Numerical approximation of bang-bang controls for the heat equation: an optimal design approach

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Abstract

This work is concerned with the numerical computation of null controls of minimal L^∞-norm for the linear heat equation with a bounded potential. Both, the cases of internal and boundary (Dirichlet and Neumann) controls are considered. Dual arguments allow to reduce the search of controls to the unconstrained minimization of a conjugate function with respect to the initial condition of a backward heat equation. However, as a consequence of the regularizing property of the heat operator, this initial (final) condition lives in a huge space, that can not be approximated with robustness. For this reason, very specific to the parabolic situation, the minimization is severally ill-posed. On the other hand, the optimality conditions for this problem show that, in general, the unique control v of minimal L^∞-norm has a bang-bang structure as he takes only two values: this allows to reformulate the problem as an optimal design problem where the new unknowns are the amplitude of the bang-bang control and the space-time regions where the control takes its two possible values. This second optimization variable is modeled through a characteristic function. Since the admissibility set for this new control problem is not convex, we obtain a relaxed formulation of it which leads to a well-posed relaxed problem and lets use a gradient descent method for the numerical resolution of the problem. Numerical experiments, for the inner and boundary controllability cases, are described within this new approach.

Keywords: Linear heat equation with potential, approximate controllability, bang-bang control, relaxation, numerical approximation.

Mathematics Subject Classification: 35L05, 49J05, 65K10.

1 Introduction

In this paper, we consider both the internal and boundary controllability problem of a linear heat equation with a bounded potential. Let us describe the problem in the distributed case for which the state equation is

$$\begin{cases} y_t - \Delta y + ay = v\,1_\omega, & (x,t) \in Q_T \\ y(\sigma,t) = 0, \quad (\sigma,t) \in \Sigma_T, & y(x,0) = y_0(x), \quad x \in \Omega. \end{cases} \tag{1}$$

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Here, we denote by Ω an open and bounded set of $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary Γ, $Q_T = \Omega \times (0,T)$, $\Sigma_T = \Gamma \times (0,T)$, $\omega \subset\subset \Omega$ is a (small) non-empty open subset of Ω, 1_ω is the associated characteristic function, $q_T = \omega \times (0,T)$, $T > 0$, $y_0 \in L^2(\Omega)$, $v \in L^\infty(q_T)$ is the *control* and y is the associated *state*. The potential $a = a(x,t) \in L^\infty(Q_T)$.

It is known (see for instance [10, 11]) that, for any $y_0 \in L^2(\Omega)$, $T > 0$ and $v \in L^\infty(q_T)$, there exists exactly one solution of (1), with $y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. For any $y_0 \in L^2(\Omega)$, the null controllability problem for (1) at time $T > 0$ consists to find a function v in $L^\infty(q_T)$, such that the associated solution to (1) satisfies

$$y(x,T) = 0, \quad x \in \Omega. \tag{2}$$

The null controllability has been proved for the heat equation in the nineties: we mention the seminal contributions [6], [9] and also more recently [1] for $L^\infty-$diffusion coefficient.

In this work, we are interested in the numerical approximation of the following optimization problem: for any $\alpha \geq 0$

$$(P_\alpha) \begin{cases} \text{Minimize} \ \ J_\alpha(v) = \|v\|_{L^\infty(q_T)} \\ \text{subject to} \ \ v \in \mathcal{C}_\alpha(y_0,T) \end{cases}$$

where $\mathcal{C}_\alpha(y_0,T) = \{v \in L^\infty(q_T) : y \text{ solves (1) and satisfies } \|y(T)\|_{L^2(\Omega)} \leq \alpha\}$. Problem (P_0) corresponds to the null controllability situation. For any $\alpha > 0$, it is shown in [5] that the unique control solution of the extremal constrained problem (P_α) is given by

$$v_\alpha = \|\varphi_\alpha\|_{L^1(q_T)} \, \text{sign}(\varphi_\alpha) \, 1_\omega, \tag{3}$$

(a quasi bang-bang control) where $\varphi_\alpha = \varphi$ solves the backward equation

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi = 0, \quad (x,t) \in Q_T, \\ \varphi(\sigma,t) = 0, \quad (\sigma,t) \in \Sigma_T, \qquad \varphi(x,T) = \varphi_{\alpha,T}(x), \quad x \in \Omega \end{cases} \tag{4}$$

and with $\varphi_{\alpha,T}$ the unique solution of the following extremal problem, dual of (P_α),

$$(D_\alpha) \begin{cases} \text{Minimize} \ \ \mathcal{J}_\alpha(\varphi) = \dfrac{1}{2}\|\varphi\|^2_{L^1(q_T)} + \alpha\|\varphi\|_{L^2(\Omega)} + \displaystyle\int_\Omega y_0(x)\varphi(x,0)dx \\ \text{subject to} \ \ \varphi \in L^2(\Omega). \end{cases}$$

In the case where the potential a vanishes and, in general, in space dimension $N = 1$ (see [2]), the control (3) is in fact of bang-bang type since the zero set of φ has zero Lebesgue measure.

For any $\alpha > 0$, the minimization of $\mathcal{J}_\alpha$ can be performed using a descent gradient method coupled with a splitting operator approach. Once the minimizer $\varphi_{\alpha,T}$ is determined, φ_α is computed from (4) and the control v_α of minimal L^∞-norm is then given by (3). We refer to [4, 7] where this method has been used (in the inner point-wise and boundary cases). Moreover, as a consequence of the null controllability property, the sequence $(v_\alpha)_{\alpha>0}$, defined by (3), is uniformly bounded w.r.t. α. However, as α goes to zero, the minimizer $\varphi_{\alpha,T}$ may be not uniformly bounded in L^2 but in a larger space, say $\mathcal{H}$ defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|\varphi\|_{L^1(q_T)}$ (we refer to [16], section 4.6 for more details on this passage). Actually, for the control of minimal L^2-norm, it is shown in [12] that the corresponding minimizer belongs to any negative Sobolev space. We also refer to [3, 14] where this phenomenon is fully discussed in the L^2-case. In particular, it is seen that the (so-called HUM) control exhibits a very oscillatory behavior near the controllability time T and that the numerical minimization of $\mathcal{J}_0$ is ill-posed. Since this phenomenon is related to the regularizing property of the heat kernel, it occurs very likely for the L^∞ case as well. But to our knowledge, this has not been studied theoretically so far. This prediction is supported by the numerical experiments reported in [4], where for α small (of the order $\mathcal{O}(10^{-2})$), the approximated bang-bang control oscillates very frequently near T.

On the other hand, (P_α) has also been solved numerically by using a penalty and regularization technique (see [7]). In this method, the cost function in (P_α) is replaced by

$$\frac{1}{2}\left\|v\right\|^2_{L^s(\Omega)} + \frac{1}{2}k\left\|y\left(T\right)\right\|^2_{L^2(\Omega)},$$

where the penalty parameter $k = k(\alpha)$ is chosen large enough so that $\|y(T)\|_{L^2(\Omega)} \leq \alpha$ holds. The use of the L^s-norm (s large enough) instead of the L^∞-norm avoids the problem of the non-differentiability of the L^∞-norm and of any power of it. Numerical simulation results within this approach and for the case of internal point-wise control are described in [7].

In this work we propose an alternative approach to solve numerically (P_α). Precisely, we take advantage of the bang-bang structure of the control and hence consider from the very beginning the control system

$$\begin{cases} y_t - \Delta y + ay = [\lambda 1_{\mathcal{O}} + (-\lambda)(1 - 1_{\mathcal{O}})]1_\omega, & (x,t) \in Q_T \\ y(\sigma,t) = 0, \quad (\sigma,t) \in \Sigma_T, & y(x,0) = y_0(x), \quad x \in \Omega. \end{cases} \tag{5}$$

Notice that here, we impose *a priori* that the control v is of bang-bang type, that is, it takes only two values, λ on $\mathcal{O} \cap q_T$ and $-\lambda$ on $(Q \backslash \mathcal{O}) \cap q_T$, respectively. λ is the amplitude of the piecewise constant control and $\mathcal{O}$ depends on (x,t) but no volume constraint nor regularity assumption are introduced on $\mathcal{O}$. Accordingly, for any $\alpha > 0$, we consider the optimization problem

$$(BB_\alpha) \begin{cases} \text{Minimize in } (\lambda, 1_{\mathcal{O}}) : \dfrac{1}{2}\lambda^2 \\ \text{subject to } (\lambda, 1_{\mathcal{O}}) \in \mathcal{D}_\alpha(y_0, T) \end{cases} \tag{6}$$

where

$$\mathcal{D}_\alpha(y_0,T) = \{(\lambda, 1_{\mathcal{O}}) \in \mathbb{R}^+ \times L^\infty\left(q_T; \{0,1\}\right) : \quad y = y(\lambda, \mathcal{O}) \text{ solves (5) and satisfies } \|y(T)\|_{L^2(\Omega)} \leq \alpha\}. \tag{7}$$

Thus, (BB_α) can be viewed as an optimal design problem where we design the space-time region where the control takes its two possible values and the optimality is related to the amplitude of the bang-bang control. We would like to emphasize that this formulation makes sense and is of a practical interest even if the control of minimal L^∞-norm is not of bang-bang type. Indeed, we address directly the problem of computing bang-bang type controls with minimal amplitude which, as indicated above, is of a major interest in practice. Up to our knowledge, this perspective has not been addressed so far.

Since the space of admissible designs is not convex, we first obtain a well-posed relaxed formulation and then show how this equivalent but new formulation allows to obtain, for any $\alpha > 0$ arbitrarily small, a robust approximation of the solution of the original problem. Precisely, in Section 2 we introduce and analyze this relaxed formulation (see Theorem 2.1). In particular, we obtain that the relaxed problem is an equivalent penalty version of (P_α) and prove that there exists a minimizing sequence of bang-bang type controls (see Remark 1 and Theorem 2.1, part 3, for precise statements). Then, we derive and discuss the first-order necessary optimality condition of the relaxed problem. From this, we recover the bang-bang structure of the control for the pure heat equation and in one space dimension.

