Relaxation of an optimal design problem for the heat equation

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Abstract

We consider the heat equation in $(0,T) \times \Omega$, $\Omega \subset \mathbb{R}^N, N \geq 1$, and address the nonlinear optimal design problem which consists in finding the distribution in Ω of two given isotropic materials which minimizes a suitable cost functional depending on the heat flux. Both the case of a time-independent design and of the time-dependent one are analyzed. Well-posed relaxations of the two problems are obtained by using two well-known approaches: the homogenization method and the classical tools of non-convex, vector, variational problems. We also implement several numerical experiments based on these relaxed formulations to support the theoretical results. Finally, we point out some differences and analogies of the two proposed methods.

Résumé

Dans le cadre de l'équation de la chaleur posée sur le cylindre borné $(0,T) \times \Omega$, $\Omega \subset \mathbb{R}^N, N \geq 1$, on adresse le problème non linéaire de la distribution optimale de deux matériaux isotropes minimisant le flux de chaleur dans Ω. Les cas d'une distribution indépendente et dépendente du temps sont traités simultanément. Des formulations relaxées bien posées dans les deux cas sont obtenues en utilisant d'une part la méthode de l'homogénéisation et d'autre part l'approche variationnelle basée sur la mesure de Young. Enfin, plusieurs expériences numériques justifient les procédures de relaxation et permettent de confirmer les résultats théoriques.

Key words: Optimal design, Heat equation, Relaxation, Homogenization, Young Measure.

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1 Introduction - Problem Formulation

Optimal design problems in which the goal is to know the best way of mixing two different materials in order to optimize some physical quantity associated with the resulting structure

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have been extensively studied during the last decades, mainly in the case where the underlying state equation is elliptic. We refer the reader to [10, 16]. Among the techniques and tools used to deal with this type of problems, homogenization and variational formulations have played a very important role (see also [1, 2, 5, 18, 21]). More recently, optimal design problems for time-dependent designs and time-dependent state equations like the wave equation have been also considered ([11, 12, 13]). In particular, in [11] a class of spatial-temporal composite materials (rank-1 and rank-2 spatial-temporal laminates) were introduced. See also [12] for some physical examples. As far as we know, the case of the heat equation has been treated only from a more applied engineering point of view (see [22] and the references there in).

In this work, we aim to analyze two versions of a nonlinear optimal design problem for the heat equation. A first *time-independent* problem is

$$\text{(P)} \quad \text{Minimize in } \mathcal{X}: \quad J(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

where the state variable $u = u(t,x)$ is the solution of the system

$$\begin{cases} \beta(x)\,u'(t,x) - \text{div }(K(x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with

$$\begin{cases} \beta(x) = \mathcal{X}(x)\,\beta_1 + (1-\mathcal{X}(x))\,\beta_2, \\ K(x) = \mathcal{X}(x)\,k_1 I_N + (1-\mathcal{X}(x))\,k_2 I_N, \end{cases}$$

and the design variable $\mathcal{X} \in L^\infty(\Omega; \{0,1\})$ satisfies the volume constraint

$$\int_\Omega \mathcal{X}(x)\,dx = L\,|\Omega| \quad \text{for some fixed} \quad 0 < L < 1. \tag{2}$$

In an attempt to treat a more general situation for time-dependent designs, we will also examine the following *time-dependent* problem

$$(\text{P}_t) \quad \text{Minimize in } \mathcal{X}: \quad J_t(\mathcal{X}) = \frac{1}{2}\int_0^T\int_\Omega K(t,x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt$$

where the state variable $u = u(t,x)$ is the solution of the system

$$\begin{cases} (\beta(t,x)u(t,x))' - \text{div }(K(t,x)\nabla u(t,x)) = f(t,x) & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3}$$

with

$$\begin{cases} \beta(t,x) = \mathcal{X}(t,x)\,\beta_1 + (1-\mathcal{X}(t,x))\,\beta_2, \\ K(t,x) = \mathcal{X}(t,x)\,k_1 I_N + (1-\mathcal{X}(t,x))\,k_2 I_N, \end{cases}$$

and the design variable $\mathcal{X} \in L^\infty((0,T)\times\Omega; \{0,1\})$ satisfies the volume constraint

$$\int_\Omega \mathcal{X}(t,x)\,dx = L\,|\Omega| \quad \text{for some fixed} \quad 0 < L < 1, \text{ a.e. } t \in (0,T). \tag{4}$$

