T;X)$. As usual we call (13) the mild solution of (LP). We note that in difference to the case where $\mathcal{T}(\cdot)$ is a C_0-semigroup, the mild solution (13) is in general not continuous at $t = 0$. A positive result for the existence and uniqueness of a classical solution of (LP) is given next.

Theorem 4.1. *Let $A \in \Theta_\omega^\gamma$ with $0 < \omega < \frac{\pi}{2}$. Suppose $f \in L^\infty(0,T;X)$ satisfies $f(t) \in D(A)$ for all $0 < t < T$, and f is Hölder continuous with exponent $\vartheta > 1+\gamma$, that is, there exist $K > 0$ and $\vartheta > 1+\gamma$ such that*

$$(14) \qquad \|f(t) - f(s)\| \;\le\; K\,|t-s|^\vartheta \qquad \textit{for all } 0 < t,s < T.$$

Then for every $u_0 \in \Omega$, the mild solution (13) is the unique classical solution of (LP).

Proof. Put $v(t) = \int_0^t \mathcal{T}(t-s)f(s)\,ds$ for all $0 < t < T$. Then $v \in C([0,T[;X)$ with $\lim_{t\to 0} v(t) = 0$. Therefore, as $u_0 \in \Omega$, $\lim_{t\to 0} u(t) = u_0$.

Fix $0 < t < T$. Writing

$$\int_0^t A\mathcal{T}(t-s)f(s)\,ds \;=\; \int_0^t A\mathcal{T}(t-s)\big(f(s) - f(t)\big)\,ds + \int_0^t \mathcal{T}(t-s)Af(t)\,ds$$

and taking into account the estimates (10) and (14), it follows easily that the integral $\int_0^t A\mathcal{T}(t-s)f(s)\,ds$ is absolutely convergent. Therefore $v(t) \in D(A)$ and consequently, $u(t) \in D(A)$ since $\mathcal{T}(t)u_0 \in D(A^\infty)$ by Theorem 3.9, part (iii).

Next we prove that v is continuously differentiable at t. Let $0 < h < T - t$. Then

$$\frac{v(t+h) - v(t)}{h} \;=\; \frac{1}{h}\int_t^{t+h} \mathcal{T}(t+h-s)f(s)\,ds \;+\; \int_0^t \frac{\mathcal{T}(t+h-s) - \mathcal{T}(t-s)}{h}\,f(s)\,ds.$$

The Dominated Convergence Theorem and Theorem 3.9 yield

$$\lim_{h\to 0} \int_0^t \frac{\mathcal{T}(t+h-s) - \mathcal{T}(t-s)}{h}\,f(s)\,ds \;=\; -\int_0^t A\mathcal{T}(t-s)f(s)\,ds.$$

Further, we have

$$\begin{aligned}\frac{1}{h}&\int_t^{t+h} \mathcal{T}(t+h-s)f(s)\,ds - f(t)\\ &= \frac{1}{h}\int_0^h \mathcal{T}(s)\big(f(t+h-s)-f(t-s)\big)\,ds\\ &\quad + \frac{1}{h}\int_0^h \mathcal{T}(s)\big(f(t-s)-f(t)\big)\,ds \;+\; \frac{1}{h}\int_0^h \big(\mathcal{T}(s)f(t)-f(t)\big)\,ds\\ &= I_1+I_2+I_3.\end{aligned}$$

From (12) and (14), it follows easily that $\|I_1\|, \|I_2\| \;\le\; Mh^{\vartheta-1-\gamma}$, where $M>0$ is a constant. Hence I_1 and I_2 converge to zero as $h\to 0$. Since $f(t)\in D(A)\subseteq \Omega$, given $\varepsilon>0$, there exists $\delta>0$ such that

$$\|\mathcal{T}(s)f(t)-f(t)\| \;\le\; \varepsilon \qquad \text{for all } 0<s<\delta$$

and therefore $\|I_3\| \;\le\; \varepsilon$ for all $0<h<\delta$.