The case where the control acts on a part of the boundary in Dirichlet and Neumann forms is also considered and the same type of results are obtained. The numerical resolution of the relaxed problem is addressed in Section 3. We describe the algorithm used to solve the relaxed problem and present several numerical experiments. In particular, the approach allows to capture the oscillatory behavior of the control, as α goes to zero.

2 Relaxation and necessary optimality conditions

2.1 The inner case

We adopt a penalty approach and for simplicity, we still use α to denote the penalty parameter. Hence we transform (BB_α) into the following problem:

$$(T_\alpha)\quad\begin{cases}\text{Minimize in }\ (\lambda,1_{\mathcal{O}}): & J_\alpha(\lambda,1_{\mathcal{O}})=\frac{1}{2}\left(\lambda^2+\frac{1}{\alpha}\|y(T)\|^2_{L^2(\Omega)}\right)\\ \text{subject to} & \\ & y_t-\Delta y+ay=\lambda\left[(2\ 1_{\mathcal{O}}-1)\right]1_\omega, \qquad (x,t)\in Q_T,\\ & y(\sigma,t)=0, \qquad (\sigma,t)\in\Sigma_T\\ & y(x,0)=y_0(x), \qquad x\in\Omega\\ & (\lambda,1_{\mathcal{O}})\in\mathbb{R}^+\times L^\infty(q_T;\{0,1\}).\end{cases}$$

Accordingly, we also consider the problem

$$(RT_\alpha)\quad\begin{cases}\text{Minimize in }\ (\lambda,s): & \overline{J}_\alpha(\lambda,s)=\frac{1}{2}\left(\lambda^2+\frac{1}{\alpha}\|y(T)\|^2_{L^2(\Omega)}\right)\\ \text{subject to} & \\ & y_t-\Delta y+ay=\left[\lambda\left(2s(x,t)-1\right)\right]1_\omega, \qquad (x,t)\in Q_T\\ & y(\sigma,t)=0, \qquad (\sigma,t)\in\Sigma_T\\ & y(x,0)=y_0(x) \qquad x\in\Omega\\ & (\lambda,s)\in\mathbb{R}^+\times L^\infty(q_T;[0,1])\end{cases}$$

From now on we consider the space $L^\infty(q_T;[0,1])$ endowed with the usual weak-$\star$ topology. We then have the following result.

THEOREM 2.1 *(RT_α) is a true relaxation of (T_α) in the following sense:*

1. *there exists one minimizer of (RT_α),*

2. *up to subsequences, every minimizing sequence, say $(\lambda_n,1_{\mathcal{O}_n})$ of (T_α) converges to some $(\lambda,s)\in\mathbb{R}^+\times L^\infty(q_T;[0,1])$ such that (λ,s) is a minimizer for (RT_α), and conversely,*

3. *if (λ,s) is a minimizer for (RT_α) and if $1_{\mathcal{O}_n}$ converges to s weak-$\star$ in $L^\infty(q_T;[0,1])$, then, up to a subsequence, $(\lambda,1_{\mathcal{O}_n})$ is a minimizing sequence for (T_α).*

Proof. Let us first prove that the functional $\overline{J}_\alpha(\lambda,s)$ is continuous. Assume that $(\lambda_n,s_n)\in\mathbb{R}^+\times L^\infty(q_T;[0,1])$ satisfies

$$\begin{cases}\lambda_n\to\lambda & \\ s_n\rightharpoonup s & \text{weak}-\star\text{ in }L^\infty(q_T;[0,1])\end{cases}\qquad\text{as }n\to\infty.$$

Since $\left[\lambda_n\left(2s_n(x,t)-1\right)\right]\rightharpoonup\left[\lambda\left(2s(x,t)-1\right)\right]$ weak-$\star$ in $L^\infty(q_T;[0,1])$ (in particular, also weakly in $L^2(q_T)$), the solution y^n of the system

$$\begin{cases}y^n_t-\Delta y^n+ay^n=\left[\lambda_n\left(2s_n(x,t)-1\right)\right]1_\omega, & (x,t)\in Q_T\\ y^n(\sigma,t)=0, & (\sigma,t)\in\Sigma_T\\ y^n(x,0)=y_0(x) & x\in\Omega\end{cases}$$

satisfies

$$\begin{cases}y^n\rightharpoonup y & \text{weakly in }L^2\left(0,T;H^1_0(\Omega)\right)\\ y^n_t\rightharpoonup y_t & \text{weakly in }L^2\left(0,T;H^{-1}(\Omega)\right),\end{cases}$$

where $y = y(x,t)$ solves

$$\begin{cases} y_t - \Delta y + ay = [\lambda (2s(x,t) - 1)] 1_\omega, & (x,t) \in Q_T \\ y(\sigma,t) = 0, & (\sigma,t) \in \Sigma_T \\ y(x,0) = y_0(x) & x \in \Omega. \end{cases}$$

By Aubin's lemma, up to a subsequence still labeled by n,

$$y^n \to y \quad \text{strongly in } L^2\left(0,T;L^2(\Omega)\right).$$

Hence, up to a subsequence,

$$y^n(t,\cdot) \to y(t,\cdot) \quad \text{strongly in } L^2(\Omega) \text{ and a.e. } t \in [0,T]. \tag{8}$$

Since $y^n(t)$ are continuous functions, convergence (8) in fact holds for all $t \in [0,T]$. In particular,

$$\overline{J}_\alpha(\lambda_n, s_n) \to \overline{J}_\alpha(\lambda, s) \quad \text{as } n \to \infty.$$

Moreover, $\overline{J}_\alpha(\lambda,s)$ is clearly coercitive. As a consequence, problem (RT_α) has a solution.

Statements 2. and 3. are a straightforward consequence of the continuity of $\overline{J}_\alpha(\lambda,s)$ and of the density of the space $L^\infty(q_T;\{0,1\})$ in $L^\infty(q_T;[0,1])$.

Remark 1 *Notice that if we denote by $v = \lambda(2s-1)$ (so that $\lambda = \|v\|_{L^\infty}$), then the state law in problem (RT_α) equals the system (1). This way, (RT_α) is the penalty version of (P_α). Moreover, as a consequence of the continuity of $\overline{J}_\alpha$ we conclude that there exists a minimizing sequence of bang-bang type controls for the penalty version of the approximate controllability problem (P_α).*

Next, we analyze the first-order necessary optimality condition for the relaxed problem (RT_α).

THEOREM 2.2 *The functional $\overline{J}_\alpha$ as defined in problem (RT_α) is Gâteaux differentiable and its directional derivative at (λ,s) in the admissible direction $\left(\widehat{\lambda},\widehat{s}\right)$ is given by*

$$\frac{\partial \overline{J}_\alpha(\lambda,s)}{\partial(\lambda,s)} \cdot (\hat{\lambda},\hat{s}) = \hat{\lambda}\left(\lambda - \int_{q_T} p(2s-1)\,dx\,dt\right) - 2\lambda \int_{q_T} p\hat{s}\,dxdt \tag{9}$$

where $p \in C\left([0,T];L^2(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right)$ solves the adjoint equation

$$\begin{cases} -p_t - \Delta p + ap = 0, \quad (x,t) \in Q_T \\ p(\sigma,t) = 0, \quad (\sigma,t) \in \Sigma_T, \qquad p(x,T) + \alpha^{-1} y(x,T) = 0, \quad x \in \Omega. \end{cases} \tag{10}$$

Proof. Let $\left(\widehat{\lambda},\widehat{s}\right) \in \mathbb{R}^+ \times L^\infty(q_T;[0,1])$ be an admissible direction, i.e., for ε small enough, $\left(\lambda + \varepsilon\widehat{\lambda}, s + \varepsilon\widehat{s}\right) \in \mathbb{R}^+ \times L^\infty(q_T;[0,1])$. Denote by $y_{\left(\lambda+\varepsilon\widehat{\lambda},s+\varepsilon\widehat{s}\right)}$ the solution of the state law as defined in (RT_α) associated with the perturbation $\left(\lambda + \varepsilon\widehat{\lambda}, s + \varepsilon\widehat{s}\right)$. Thanks to the linearity of the heat equation it is easy to see that

$$y_{\left(\lambda+\varepsilon\widehat{\lambda},s+\varepsilon\widehat{s}\right)} = y_{(\lambda,s)} + \varepsilon\widehat{y} + \varepsilon^2\widetilde{y}$$

where $y_{(\lambda,s)}$ is the state associated with the control (λ,s), $\widehat{y}$ is a solution to

$$\begin{cases} \widehat{y}_t - \Delta\widehat{y} + a\widehat{y} = 2\left[\lambda\widehat{s} + \widehat{\lambda}(s-1)\right] 1_\omega, \quad (x,t) \in Q_T \\ \widehat{y}(\sigma,t) = 0 \quad (\sigma,t) \in \Sigma_T, \qquad \widehat{y}(x,0) = 0 \quad x \in \Omega, \end{cases} \tag{11}$$

and $\widetilde{y}$ solves

$$\begin{cases} \widetilde{y}_t - \Delta\widetilde{y} + a\widetilde{y} = \widehat{\lambda}\widehat{s}1_\omega, & (x,t) \in Q_T \\ \widetilde{y}(\sigma,t) = 0, \quad (\sigma,t) \in \Sigma_T, & \widetilde{y}\,(x,0) = 0 \quad x \in \Omega. \end{cases}$$

A straightforward computation shows that

$$\frac{\partial \overline{J}_\alpha(\lambda,s)}{\partial(\lambda,s)} \cdot (\hat{\lambda},\hat{s}) = \lim_{\varepsilon\to 0} \frac{\overline{J}_\alpha\left(\lambda+\varepsilon\widehat{\lambda}, s+\varepsilon\widehat{s}\right) - \overline{J}_\alpha\,(\lambda,s)}{\varepsilon} = \lambda\widehat{\lambda} + \frac{1}{\alpha}\int_\Omega y_{(\lambda,s)}\,(T)\,\widehat{y}\,(T)\,dx. \tag{12}$$

On the other hand, taking into account the initial condition $\widehat{y}\,(0) = 0$ and the final condition $p\,(T) = -\alpha^{-1}y\,(T)$, from the weak form of the system (11) it is deduced that

$$\frac{1}{\alpha}\int_\Omega y_{(\lambda,s)}\,(T)\,\widehat{y}\,(T)\,dx = -\widehat{\lambda}\int_{q_T}(2s-1)\,dxdt - 2\lambda\int_{q_T} p\widehat{s}dxdt$$

for $p \in \mathcal{W}\,(0,T) = \left\{v \in L^2\left(0,T;H_0^1\,(\Omega)\right) : v_t \in L^2\left(0,T;H^{-1}\,(\Omega)\right)\right\}$ solution of (10). Replacing this expression into (12) we obtain (9).