In both cases, we assume that $T > 0$ is a final time and $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded domain composed of two homogeneous, isotropic materials with mass densities $\rho_i > 0$, specific heats $c_i > 0$, and thermal conductivities $k_i > 0$, $i = 1, 2$ such that $k_1 \neq k_2$. We have put $\beta_i = \rho_i c_i$, $i = 1, 2$. I_N denotes the identity matrix of order N, f is a heat source, u_0 the initial temperature, and $u(t, x)$ the temperature at time t and position x. The design variable $\mathcal{X}$ is a characteristic function which indicates the region occupied by the first material (β_1, k_1). As a consequence, the condition (4) constraints the amount of this material that we have at our disposal.

As for the physical meaning of the cost function $J(\mathcal{X})$, it is a measure of the heat flux during the period of time $(0, T)$. Therefore, the design problem (P) consists in finding the optimal distribution of two different materials in order to minimize the gradient part of the energy for the heat equation. We recall that the energy at time T corresponding to the solution of (1) is defined by

$$E(T) = \frac{1}{2}\int_{\Omega} \beta(x) u^2(T, x)\, dx + \frac{1}{2}\int_0^T\int_{\Omega} K(x) \nabla u(t, x) \cdot \nabla u(t, x)\, dxdt. \tag{5}$$

The same optimal design problem, but with a cost function depending only on the temperature u, was considered in [22] where the numerical simulations suggest the non-existence of optimal designs in the class of characteristic functions. The optimal design is then found in the form of a composite material. For the steady-state case, a counterexample on the non-existence of solutions may be found in [1, p. 206-211]. Relaxation is the appropriate way of dealing mathematically with this type of situations. This basically consists in replacing the original problem by another suitable one which has (at least) a minimizer and, in addition, the optimal cost associated with this new problem coincides with the infimum of the original one. The process is successfully completed whenever we are able to find out the behavior of some minimizing sequences of the original problem from the information codified in the minimizers of the relaxed one.

As indicated above, the homogenization method and the classical tools of non-convex variational problems (in particular, Young measures) are, for the moment, two of the most popular approaches in the mathematical literature to analyze this type of optimal design problems. For solving (P) we use in a standard way the homogenization method which seems to be very suitable to deal with time-independent designs. For the contrary, the time-dependent case (P_t) is analyzed by using the second method (in particular, the concepts of quasi-convexification and div-curl Young measures). This approach is very well adapted to the new ingredient of time-dependence of designs. Our study will be not limited to theoretical results. We shall implement several numerical experiments based on both procedures in the two dimensional case. With the analysis of the two versions of the same problem (time-independent and time-dependent), we stress in this scenario the complementariety of the two approaches: homogenization for the time-independent optimal design problem, and Young measures for the time-dependent situation.

The rest of the paper is organized as follows. In Section 2, after an overview of standard results in Homogenization theory, we associate with problem (P) a well-posed relaxation (RP) using this approach (see Theorem 2.4). Then, in Section 3, we use a variational formulation and the notion of div-curl Young measure introduced recently in [20] to derive a relaxation (RP_t) of (P_t) (see Theorem 3.1). A deeper analysis of (RP_t) then leads to conjecture **that the two relaxed problems (RP) and ($\mathbf{RP}_t$) share the property of time-independence of the optimal local volume fraction, and that the harmonic mean plays a prominent role in ($\mathbf{RP}_t$)**. However, in (RP) the microgeometry of the optimal composite is time-independent, but this is not the case for (RP_t) where optimal composites are found in the form of time-dependent first-order laminates. In addition, we conjecture (as has been just indicated) that this optimal composite is given by the harmonic mean of the two phases. Several numerical experiments in Section 4 support this conjecture. Finally, we would like to emphasize that this work is but a first step towards a better understanding of design problems for parabolic equations, and the relationship between these two points of view. There is still a lot of work to be done. Some interesting open questions are listed in Section 5.

2 The Homogenization Method

We obtain in this section a suitable relaxation for the optimal design problem (P). We focus on the homogenization method. In order to make this section easier to read we first collect some well-known results. Relaxation will follow directly from these results. Throughout this section, we denote by $\mathcal{X}_n \in L^\infty(\Omega;\{0,1\})$, $n = 1, 2, \cdots$, a sequence of characteristic functions and by $K_n \in \mathcal{M}^{N\times N}$ a sequence of tensors of the form

$$K_n = \mathcal{X}_n(x)\, k_1 I_N + (1 - \mathcal{X}_n(x))\, k_2 I_N, \tag{6}$$

with $k_1, k_2 > 0$.