This means v is differentiable from the right at t and the right derivative $v'_+(t)$ satisfies

$$v'_+(t) \;=\; f(t) - \int_0^t A\mathcal{T}(t-s)f(s)\,ds \;=\; f(t)-Av(t).$$

It remains to show that v'_+ is continuous. As f is continuous at t we only have to prove that the function

$$\begin{aligned}\int_0^t A\mathcal{T}(t-s)f(s)\,ds \;&=\; \int_0^t A\mathcal{T}(t-s)\big(f(s)-f(t)\big)\,ds + \int_0^t A\mathcal{T}(t-s)f(t)\,ds\\ &=\; w_1(t) \;+\; w_2(t)\end{aligned}$$

is continuous at t. Since

$$w_2(t) \;=\; \int_0^t \frac{d}{ds}\big(\mathcal{T}(t-s)f(t)\big)\,ds \;=\; -\big(\mathcal{T}(t)-I\big)f(t),$$

w_2 is continuous at t. Take $0<h<T-t$. Then

$$\begin{aligned}w_1(t+h)-w_1(t) \;&=\; \int_t^{t+h} A\mathcal{T}(t+h-s)\big(f(s)-f(t+h)\big)\,ds\\ &\quad+\; \int_0^t A\mathcal{T}(t+h-s)\big(f(t)-f(t+h)\big)\,ds\\ &\quad+\; \int_0^t A\big[\mathcal{T}(t+h-s)-\mathcal{T}(t-s)\big]\big(f(s)-f(t)\big)\,ds.\end{aligned}$$

Estimating the integrands using (10) and (14), it follows that the first two integrals on the right converge to zero as $h\to 0^+$. Moreover, since

$$\|A\mathcal{T}(t+h-s)\big(f(s)-f(t)\big)\| \;\le\; C\,(t+h-s)^{-\gamma-2}(t-s)^{\vartheta} \;\le\; C\,(t-s)^{-\gamma-2+\vartheta}$$

for all $0 < h < T - t$, the Dominated Convergence Theorem yields

$$\lim_{h\to 0^+} \int_0^t A\big[\mathcal{T}(t+h-s) - \mathcal{T}(t-s)\big]\big(f(s) - f(t)\big)\, ds \;=\; 0.$$

This gives $w_1(t+h) - w_1(t) \to 0$ as $h \to 0^+$. Next, assume $0 < h < t$. Then

$$\begin{aligned} &w_1(t-h) - w_1(t) \\ &= \int_0^{t-h} \Big[A\mathcal{T}(t-h-s)\big(f(s) - f(t-h)\big) - A\mathcal{T}(t-s)\big(f(s) - f(t)\big)\Big]\, ds \\ &\quad - \int_{t-h}^t A\mathcal{T}(t-s)\big(f(s) - f(t)\big)\, ds. \end{aligned}$$

Obviously, the second integral converges to zero as $h \to 0^+$. The first integral can be written as

$$\begin{aligned} &\int_0^{t-h} \Big[A\mathcal{T}(t-h-s)\big(f(s) - f(t-h)\big) - A\mathcal{T}(t-s)\big(f(s) - f(t)\big)\Big]\, ds \\ &\quad = \int_0^h A\mathcal{T}(s)\big(f(t-h-s) - f(t-h)\big)\, ds \\ &\qquad + \int_h^{t-h} A\mathcal{T}(s)\big(f(t-h-s) - f(t-s) + f(t) - f(t-h)\big)\, ds \\ &\qquad + \int_{t-h}^t A\mathcal{T}(s)\big(f(t-s) - f(t)\big)\, ds. \end{aligned}$$

By reasoning as above, it can be shown that these three integrals converge to zero as $h \to 0^+$. Hence $w_1(t-h) - w_1(t) \to 0$ as $h \to 0^+$. This completes the proof. $\square$