Corollary 2.1 *Let $(\lambda^\star,s^\star) \in \mathbb{R}^+ \times L^\infty\,(q_T;[0,1])$ be an optimal solution of (RT_α). Then $s^\star$ takes the form*

$$s^\star\,(x,t) = \begin{cases} 0 & \text{if} \quad p\,(x,t) < 0 \\ 1 & \text{if} \quad p\,(x,t) > 0 \end{cases} \tag{13}$$

and $\lambda^\star = \|p\|_{L^1(q_T)}$. Consequently, if $N=1$ or if the potential $a=0$ for $N>1$, then $s^\star$ is a characteristic function and therefore problem (T_α) is well-posed, i.e., the control is of bang-bang type.

Proof. Let $(\lambda^\star,s^\star) \in \mathbb{R}^+ \times L^\infty\,(q_T;[0,1])$ be an optimal solution of (RT_α). From (9) it follows that

$$(\lambda-\lambda^\star)\left(\lambda^\star - \int_{q_T} p(2s^\star-1)\,dx\,dt\right) - 2\lambda^\star\int_{q_T} p(s-s^\star)\,dxdt \geq 0 \tag{14}$$

for all $(\lambda,s) \in \mathbb{R}^+ \times L^\infty\,(q_T;[0,1])$. In particular, if $\lambda = \lambda^\star$, then

$$\int_{q_T} ps^\star dxdt \geq \int_q psdxdt \quad \forall s \in L^\infty\,(q_T;[0,1])\,.$$

An standard localization argument (see for instance [15, pages 67-69]) shows that this variational inequality is equivalent to the point-wise variational inequality

$$p\,(x,t)\,s^\star\,(x,t) \geq p\,(x,t)\,s\,(x,t) \quad \forall s \in L^\infty\,(q_T;[0,1])\,,\ \text{for a.e. } (x,t) \in q_T.$$

From this we easily obtain (13).

Now consider the case of the pure heat equation, i.e., $a=0$. Using the fact that thanks to the analyticity of p the zero set of p has zero Lebesgue measure, we conclude that $s^\star$ is a characteristic function. The same holds if $a \neq 0$ and $N=1$ (see [2]).

Finally, if we put $s = s^\star$ in (14), then

$$\lambda^\star = \int_{q_T} p\,(2s^\star-1)\,dxdt = \|p\|_{L^1(q_T)}\,,$$

where the last equality is a consequence of (13). Remark that from this last equality and (13), the optimal control $\lambda^\star(2s^\star-1)$ has exactly the structure given by (3).

Remark 2 *Notice that even in the case where (T_α) is well-posed, the relaxed formulation (RT_α) is not useless. Indeed, at the numerical level, since the admissibility set for (RT_α) is convex (contrary to what happens in (T_α)), it is allowed to make variations in this space and therefore we may implement a descent algorithm to solve (RT_α), and consequently also (T_α). Moreover, Theorem 2.1, part 3, provides a constructive way of computing a minimizing sequence of bang-bang type controls for the general case of the heat equation with a potential in dimension $N>1$.*

2.2 The boundary case

In this section we address the situation in which the control acts on a part of the boundary. We consider both the cases of Dirichlet and Neumann type controls.

2.2.1 Dirichlet-type controls

For a fixed $\alpha > 0$, we focus on the control system

$$\left\{ \begin{array}{ll} y_t - \Delta y + ay = 0, & (x,t) \in Q_T \\ y(\sigma,t) = f(\sigma,t)\, 1_{\Sigma_0} & (\sigma,t) \in \Sigma_T \\ y(x,0) = y_0(x) & x \in \Omega \end{array} \right. \tag{15}$$

and look for the control f, with support in $\Sigma_0 = \Gamma_0 \times (0,T)$, which satisfies

$$\|y(T)\|_{H^{-1}(\Omega)} \leq \alpha. \tag{16}$$

It is important to notice that we have moved from the L^2-norm for the final state to the $H^{-1}-$norm because, as noticed in ([10, p. 217]), if we take $f \in L^2(\Sigma_0)$ and $y_0 \in L^2(\Omega)$, then, in general, we do not have $y(T) \in L^2(\Omega)$.

Under some technical assumptions, some positive results concerning the existence of a solution for the approximate controllability problem (15)-(16) are obtained in [5].

Similarly to the inner situation, we consider the optimization problem

$$(B_\alpha) \left\{ \begin{array}{lll} \text{Minimize in } (\lambda, 1_{\mathcal{O}}): & J_\alpha(\lambda, 1_{\mathcal{O}}) = \frac{1}{2}\left(\lambda^2 + \frac{1}{\alpha}\|y(T)\|^2_{H^{-1}(\Omega)}\right) & \\ \text{subject to} & & \\ & y_t - \Delta y + ay = 0, & (x,t) \in Q_T \\ & y(\sigma,t) = \lambda\,[2\,1_{\mathcal{O}} - 1]\, 1_{\Sigma_0} & (\sigma,t) \in \Sigma_T \\ & y(x,0) = y_0(x), & x \in \Omega \\ & (\lambda, 1_{\mathcal{O}}) \in \mathbb{R}^+ \times L^\infty(\Sigma_0; \{0,1\}) & \end{array} \right.$$

with $y_0 \in L^2(\Omega)$.

We also consider the new problem

$$(RB_\alpha) \left\{ \begin{array}{lll} \text{Minimize in } (\lambda, s): & \overline{J}_\alpha(\lambda, s) = \frac{1}{2}\left(\lambda^2 + \frac{1}{\alpha}\|y(T)\|^2_{H^{-1}(\Omega)}\right) & \\ \text{subject to} & & \\ & y_t - \Delta y + ay = 0, & (x,t) \in Q_T \\ & y(\sigma,t) = \lambda\,[2s(\sigma,t) - 1]\, 1_{\Sigma_0} & (\sigma,t) \in \Sigma_T \\ & y(x,0) = y_0(x), & x \in \Omega \\ & (\lambda, s) \in \mathbb{R}^+ \times L^\infty(\Sigma_0; [0,1]). & \end{array} \right.$$

Then, we have:

THEOREM 2.3 *(RB_α) is a relaxation of (B_α) in the same terms as stated in Theorem 2.1.*

Before proving this result, we recall some results concerning the properties of the solution to the following non-homogeneous system:

$$\left\{ \begin{array}{ll} y_t - \Delta y + ay = 0, & (x,t) \in Q_T \\ y(\sigma,t) = f(\sigma,t) & (\sigma,t) \in \Sigma_T \\ y(x,0) = y_0(x) & x \in \Omega, \end{array} \right. \tag{17}$$

with $f \in L^\infty(\Sigma_T)$ and $y_0 \in L^2(\Omega)$.

Following [10, pp. 208-221] or [11, Vol. II, p.86], a weak solution of (17) is a function $y \in L^2(Q_T)$ which satisfies

$$\int_{Q_T} y(-\varphi_t - \Delta\varphi + a\varphi)dxdt = \int_\Omega y_0(x)\varphi(x,0)dx - \int_{\Sigma_T} f\partial_\nu\varphi d\Sigma_T$$

for all $\varphi \in X^1(Q_T) = \left\{ v \in H^{2,1}(Q_T) : v = 0 \quad \text{on} \quad \Sigma_T \quad \text{and} \quad v(x,T) = 0, x \in \Omega \right\}$. As usual, $\partial_\nu\varphi$ denotes the directional derivative of φ in the direction of the outward unit normal vector to Γ. Then, it is proved (see also [5, Prop. 5.1]) that there exists a unique weak solution of system (17) which has the regularity

$$y \in H^{1/2,1/4}(Q_T) = L^2\left(0,T;H^{1/2}(\Omega)\right) \cap H^{1/4}\left(0,T;L^2(\Omega)\right).$$

Moreover, the estimate

$$\|y\|_{H^{1/2,1/4}(Q_T)} \le c\left(\|f\|_{L^\infty(\Sigma_T)} + \|y_0\|_{L^2(\Omega)}\right)$$

holds. In particular,

$$\|y\|_{L^2(0,T;L^2(\Omega))} \le c\left(\|f\|_{L^\infty(\Sigma_T)} + \|y_0\|_{L^2(\Omega)}\right). \tag{18}$$

On the other hand, standard arguments show that the weak solution of (17) also satisfies $y_t \in L^2\left(0,T;H^{-2}(\Omega)\right)$ and

$$\|y_t\|_{L^2(0,T;H^{-2}(\Omega))} \le c\left(\|f\|_{L^\infty(\Sigma_T)} + \|y_0\|_{L^2(\Omega)}\right). \tag{19}$$

Finally, we also notice that $y \in C\left([0,T];H^{-1}(\Omega)\right)$, see [11, Vol. I, Th. 3.1]. As a consequence, the cost functionals J_α and $\overline{J}_\alpha$ above are well-defined.