2.1 General results on homogenization

The material of this subsection has been taken from [1, Chap. 1 and 2] and [3].

Homogenization is based on the concept of $H-$convergence. Precisely, a sequence of tensors $\{K_n(x)\}_{n\in\mathbb{N}}$ $H-$converges to the tensor $K^* \in L^\infty\left(\Omega;\mathcal{M}^{N\times N}\right)$ if for any $f \in H^{-1}(\Omega)$ the sequence of solutions $u_n \in H_0^1(\Omega)$ of

$$\begin{cases} -\text{div }(K_n \nabla u_n) = f & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

satisfies

$$\begin{cases} u_n \rightharpoonup u & \text{weak in } H_0^1(\Omega), \\ K_n \nabla u_n \rightharpoonup K^* \nabla u & \text{weak in } \left(L^2(\Omega)\right)^N, \end{cases}$$

where u is the solution of the homogenized system

$$\begin{cases} -\text{div }(K^* \nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We shall write $K_n \overset{\text{H}}{\to} K^*$ to indicate this kind of convergence.

Assume now that there exists $\theta \in L^\infty(\Omega;[0,1])$ and $K^* \in L^\infty\left(\Omega;\mathcal{M}^{N\times N}\right)$ such that

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \text{weak } \star \text{ in } L^\infty(\Omega), \\ K_n \overset{\text{H}}{\to} K^*. \end{cases}$$

The $H-$limit K^* is said to be the homogenized or effective tensor of two isotropic materials obtained by mixing k_1 and k_2 in proportions θ and $1-\theta$, respectively, with a microstructure defined by $\mathcal{X}_n$.

As we will see later on, it is very important to identify all possible homogenized tensors obtained by mixing two given materials with all possible micro-structures. This is the so-called $G-$closure problem. Precisely, we have the following definition.

Definition 2.1 *Given $\theta \in L^{\infty}(\Omega;[0,1])$, the $G_\theta-$closure of two isotropic materials is defined as the set of tensors $K^* \in L^{\infty}\left(\Omega;\mathcal{M}^{N\times N}\right)$ such that there exist $\mathcal{X}_n \in L^{\infty}(\Omega;\{0,1\})$ and K_n of the form (6) satisfying*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \textit{weak } \star \textit{ in } L^{\infty}(\Omega), \\ K_n \xrightarrow{H} K^*. \end{cases}$$

Fortunately, for the case of two isotropic materials, the $G_\theta-$closure is well-known.

Theorem 2.2 *Given $\theta \in L^{\infty}(\Omega;[0,1])$, the $G_\theta-$closure of two isotropic materials $k_i > 0$, $i = 1, 2$, is the set of all symmetric matrices with eigenvalues $\lambda_1, \cdots, \lambda_N$ satisfying*

$$\begin{cases} \lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+, \qquad 1 \leq j \leq N, \\ \sum_{j=1}^N \frac{1}{\lambda_j - k_1} \leq \frac{1}{\lambda_\theta^- - k_1} + \frac{N-1}{\lambda_\theta^+ - k_1}, \\ \sum_{j=1}^N \frac{1}{k_2 - \lambda_j} \leq \frac{1}{k_2 - \lambda_\theta^-} + \frac{N-1}{k_2 - \lambda_\theta^+}, \end{cases}$$

where $\lambda_\theta^- = \left(\frac{\theta}{k_1} + \frac{1-\theta}{k_2}\right)^{-1}$ is the harmonic mean and $\lambda_\theta^+ = \theta k_1 + (1-\theta) k_2$ the arithmetic mean of (k_1, k_2).

We conclude this section with an homogenization result for the heat equation (see [3, Th. 7.1] for the proof). We also refer to [3, Th. 6.1] for the existence and uniqueness of solutions for system (1).