Next, we consider the semilinear abstract Cauchy problem

$$\text{(SLP)} \qquad \begin{cases} u'(t) + Au(t) \;=\; f(t, u(t)), & 0 < t < T, \\ u(0) \;=\; u_0, \end{cases}$$

where $f : \,]0, T[\, \times X \to X$ and $u_0 \in X$ are given. As in the linear case, by a classical solution of (SLP), we mean a function $u \in C([0,T[\,; X) \cap C^1(]0,T[\,; X)$ which takes values in $D(A)$ for all $0 < t < T$ and satisfies (SLP). As usual, a solution of the integral equation

$$\text{(IE)} \qquad u(t) \;=\; \mathcal{T}(t)u_0 \;+\; \int_0^t \mathcal{T}(t-s) f(s, u(s))\, ds \qquad \text{for all } 0 < t < T,$$

is called a mild solution of (SLP).

The following result is the analogue to [8, Theorem 2.4], where it is assumed that A is sectorial.

Theorem 4.2. *Let $A \in \Theta_\omega^\gamma$ with $0 < \omega < \frac{\pi}{2}$. Suppose $f \in L^\infty(]0,T[\, \times X; X)$ and there exists $K > 0$ such that*

$$(15) \qquad \|f(t,x) - f(t,y)\| \;\le\; K\|x - y\| \qquad \textit{for all } 0 < t < T \textit{ and } x, y \in X.$$

Then for all $u_0 \in \Omega$, there exists $\tau > 0$, sufficiently small, such that (SLP) has a unique mild solution $u \in C([0,\tau]\,;X)$.

Proof. Fix $\delta > 0$. Since $u_0 \in \Omega$, there is $\tau_1 > 0$ such that

$$\|\mathcal{T}(t)u_0 - u_0\| \;\leq\; \frac{\delta}{2} \qquad \text{for all } 0 < t \leq \tau_1.$$

Let $B = \sup\{\|f(t,u_0)\| \mid 0 < t < \tau_1\}$ and $\tau \;=\; \min\{\tau_1\,,\,\left(\frac{-\gamma\delta}{2C(K\delta+B)}\right)^{-1/\gamma}\}$, where C and K are the constants that appear in (12) and (15), respectively.

Consider the Banach space $Y = C([0,\tau]\,;X)$ with norm $\|v\|_Y = \sup_{0\leq t\leq\tau}\|v(t)\|$, and its closed and bounded subset

$$S \;=\; \{v \in Y \mid v(0) = u_0 \text{ and } \|v - u_0\|_Y \leq \delta\}.$$

The mapping F on Y given by

$$Fu(t) \;=\; \mathcal{T}(t)u_0 \;+\; \int_0^t \mathcal{T}(t-s)f(s,u(s))\,ds \qquad \text{for all } 0 \leq t \leq \tau,$$

satisfies $Fu \in Y$ and $Fu(0) = u_0$ for all $u \in Y$. Similar to the proof of [8, Theorem 2.4], one shows that F maps S into S and that $F : S \to S$ is a contraction. Hence from the Contractive Mapping Theorem, it follows that F has a unique fixed point $u \in S$ which is the unique, continuous in $[0,\tau]$, solution of (IE). □

Theorem 4.3. *Let $A \in \Theta^\gamma_\omega$ with $0 < \omega < \frac{\pi}{2}$. Suppose $f \in L^\infty(]0,T[\times X;X)$ and there are constants $L > 0$ and $\vartheta > 1+\gamma$ such that*

$$\|f(t,x) - f(s,y)\| \;\leq\; L\left(|t-s|^\vartheta + \|x-y\|\right) \tag{16}$$

for all $0 < t,s < T$ and $x,y \in X$. Let $u_0 \in X$ such that there exists a local mild solution u of (SLP), defined on $[0,\tau[$, which satisfies $u \in C^\beta(]0,\tau[\,;X)$ for some $\beta > 1+\gamma$, and $f(t,u(t)) \in D(A)$ for all $0 < t < \tau$. Then u is the unique classical solution of (SLP) defined on $[0,\tau[$.