Proof of Theorem 2.3. The proof follows the same lines as in Theorem 2.1 so that we only indicate the main differences. To prove the continuity of $\overline{J}_\alpha$, let us take $(\lambda_n, s_n) \in \mathbb{R}^+ \times L^\infty(\Sigma_0;[0,1])$ such that

$$\begin{cases} \lambda_n \to \lambda \\ s_n \rightharpoonup s \quad \text{weak} - \star \text{ in } L^\infty(\Sigma_0;[0,1]). \end{cases}$$

From estimates (18) and (19) it follows that the corresponding weak solution of

$$\begin{cases} y_t^n - \Delta y^n + ay^n = 0, & (x,t) \in Q_T \\ y^n(\sigma,t) = \lambda_n(2s_n(\sigma,t) - 1)1_{\Sigma_0} & (\sigma,t) \in \Sigma_T \\ y^n(x,0) = y_0(x) & x \in \Omega, \end{cases}$$

satisfies

$$\begin{cases} y^n \rightharpoonup y \quad \text{weakly in } L^2\left(0,T;L^2(\Omega)\right) \\ y_t^n \rightharpoonup y_t \quad \text{weakly in } L^2\left(0,T;H^{-2}(\Omega)\right), \end{cases}$$

where $y = y(x,t)$ solves

$$\begin{cases} y_t - \Delta y + ay = 0, & (x,t) \in Q_T \\ y(\sigma,t) = \lambda(2s(\sigma,t) - 1)1_{\Sigma_0} & (\sigma,t) \in \Sigma_T \\ y(x,0) = y_0(x) & x \in \Omega. \end{cases}$$

Again by Aubin's lemma, up to a subsequence

$$y^n \to y \quad \text{strongly in } L^2\left(0,T;H^{-1}(\Omega)\right).$$

Hence, as in the inner case, we have

$$\overline{J}_\alpha(\lambda_n,s_n) \to \overline{J}_\alpha(\lambda,s) \quad \text{as } n \to \infty.$$

The rest of the proof runs as in Theorem 2.1.

Remark 3 *We notice that if the initial condition $y_0 \in L^\infty(\Omega)$, then the solution of system (17) satisfies $y \in L^\infty(Q_T)$ (see [10, p.221]). In particular, $y(T) \in L^2(\Omega)$ and therefore the $H^{-1}(\Omega)$ −norm in the costs J_α and $\overline{J}_\alpha$ may be replaced by the $L^2(\Omega)$ −norm.*

From now on, we assume that y_0 belongs to $L^\infty(\Omega)$ and then replace in J_α and $\overline{J}_\alpha$ the term $\alpha^{-1}\|y(T)\|^2_{H^{-1}(\Omega)}$ by $\alpha^{-1}\|y(T)\|^2_{L^2(\Omega)}$. Similarly to Theorem 2.2 and Corollary 2.1 we have:

Theorem 2.4 *For $y_0 \in L^\infty(\Omega)$ the functional $\overline{J}_\alpha$ as defined above is Gâteaux differentiable and its directional derivative at (λ, s) in the admissible direction $\left(\widehat{\lambda}, \widehat{s}\right)$ is given by*

$$\frac{\partial \overline{J}_\alpha(\lambda, s)}{\partial(\lambda, s)} \cdot (\hat{\lambda}, \hat{s}) = \hat{\lambda}\left(\lambda + \int_{\Sigma_0} \partial_\nu p(2s-1)\, d\Sigma_0\right) + 2\lambda \int_{\Sigma_0} \partial_\nu p \hat{s}\, d\Sigma_0 \tag{20}$$

where p solves the backward equation (10).

Corollary 2.2 *Let $(\lambda^\star, s^\star) \in \mathbb{R}^+ \times L^\infty(\Sigma_0; [0,1])$ be an optimal solution of (RB_α). Then $s^\star$ takes the form*

$$s^\star(\sigma, t) = \begin{cases} 0 & \text{if} \quad \partial_\nu p(\sigma, t) < 0 \\ 1 & \text{if} \quad \partial_\nu p(\sigma, t) > 0 \end{cases} \tag{21}$$

and $\lambda^\star = \|\partial_\nu p\|_{L^1(\Sigma_0)}$. As a consequence, if $N = 1$ or if the potential $a = 0$ for $N > 1$, then $s^\star$ is a characteristic function and therefore problem (B_α) is well-posed, i.e., the control is of bang-bang type.

2.2.2 Neumann-type controls

Consider the system

$$\begin{cases} y_t - \Delta y + ay = 0 & \text{in} \quad Q_T \\ \partial_\nu y = g & \text{on} \quad \Sigma_T \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{22}$$

It is well-known (see for instance [11] or [15]) that for $y_0 \in L^2(\Omega)$ and $g \in L^2\left(0,T; H^{-1/2}(\Gamma)\right)$ the system (22) has a unique solution $y \in L^2\left(0,T; H^1(\Omega)\right) \cap C\left([0,T]; L^2(\Omega)\right)$ which satisfies the variational formulation

$$\frac{d}{dt}\int_\Omega y(x,t)\, v(x)\, dx + \int_\Omega \left[\nabla y(x,t)\, \nabla v(x) + a(x,t)\, y(x,t)\, v(x)\right] dx = \ < g(t), v >_\Gamma \quad \forall v \in H^1(\Omega),$$

where $< \cdot, \cdot >_\Gamma$ stands for the duality product in $H^{1/2}(\Gamma)$. Moreover,

$$\|y\|_{C([0,T];L^2(\Omega))} + \|y\|_{L^2(0,T;H^1(\Omega))} \leq C\left(\|y_0\|_{L^2(\Omega)} + \|g\|_{L^2\left(0,T;H^{-1/2}(\Gamma)\right)}\right).$$

With the same notation as in the preceding section, we consider the two problems

$$(NB_\alpha)\begin{cases} \text{Minimize in } (\lambda, 1_{\mathcal{O}}): \quad J_\alpha(\lambda, 1_{\mathcal{O}}) = \frac{1}{2}\left(\lambda^2 + \frac{1}{\alpha}\|y(T)\|^2_{L^2(\Omega)}\right) \\ \text{subject to} \\ \qquad y_t - \Delta y + ay = 0 \quad \text{in} \quad Q_T \\ \qquad \partial_\nu y(\sigma, t) = \lambda\left[2 1_{\mathcal{O}}(\sigma, t) - 1\right] 1_{\Sigma_0} \quad \text{on} \quad \Sigma_T \\ \qquad y(0) = y_0 \quad \text{in} \quad \Omega \\ \qquad (\lambda, 1_{\mathcal{O}}) \in \mathbb{R}^+ \times L^\infty(\Sigma_T; \{0,1\}) \end{cases}$$

and

$$(RNB_\alpha)\begin{cases} \text{Minimize in } (\lambda, s): \quad \overline{J}_\alpha(\lambda, 1_{\mathcal{O}}) = \frac{1}{2}\left(\lambda^2 + \frac{1}{\alpha}\|y(T)\|^2_{L^2(\Omega)}\right) \\ \text{subject to} \\ \qquad y_t - \Delta y + ay = 0 \quad \text{in} \quad Q_T \\ \qquad \partial_\nu y(\sigma, t) = \lambda\left[2s(\sigma, t) - 1\right] 1_{\Sigma_0} \quad \text{on} \quad \Sigma_T \\ \qquad y(0) = y_0 \quad \text{in} \quad \Omega \\ \qquad (\lambda, s) \in \mathbb{R}^+ \times L^\infty(\Sigma_T; [0,1]). \end{cases}$$

The same type of arguments as the ones used in the two preceding cases lets prove that (RNB_α) is a relaxation of (NB_α). Also, a direct computation shows that the functional $\overline{J}_\alpha$ is Gâteaux differentiable and its directional derivative at (λ, s) in the admissible direction $\left(\widehat{\lambda}, \widehat{s}\right)$ is given by

$$\frac{\partial \overline{J}_\alpha(\lambda, s)}{\partial(\lambda, s)} \cdot \left(\widehat{\lambda}, \widehat{s}\right) = \widehat{\lambda}\left(\lambda + \int_{\Sigma_0} p(2s-1)\, d\Sigma_0\right) + 2\lambda \int_{\Sigma_0} \widehat{s} p d\Sigma, \tag{23}$$

where p solves the system

$$\begin{cases} -p_t - \Delta p + ap = 0 & \text{in} \quad Q_T \\ \partial_\nu p = 0 & \text{on} \quad \Sigma_T \\ p(T) = \frac{1}{\alpha} y(T) & \text{in} \quad \Omega. \end{cases} \tag{24}$$

By using (23), it is not hard to show that if $(\lambda^\star, s^\star)$ is a solution of (RNB_α), then

$$s^\star(\sigma, t) = \begin{cases} 0 & \text{if} \quad p(\sigma, t) > 0 \\ 1 & \text{if} \quad p(\sigma, t) < 0 \end{cases}$$

and $\lambda^\star = \|p\|_{L^1(\Sigma_0)}$

3 Algorithm - Numerical experiments

3.1 Algorithm and numerical approximation

Let us provide some details on the inner situation and in the one dimensional space case. From now on, we take $\Omega = (0, 1)$.

First notice that we may remove the constraint $\lambda \in \mathbb{R}^+$ since, if (λ, s) solves (RT_α), then $(-\lambda, 1-s)$ is also a solution. Hence, the expression (9) provides the following iterative descent algorithm :

$$\begin{cases} (\lambda^0, s^0) \in \mathbb{R} \times L^\infty(Q_T, [0,1]), \\ \lambda^{n+1} = \lambda^n - a_n\left(\lambda^n - \displaystyle\int_{q_T} p^n(2s^n - 1)\, dx\, dt\right), \quad n \geq 0, \\ s^{n+1} = P_{[0,1]}(s^n + b_n \lambda^n p^n), \quad n \geq 0 \end{cases} \tag{25}$$

where p^n solves (10), $P_{[0,1]}(x) = \max(0, \min(1, x))$ denotes the projection of any x onto $[0,1]$ and a_n, b_n denote the optimal descent step which is obtained as the solution of the extremal problem :

$$\min \overline{J}_\alpha(\lambda^{n+1}(a), s^{n+1}(b)) \quad \text{over} \quad a, b \in \mathbb{R}^+. \tag{26}$$

Problem (26) is solved by using line search techniques. The gradient algorithm is stopped as soon as

$$|\lambda^n - \int_{q_T} p^n(2s^n - 1) dx dt| \leq \sigma \tag{27}$$

for some given tolerance $\sigma > 0$ small enough. In the sequel, $\sigma := 10^{-3}$.