Theorem 2.3 *Let $\mathcal{X}_n \in L^{\infty}(\Omega;\{0,1\})$ and let K_n be of the form (6). Assume that*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \textit{weak } \star \textit{ in } L^{\infty}(\Omega), \\ K_n \xrightarrow{H} K^*. \end{cases}$$

Consider the system

$$\begin{cases} \beta_n(x)\, u_n'(t,x) - div\ (K_n(x)\nabla u_n(t,x)) = f(t,x) & in \quad (0,T)\times\Omega, \\ u_n = 0 & on \quad (0,T)\times\partial\Omega, \\ u_n(0,x) = u_0(x) & in \quad \Omega, \end{cases}$$

where $\beta_n = \mathcal{X}_n\beta_1 + (1-\mathcal{X}_n)\beta_2$, with $\beta_1, \beta_2 > 0$, $f \in L^2((0,T)\times\Omega)$ and $u_0 \in L^2(\Omega)$. Then

$$\int_0^T\int_\Omega K_n(x)\nabla u_n(t,x)\cdot\nabla u_n(t,x)\,dxdt \to \int_0^T\int_\Omega K^*(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt, \tag{7}$$

u being the solution of the limit system

$$\begin{cases} \beta(x)\, u'(t,x) - div\ (K^*(x)\nabla u(t,x)) = f(t,x) & in \quad (0,T)\times\Omega, \\ u = 0 & on \quad (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & in \quad \Omega, \end{cases} \tag{8}$$

with $\beta = \theta\beta_1 + (1-\theta)\beta_2$.

2.2 Relaxation by the homogenization method

As indicated in the introduction, problem (P) is usually ill-posed in the sense that there are no minimizers in the space of *classical designs*

$$\mathbf{CD} = \{\mathcal{X} \in L^\infty(\Omega;\{0,1\}) : \mathcal{X} \text{ satisfies } (4)\}.$$

The idea of relaxation basically consists in considering a larger class of admissible designs with the hope that the optimal design problem to be well-posed in this new class of designs. Having this in mind and based on Theorem 2.2 we introduce the space of *relaxed designs*

$$\mathbf{RD} = \left\{(\theta, K^*) \in L^\infty\left(\Omega;[0,1]\times\mathcal{M}^{N\times N}\right) : K^*(x) \in G_{\theta(x)} \text{ a.e. } x \in \Omega \text{ and } \theta \text{ satisfies } (4)\right\},$$

where $G_{\theta(x)}$ is as in Theorem 2.2.

From Theorem 2.3 is then natural to consider, for $(\theta, K^*) \in \mathbf{RD}$**,** the *relaxed cost*

$$J^*(\theta, K^*) = \frac{1}{2}\int_0^T\int_\Omega K^*(x)\nabla u(t,x)\cdot\nabla u(t,x)\,dxdt \tag{9}$$

where u is the solution of (8), and then to introduce the optimal design problem

$$\text{(RP)} \quad \text{Minimize in } (\theta, K^*) \in \mathbf{RD}: \quad J^*(\theta, K^*).$$

We have the following main result.

THEOREM 2.4 *(RP) is a relaxation of (P) in the sense that*

(i) *there exists at least one minimizer for (RP) in the space* $\mathbf{RD}$*,*

(ii) *up to a subsequence, every minimizing sequence of classical designs $\mathcal{X}_n$ converges, weakly $\star$ in $L^\infty(\Omega;[0,1])$, to a relaxed density θ, and its associated sequence of tensors*

$$K_n = \mathcal{X}_n k_1 I_N + (1-\mathcal{X}_n)k_2 I_N$$

$H-$converges to an effective tensor K^* such that (θ, K^*) is a minimizer for (RP), and

(iii) *conversely, every relaxed minimizer $(\theta, K^*) \in \mathbf{RD}$ of (RP) is attained by a minimizing sequence $\mathcal{X}_n$ of (P) in the sense that*

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \textit{weak } \star \textit{ in } L^\infty(\Omega), \\ K_n \xrightarrow{H} K^*. \end{cases}$$

Proof. The proof of this result follows the same lines as in the static case (see [1, p.p. 213-215]). Anyway, we include it here for completeness.

Let $\mathcal{X}_n$ be a minimizing sequence for (P). Since $\|\mathcal{X}_n\|_{L^\infty(\Omega)} \le 1$, there exists a subsequence, still denoted by $\mathcal{X}_n$, such that

$$\mathcal{X}_n \rightharpoonup \theta_\infty \quad \text{weak } \star \text{ in } L^\infty(\Omega).$$