A sufficient condition for the mild solution u to belong to $C^\beta(]0,\tau[\,;X)$ for some $\beta > 1+\gamma$, is that the following hold.

(i) $-1 < \gamma < -\frac{1}{2}$.
(ii) $Af(\cdot,u(\cdot)) \in L^\infty(]0,\tau[\,;X)$.
(iii) $u_0 \in D(A)$ *and* $Au_0 \in \Omega$.

Proof. Consider the linear abstract Cauchy problem

$$\begin{cases} v'(t) + Av(t) \;=\; f(t,u(t)), & 0 < t < \tau, \\ v(0) \;=\; u_0. \end{cases}$$

As f is bounded, the unique mild solution of this problem is given by

$$v(t) \;=\; \mathcal{T}(t)u_0 \;+\; \int_0^t \mathcal{T}(t-s)f(s,u(s))\,ds \qquad \text{for all } 0 < t < \tau.$$

Hence $v = u$. In order to prove that v is a classical solution, by Theorem 4.1, we only have to show that $f(\cdot, u(\cdot))$ is Hölder continuous in $]0,\tau[$ with an exponent greater than $1+\gamma$. But, this follows easily from (16).

Next, we prove that a mild solution u belongs to $C^{-\gamma}(]0,\tau[\,;X)$ if it satisfies (i)-(iii). Note that (i) implies $-\gamma > 1+\gamma$. Let $0 < t < \tau$ and $0 < h < \tau - t$. By the Vector-Valued Mean Value Theorem (see, for example, [6, p. 158]),

$$\begin{aligned} \|u(t+h) - u(t)\| \ &\leq\ \|\mathcal{T}(t+h)u_0 - \mathcal{T}(t)u_0\| \ +\ \int_t^{t+h} \|\mathcal{T}(t+h-s)f(s,u(s))\|\,ds \\ &\quad +\ \int_0^t \|\big(\mathcal{T}(t+h-s) - \mathcal{T}(t-s)\big)f(s,u(s))\|\,ds \\ &\leq\ h \sup_{0<s<\tau} \|\mathcal{T}(s)Au_0\| \ +\ M_1\, t^{-\gamma} h \ +\ M_2\, h^{-\gamma} \\ &\leq\ M\, h^{-\gamma}, \end{aligned}$$

where the constants M_1, M_2 and M do not depend on t or on h. Here we used that as a consequence of (iii), $\sup_{0<s<\tau} \|\mathcal{T}(s)Au_0\| < \infty$. This completes the proof. □

Finally, consider the incomplete second-order abstract Cauchy problem

$$\text{(IACP)} \quad \begin{cases} u''(t) \ = \ Au(t), & t > 0, \\ u(0) \ = \ u_0, \\ \sup_{t>0} \|u(t)\| \ < \ \infty, \end{cases}$$

where $u_0 \in X$ is given. By a classical solution of this problem, we understand a function $u \in C^2(]0,\infty[\,;X) \cap C([0,\infty[\,;X)$ which takes values in $D(A)$ for all $t > 0$ and satisfies (IACP). It is evident that if the initial datum u_0 belongs to the continuity set of the semigroup $\mathcal{T}_{1/2}(\cdot)$ associated with $A^{1/2}$, then the function $u(t) = \mathcal{T}_{1/2}(t)u_0$ for $t > 0$, is a classical solution of (IACP). In the same way as in [16] it is proved that this solution is unique.

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DEPARTAMENTO DE MATEMÁTICA APLICADA Y ESTADÍSTICA, UNIVERSIDAD POLITÉCNICA DE CARTAGENA, C/ PASEO ALFONSO XIII, 30320 CARTAGENA, SPAIN

E-mail address: `f.periago@upct.es`

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF NEW SOUTH WALES, UNSW SYDNEY NSW 2052, AUSTRALIA

E-mail address: `bernd@maths.unsw.edu.au`