As for the numerical discretization, we use the two-step Gear scheme (of second order) for the time integration coupled with a P_1 finite element approach for the spatial approximation. Precisely, for large integer N_x, we consider the N_x points $x_i \in [0,1]$ such that $x_1 = 0$, $x_i < x_{i+1}$ and $x_{N_x} = 1$. We note for $i \in \{1, N_x - 1\}$, $\Delta x_i = x_{i+1} - x_i$ and $\Delta x = \max \Delta x_i$. We note by $\mathcal{P}_{\Delta x}$ the corresponding partition of $\Omega = [0,1]$ and by $\mathcal{P}_{\Delta t}$ the corresponding partition of $[0, T]$, obtained in the same way. Finally, set $h = (\Delta x, \Delta x)$ and $\mathcal{Q}_h$ the quadrangulation of Q_T associated to h so that in particular $\overline{Q}_T = \bigcup_{K \in \mathcal{Q}_h} K$.

The following (conformal) finite element approximation of $L^2(0, T; H_0^1(0,1))$ is introduced:

$$X_{0h} = \{\, \varphi_h \in C^0(Q_T) : \varphi_h|_K \in (\mathbb{P}_{1,x} \otimes \mathbb{P}_{1,t})(K) \ \ \forall K \in \mathcal{Q}_h, \varphi_h(0,t) = \varphi_h(1,t) = 0 \ \ \forall t \in (0,T) \,\}.$$

Here $\mathbb{P}_{m,\xi}$ denotes the space of polynomial functions of order m in the variable ξ. Accordingly, the functions in X_{0h} reduce on each quadrangle $K \in \mathcal{Q}_h$ to a linear polynomial in both x and t. The space X_{0h}, conformal approximation of $L^2(Q_T)$ is a finite-dimensional subspace of $L^2(0,T;H_0^1(0,1))$. Moreover, the functions $\varphi_h \in X_{0h}$ are uniquely determined by their values at the nodes (x_i,t_j) of $\mathcal{Q}_h$ such that $0 < x_i < 1$.

Let us now introduce other finite dimensional spaces. First, we set

$$\Phi_{\Delta x} = \{\, z \in C^0([0,1]) : z|_k \in \mathbb{P}_{1,x}(k) \quad \forall k \in \mathcal{P}_{\Delta x} \,\}.$$

Then, $\Phi_{\Delta x}$ is a finite dimensional subspace of $L^2(0,1)$ and the functions in $\Phi_{\Delta x}$ are uniquely determined by their values at the nodes of $\mathcal{P}_{\Delta x}$.

Secondly, since the variable $\lambda(2s-1) \in L^\infty(Q_T)$ appears in the right hand side of the forward problem in y, it is natural to approximate $\lambda(2s-1) \in L^\infty(Q_T)$ by a piecewise constant function. Thus, let M_h be the space defined by

$$M_h = \{\, \mu_h \in L^\infty(Q_T) : \mu_h|_K \in (\mathbb{P}_{0,x} \otimes \mathbb{P}_{0,t})(K) \quad \forall K \in \mathcal{Q}_h \,\}.$$

M_h is a finite dimensional subspace of $L^\infty(Q_T)$ and the functions in M_h are uniquely determined by their (constant) values on the quadrangles $K \in \mathcal{Q}_h$.

Therefore, for any $s_h \in M_h$ and any $\lambda_h \in \mathbb{R}$, the approximation $y_h \in X_{0h}$ of the solution of (RT_α) is given as follows :

(i) Consider the times $t_j = j\Delta t$ and set $y_h|_{t=0} = \pi_{\Delta x}(y_0) \in \Phi_{\Delta x}$.

(ii) Then, $y_h|_{t=t_1}$ is the solution to the linear problem

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{\Delta t}(\Psi - y_h|_{t=0}) z\,dx + \frac{1}{2}\int_0^1 (\Psi_x z_x + \pi_{\Delta x} A(x,t_1)\Psi z)\,dx \\ \displaystyle\qquad + \frac{1}{2}\int_0^1 ((y_h|_{t=0})_x z_x + \pi_{\Delta x} A(x,t_{N_t}) y_h|_{t=0} z)\,dx \\ \displaystyle\qquad = \frac{1}{2}\lambda_h \int_0^1 ((2s_h(x,t_1)-1) + (2s_h(x,t_0)-1) z(x)\,dx \quad \forall z \in \Phi_{\Delta x}, \quad \Psi \in \Phi_{\Delta x}. \end{cases}$$

(iii) Finally, for given $n = 1,\ldots,N-1$, $\Psi^\star = y_h|_{t=t_{n-1}}$ and $\overline{\Psi} = y_h|_{t=t_n}$, $y_h|_{t=t_{n+1}}$ is the solution to the linear problem

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{2\Delta t}(3\Psi - 4\overline{\Psi} + \Psi^\star) z\,dx + \int_0^1 (\Psi_x z_x + \pi_{\Delta x}(A(x,t_{n-1}))\Psi z)\,dx \\ \displaystyle\quad = \int_0^1 \mu_h(x,t_{n-1}) z(x)\,dx \quad \forall z \in \Phi_{\Delta x}, \quad \Psi \in \Phi_{\Delta x}. \end{cases}$$

Here, $\pi_{\Delta x}$ denotes the projection over $\Phi_{\Delta x}$.

We are thus using the two-step *implicit Gear* algorithm as a numerical tool to solve numerically the problem (RT_α). As advocated in [4], where the influence of the time discretization is highlighted, it has been observed that this second order scheme ensures a better behavior of the underlying gradient algorithm than, for instance, the implicit Euler scheme. At the finite dimensional level, algorithm (25) reads as follows :

$$\begin{cases} (\lambda_h^0, s_h^0) \in \mathbb{R} \times M_h, \\ \displaystyle\lambda_h^{n+1} = \lambda_h^n - a_{nh}\left(\lambda_h^n - \int_{q_T} p_h^n(2s_h^n - 1)\,dx\,dt\right), \quad n \geq 0, \\ s_h^{n+1} = P_{[0,1]}(s_h^n + b_{nh}\lambda_h^n p_h^n), \quad n \geq 0 \end{cases} \tag{28}$$

where p_h^n is an approximation of the backward problem (10), obtained by using P_1 finite element in space and the Gear scheme for the time integration, as described above.

3.2 Numerical experiments

3.2.1 Distributed case

We now provide some numerical results in the one dimensional space case and discuss mainly the influence of the penalty parameter α. We take $\Omega = (0,1)$ and first consider the data $\omega = (0.25, 0.75)$, $a \equiv 0$ and $y_0(x) = \sin(2\pi x)$. In order to have a better control of the diffusion, we also replace the operator $-\Delta$ by $-c\Delta$ with c lower than one. This does not modify the theoretical part.

The descent algorithm is initialized with $\lambda = 1$ and $s = 1/2$ over Q_T. We take $\sigma := 10^{-3}$ as stopping criterion parameter.

We take a uniform partition $\mathcal{P}_{\Delta x}$ for Ω : $x_{i+1} - x_i = 1/N_x$ with $N_x = 400$. On the other hand, in order to describe correctly the oscillations of the density near T, we take a non uniform partition $\mathcal{P}_{\Delta t}$ of the time interval $(0,T)$: precisely we define

$$t_1 = 0; \quad t_{j+1} - t_j = \frac{T}{e^{pT} - 1}(e^{\frac{pT}{N_t}} - 1)e^{\frac{p(N_t+1-j)}{N_t}T} \quad j = 1, ..., N_t$$

where $N_t \in \mathbb{N}$ is the number of sub-interval of the partition $\mathcal{P}_{\Delta t}$ and any $p \in \mathbb{N}$. The points t_j, distributed along $(0,T)$, are thus exponentially concentrated near T. This property is amplified for increasing values of p. Here $p := 6$ and $N_t = 400$.

Table 1 collects the value of λ_h and $\|y_h(\cdot, T)\|_{L^2(0,1)}$ with respect to the penalty parameter α. We take $c := 1/10$.

We check that the L^∞-norm λ_h of the control increases as α goes to zero : in other words, the amount of work needed to get closer to the zero target at time T is more important. However, as a consequence of the null controllability of (1) with $v \in L^\infty(q_T)$, we check that λ_h is uniformly bounded by above with respect to α.

The value α also affects the shape of the bang-bang control. Figure 1 depicts the iso-values of the optimal density for $\alpha = 10^{-2}, 10^{-4}, 10^{-6}$ and $\alpha = 10^{-8}$. According to the symmetry of ω and of the initial datum y_0, we obtain symmetric density over Q_T. For $\alpha = 10^{-2}$, the density is constant in time and related to the sign of y_0: precisely, for all t, $s_h(x,t) = 0$ if $y_0(x) > 0$ and $s_h(x,t) = 1$ if $y_0(x) < 0$. However, for α small enough, for instance here, $\alpha = 10^{-3}$, the optimal density exhibits some variations with respect to the variable t. These variations are mainly located at the end of the time interval. Moreover, as α decreases, the number of theses oscillations, that is, the number of theses changes of sign of v_h increases so that, at the null controllability limit ($\alpha = 0$) one may expected an oscillatory behavior of the bang-bang control both in space and time, in an arbitrarily close neighborhood of $(0,1) \times \{T\}$. This is in agreement with our observations in the L^2 case (see [14]). Of course, as in the L^2-case, this behavior may only be captured with an arbitrarily finer mesh. The increasing number of iterates needed to satisfy the criterion (27) as α decreases is also a consequence of these oscillations near T.

In Figure 1 we also observe that (except for $\alpha = 10^{-2}$) the optimal density s_h is not strictly a bi-valued $0-1$ (as it should be almost everywhere) and takes some intermediate values: this is only due to the numerical approximation and to the very low variation of the cost function with respect to the density near the minimum (a bi-valued density is obtained after a very large number of iterates). Figure 2 displays the sign of the adjoint solution p (see 10) in q_T and also clearly exhibits the variation of the bang-bang control: recall that from (13), s and p are related through the relation $2s - 1 = \text{sign}(p)$.

More interesting is the fact that, whatever be the initial value $(\lambda_h^0, s_h^0) \in \mathbb{R}^+ \times L^\infty(Q_T, [0,1])$ guess we consider as the starting point for the algorithm, we always get the same limit. This is of course in agreement with the fact the control v of minimal L^∞-norm is unique, and so the couple (λ, s) defined by $v = \lambda(2s-1)1_\omega$ is.

We now consider both a bounded but discontinuous potential a and a discontinuous initial datum. Precisely, for $\Omega = (0,1)$ and $\omega = (0.2, 0.6)$, we take

α	10^{-1}	10^{-2}	10^{-4}	10^{-6}	10^{-8}
$\Vert y_h(T)\Vert_{L^2(\Omega)}$	8.96×10^{-2}	5.34×10^{-2}	6.24×10^{-3}	4.37×10^{-4}	9.17×10^{-5}
λ_h	0.087	0.471	1.309	1.831	1.948
$\sharp$ iterates	11	213	561	1032	4501

Table 1: $N_x = N_t = 400$ - $y_0(x) = \sin(2\pi x)$ - $c = 0.1$

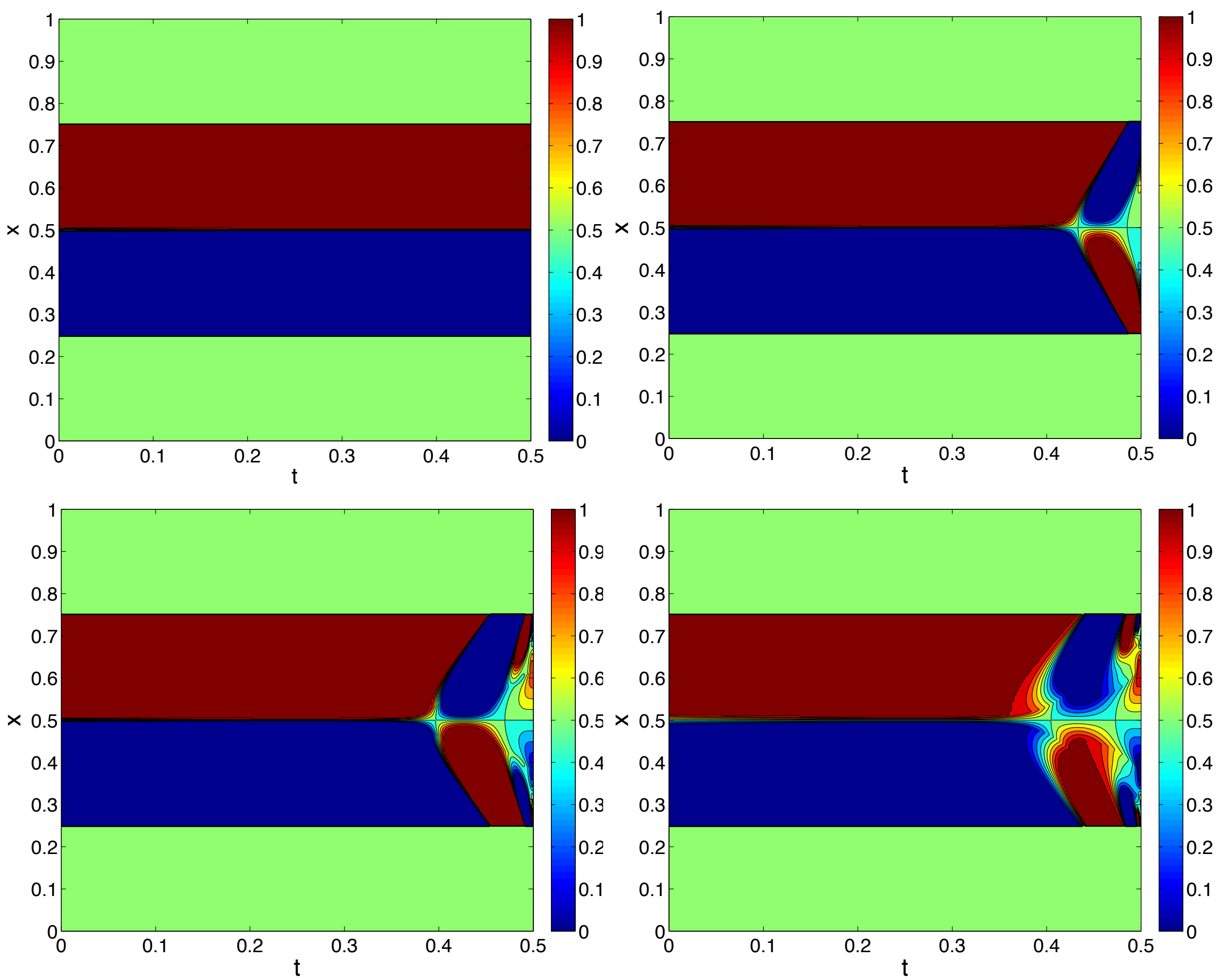

Figure 1: $N_x = N_t = 400$ - $y_0(x) = \sin(2\pi x)$ - $\omega = (0.25, 0.75)$ - $c = 0.1$ - From left to right and from top to button, iso-values of the density function s_h in Q_T for $\alpha = 10^{-2}, 10^{-4}, 10^{-6}$ and $\alpha = 10^{-8}$.

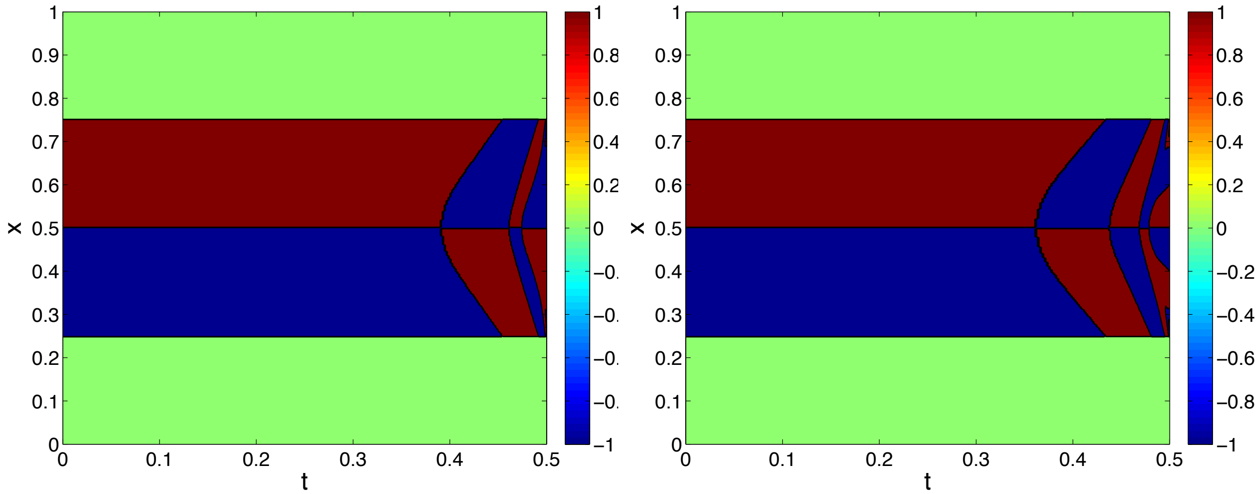

Figure 2: $N_x = N_t = 400$ - $y_0(x) = \sin(2\pi x)$ - $\omega = (0.25, 0.75)$ - $c = 0.1$ - Sign of the adjoint state p for $\alpha = 10^{-6}$ (left) and $\alpha = 10^{-8}$ (right).

$$a(x) = \begin{cases} 1, & 0 \le x \le 0.5 \\ -3, & 0.5 < x \le 1 \end{cases}, \quad y_0(x) = \begin{cases} 0, & x \in [0, 0.1] \cup [0.9, 1] \\ 1., & 0.1 < x < 0.9. \end{cases} \tag{29}$$

The over data remain unchanged. For $\alpha = 10^{-6}$, Figure 3 depicts the iso-values of the optimal density s_h together with the sign of the corresponding adjoint solution p over Q_T. The convergence leads to $\lambda_h \approx 14.35$ for which $\|y_h(T)\|_{L^2(0,1)} \approx 4.11 \times 10^{-6}$. Here, of course, the symmetry of the density is lost. As in the preceding test, we observe variations of the density near T, up to the error due to the numerical approximation.

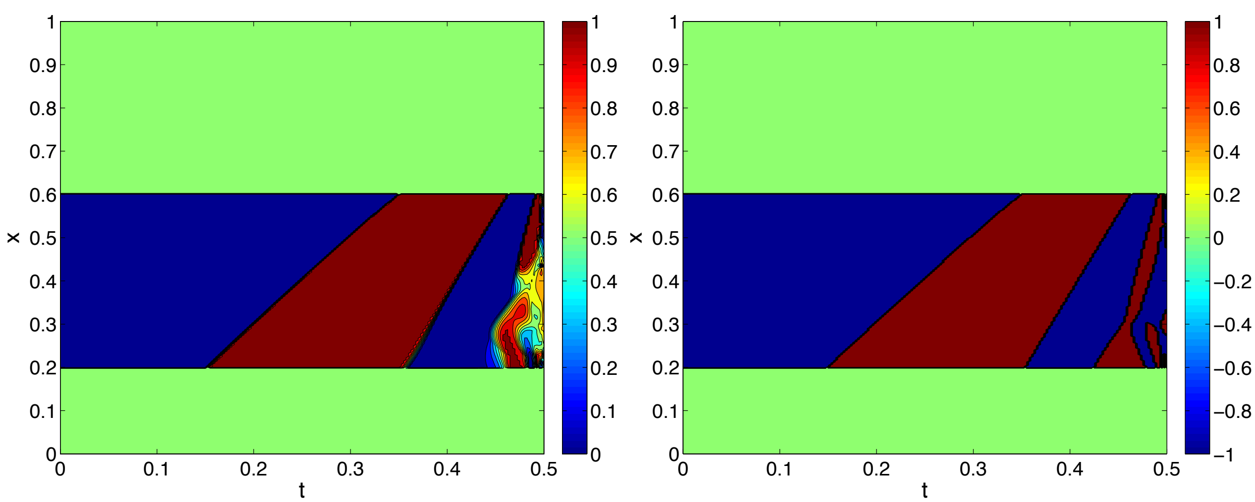

Figure 3: $Nx = 400$ - $N_t = 400$ - $\omega = (0.2, 0.6)$ - $\alpha = 10^{-6}$ - Discontinuous data as in (29)- Iso-values of the density function s_h in Q_T (left) and sign of p (right).

3.2.2 Boundary case

In the boundary situation, the density s is simply a time function. This allows to observe clearer the high singular character of the bang-bang control as α goes to zero, that is, when one wants to recover the null control of minimal L^∞-norm. The procedure is very similar, except that we also consider a non uniform partition $\mathcal{P}_{\Delta x}$ (concentrated on $x=1$) in order to better describe the final state $p(T)$ of the adjoint system (oscillating near $x=1$).

$$x_1 = 0; \quad x_{i+1} - x_i = \frac{1}{e^p - 1}(e^{\frac{p}{N_x}} - 1)e^{\frac{p(N_x+1-i)}{N_x}} \quad i = 1, ..., N_x.$$

with here $p := 3$ and still $N_x = 400$.

The minimization procedure is similar. For the Dirichlet boundary control considered in Section 2.2.1, fromTheorem 2.4, we may consider the following descent algorithm.

$$\begin{cases} (\lambda_h^0, s_h^0) \in \mathbb{R} \times N_h, \\ \lambda_h^{n+1} = \lambda_h^n - a_{nh}\left(\lambda_h^n + \int_{\Sigma_0} \partial_\nu p_h^n (2s_h^n - 1)\, d\Sigma_0\right), \quad n \geq 0, \\ s_h^{n+1} = P_{[0,1]}(s_h^n - b_{nh}\lambda_h^n \partial_\nu p_h^n(1,\cdot)), \quad n \geq 0 \end{cases} \tag{30}$$

where p_h^n is as above an approximation of the backward problem (10) and N_h the space defined by

$$N_h = \{\, \mu_h \in L^\infty([0,T]) : \mu_h|_k \in \mathbb{P}_{0,t})(K) \quad \forall k \in \mathcal{P}_{\Delta t}\,\}.$$

As in the inner situation, the residue

$$\left|\lambda_h^n + \int_{\Sigma_0} \partial_\nu p_h^n (2s_h^n - 1)\, d\Sigma_0\right|$$

is used as stopping criterion for the algorithm. For the Neumann boundary control discussed in Section 2.2.2, the algorithm is given by

$$\begin{cases} (\lambda_h^0, s_h^0) \in \mathbb{R} \times N_h, \\ \lambda_h^{n+1} = \lambda_h^n - a_{nh}\left(\lambda_h^n + \int_{\Sigma_0} p_h^n (2s_h^n - 1)\, d\Sigma_0\right), \quad n \geq 0, \\ s_h^{n+1} = P_{[0,1]}(s_h^n + b_{nh}\lambda_h^n p_h^n(1,\cdot)), \quad n \geq 0 \end{cases} \tag{31}$$

where p_h^n is an approximation of the solution p of (24).

Let us discuss the Neumann boundary case. We take $y_0(x) = \sin(\pi x)$, $T = 1/2$, $c = 1/10$, $a := 0$. The stopping criterion is related to the absolute value $|\lambda - \int_0^1 p(1,t)\,(2s-1)\,dt|$. We stop the algorithm as soon as this value is lower than 10^{-3}. Table 2 reports the L^∞-norm λ_h and the $L^2(0,1)$-norm of $y_h(T)$ for various values of the penalty parameter α. The amplitude λ_h of the bang-bang control increases as $\alpha \to 0$, and is significantly bigger than in the inner case. This is due to the fact that the control acts only on a single point of $\overline{\Omega}$.

α	10^{-2}	10^{-4}	10^{-6}	10^{-8}
$\|y_h(\cdot,T)\|_{L^2(\Omega)}$	2.96×10^{-1}	1.59×10^{-1}	5.56×10^{-2}	9.31×10^{-3}
λ_h	1.488	10.181	29.121	34.03
$\sharp$ iterates	33	512	6944	20122

Table 2: $(\lambda_h, y_h(T))$ with respect to α in the Neumann boundary case.

Figure 4 depicts the optimal density s_h with respect to the time variable. We observe here that the density is almost everywhere a characteristic function with an increasing number of change of sign as t goes to T^-. Figure 5 depicts the trace, both in time and space, of the approximate controlled solution y_h while Figure 6 displays y_h along Q_T.

Let us insist again on that oscillatory behavior and display the density for $\alpha = 10^{-6}$. Figures 7, 8 and 9 depict the optimal density $s_h(t)$ for $t \in [0., 0.5]$, $t \in [0.4, 0.5]$ and $t \in [0.48, 0.5]$ respectively. These figures indicates that the frequency of these sign changes increases as t is close to T. Moreover, the number of these sign' changes increases as $\alpha \to 0$. In spite of this singular phenomenon, we observe again the invariance of the limit (λ_h, s_h) of the algorithm with respect to the initialization, consequence of the uniqueness of the minimal L^∞- norm control.

At the limit in $\alpha \to 0$, we expect a bounded amplitude λ_h but an arbitrarily large number of oscillations near T, in full agreement with our observations for the L^2 situation in [14]. The figures also confirm, in agreement with the optimality conditions for $\overline{J}_\alpha$ that, the optimal density is almost everywhere a characteristic function, so that no relaxation is needed. The only point where the density is not 0 or 1 is at the final time: From Figure 9, $s_h(T)$ is close to 1/2, intermediate value which reinforces the property, that at the limit in α, the null control highly oscillates at $t = T^-$.

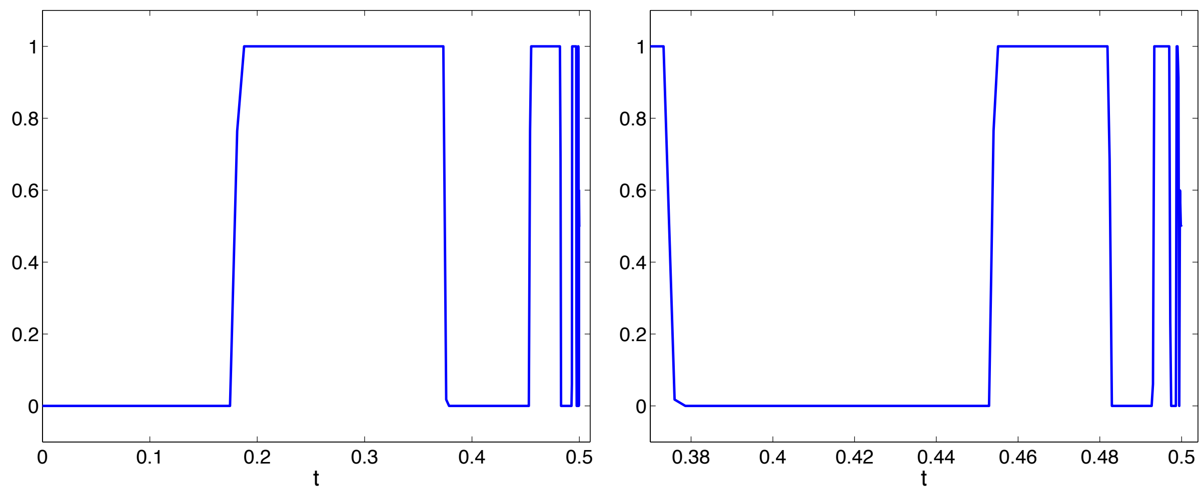

Figure 4: Neumann case - The optimal density $s_h(t)$ for $t \in [0, T]$ - $\alpha = 10^{-4}$.

For Dirichlet boundary control, the situation is very similar: we simply report here in Table 3 the optimal couple $(\lambda_h, \|y_h(T)\|_{L^2(0,1)})$ with respect to α.

α	10^{-2}	10^{-4}	10^{-6}	10^{-8}
$\|y_h(\cdot,T)\|_{L^2(\Omega)}$	9.21×10^{-2}	3.28×10^{-3}	7.31×10^{-4}	2.34×10^{-5}
λ_h	0.98	5.12	7.30	9.02
$\sharp$ iterates	21	319	4912	9301

Table 3: $(\lambda_h, \|y_h(T)\|_{L^2(0,1)})$ with respect to α in the Dirichlet boundary case.

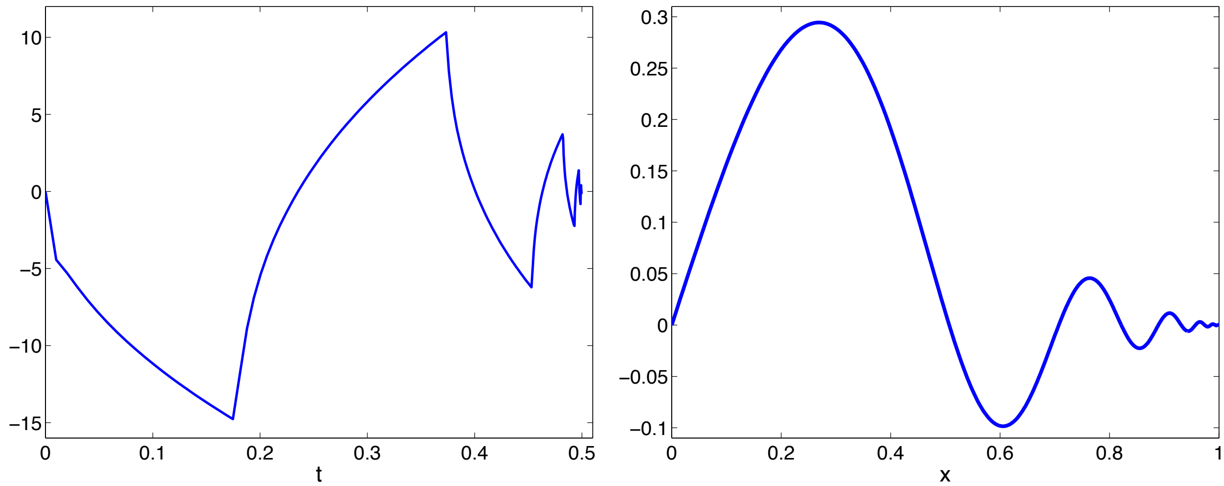

Figure 5: Neumann case - **Left** : $y_h(1,t)$ for $t \in (0,T)$; **Right**: $y_h(x,T)$ for $x \in (0,1)$ - $\alpha = 10^{-4}$.

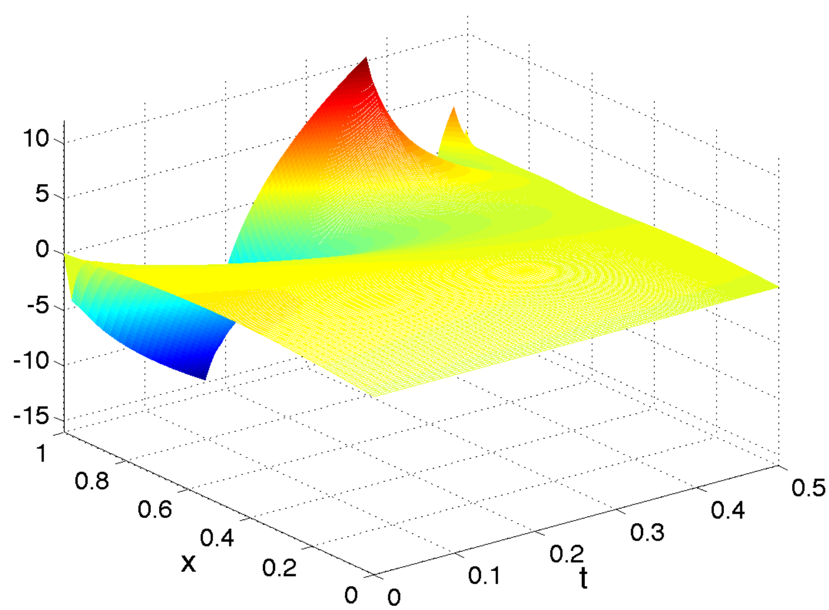

Figure 6: Neumann case - Approximated controlled solution y_h over Q_T

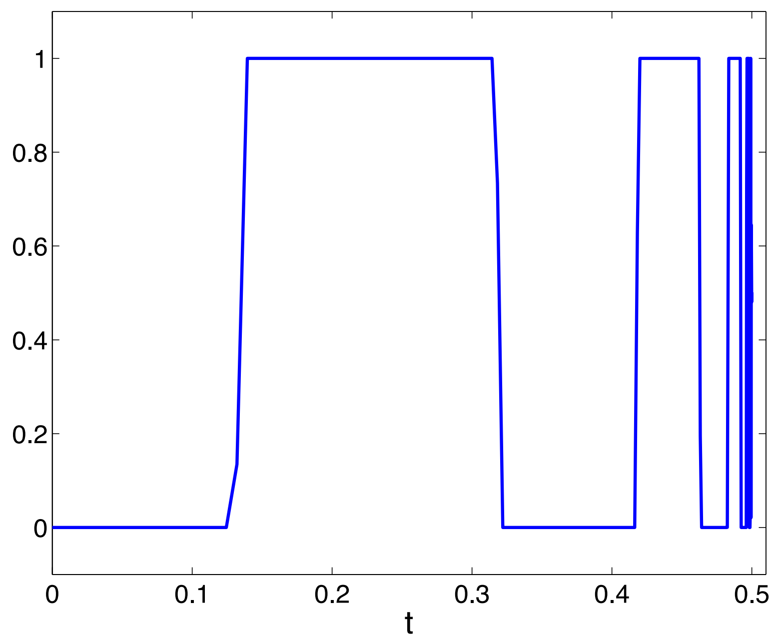

Figure 7: Neumann case - The optimal density $s_h(t)$ for $t \in [0, T]$ - $\alpha = 10^{-6}$.

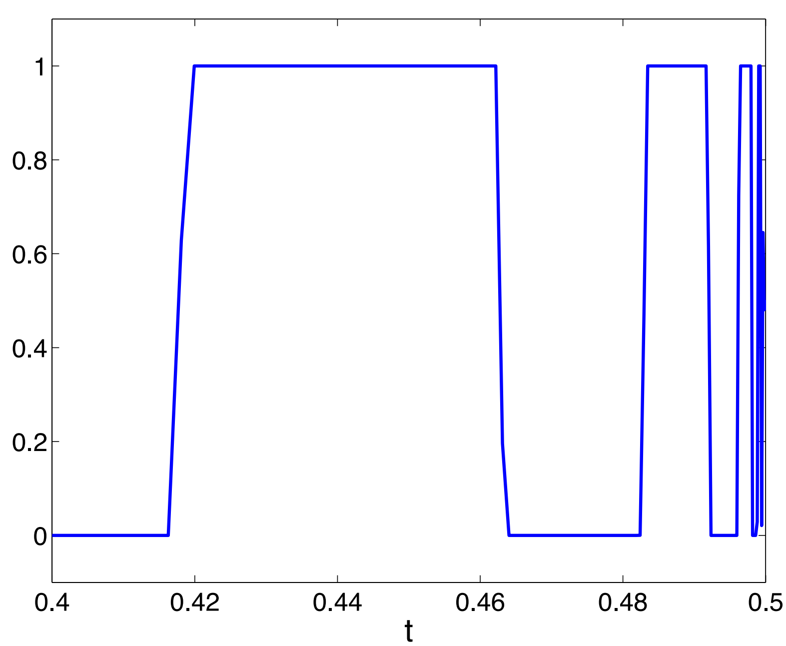

Figure 8: Neumann case - The optimal density $s_h(t)$ for $t \in [0.4, 0.5]$ - $\alpha = 10^{-6}$.

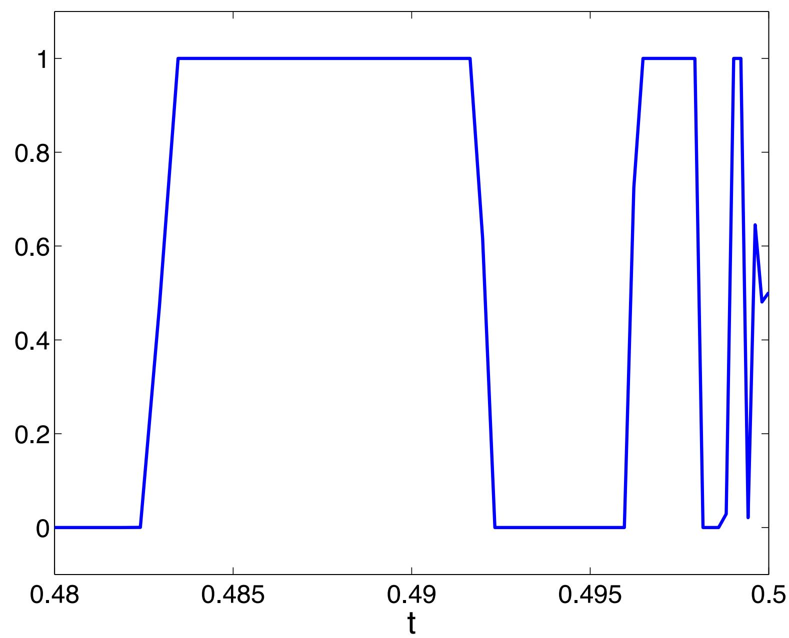

Figure 9: Neumann case - The optimal density $s_h(t)$ for $t \in [0.48, 0.5]$ for $\alpha = 10^{-6}$.

4 Conclusion and final remarks

The reformulation of the L^∞-controllability problem in terms of an optimal design one allows to get a well-posed relaxed formulation and then a simple minimization procedure. In spite of the well-posedness of this formulation, these bang-bang controls highly oscillate near the final time. While the amplitude of approximate controls is bounded by above with respect to the penalty parameter α, one may suspect that the number of these oscillations is not. This feature, closely related to the heat kernel regularization property, renders severally ill-posed the numerical approximation of the null control of L^∞ norm. Other procedures may be adopted, for instance taking into account the fact that the relaxed cost is quadratic with respect to the amplitude λ; however, they all share high degree of ill-posedeness. In that respect, the minimization of the relaxed cost is not easier than the minimization of the conjugate function (derived from duality arguments) commonly used. The subtle difference, strongly enhanced for small values of α, comes from the fact that the density s belongs to L^∞ while the dual variable degenerates in an abstract space.

On the other hand, it is also natural to consider the problem of finding the best location of the support of the control of minimal L^∞-norm. This problem has been recently addressed by the authors in the L^2-case (see [13]).

Finally, let us mention the wave type equation where the situation is different: in that case, existence of bang-bang control does not hold in general, as it depends on the initial data (see [8]). This means that, by applying the same procedure, some data may exhibit relaxation, that is, an optimal density taking values in (0,1).

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