Moreover, since $\mathcal{X}_n$ satisfies the volume constraint (4) and $\mathcal{X}_n \rightharpoonup \theta_\infty$ weak $\star$,

$$\int_\Omega \theta_\infty(x)\,dx = L|\Omega|.$$

On the other hand, thanks to the compactness of the sequence of tensors K_n with respect to $H-$convergence, up to a subsequence, there exists $K_\infty \in L^\infty\left(\Omega; \mathcal{M}^{N\times N}\right)$ such that $K_n \overset{\text{H}}{\to} K_\infty$. From Theorem 2.3 it follows that

$$\lim_{n\to\infty} J\left(\mathcal{X}_n\right) = J^*\left(\theta_\infty, K_\infty\right).$$

This proves that

$$m = \inf_{\mathcal{X}} J\left(\mathcal{X}\right) = J^*\left(\theta_\infty, K_\infty\right). \tag{10}$$

Now let (θ, K^*) be a relaxed design. By the definition of the set G_θ, there exists $\mathcal{X}_n \in L^\infty\left(\Omega; \{0,1\}\right)$ such that

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \text{weak } \star \text{ in } L^\infty\left(\Omega\right), \\ K_n \overset{\text{H}}{\to} K^*. & \end{cases}$$

In particular,

$$\lim_{n\to\infty} \int_\Omega \mathcal{X}_n\left(x\right) dx = \int_\Omega \theta\left(x\right) dx = L\left|\Omega\right|,$$

but in principle each individual $\mathcal{X}_n$ does not satisfy the volume constraint (4). Nevertheless, this difficulty may be overcome (see Proposition 2.1). So, assume that $\mathcal{X}_n$ is admissible for (P). By using again Theorem 2.3,

$$J^*\left(\theta, K^*\right) = \lim_{n\to\infty} J\left(\mathcal{X}_n\right) \geq m.$$

Combining this inequality with (10) we obtain that $(\theta_\infty, K_\infty)$ is a minimizer for (RP). This proves (i) and (ii).

Finally, to prove (iii), let $(\theta, K^*) \in \mathbf{RD}$ be a minimizer for (RP). From the definition of G_θ it follows that there exists $\mathcal{X}_n \in L^\infty\left(\Omega; \{0,1\}\right)$, which may be assumed to satisfy (4), such that

$$\begin{cases} \mathcal{X}_n \rightharpoonup \theta & \text{weak } \star \text{ in } L^\infty\left(\Omega\right) \\ K_n \overset{\text{H}}{\to} K^*, & \end{cases}$$

where K_n is the sequence of tensors defined by (6). As before, we also have $J\left(\theta, K^*\right) = \lim_{n\to\infty} J\left(\mathcal{X}_n\right)$. Obviously, this implies that $\mathcal{X}_n$ is minimizing for (P). ■

Proposition 2.1 *Let $\mathcal{X}_n \in L^\infty\left(\Omega; \{0,1\}\right)$ be such that*

$$\begin{cases} (i) & \mathcal{X}_n \rightharpoonup \theta \quad \textit{weak}\star \ \textit{ in } L^\infty\left(\Omega\right), \\ (ii) & \lim_{n\to\infty} \int_\Omega \mathcal{X}_n\left(x\right) dx = L\left|\Omega\right|, \ \textit{ and} \\ (iii) & K_n \overset{H}{\to} K, \ \textit{ where } K_n \textit{ is as in (6).} \end{cases}$$

Then there exists $\overline{\mathcal{X}}_n \in L^\infty\left(\Omega; \{0,1\}\right)$ such that

$$\begin{cases} (a) & \overline{\mathcal{X}}_n \rightharpoonup \theta \quad \textit{weak}\star \ \textit{ in } L^\infty\left(\Omega\right), \\ (b) & \int_\Omega \overline{\mathcal{X}}_n\left(x\right) dx = L\left|\Omega\right| \ \textit{ for all } n \in \mathbb{N}, \textit{ and} \\ (c) & \overline{K}_n \overset{H}{\to} K, \ \textit{ where } \overline{K}_n \textit{ is as in (6) for } \overline{\mathcal{X}}_n. \end{cases}$$

Proof. From (ii) we may construct a sequence of characteristic functions $\overline{\mathcal{X}}_n$ such that (b) holds and, in addition, the sequence of sets

$$\Omega_n = \left\{x \in \Omega : \mathcal{X}_n (x) \neq \overline{\mathcal{X}}_n (x)\right\}$$

satisfies

$$|\Omega_n| \to 0 \quad \text{as } n \to \infty. \tag{11}$$

From this, it is not difficult to see that, up to a subsequence, not relabelled, we have the convergence stated in (a).

Finally, let us denote by $\overline{K}$ the H-limit of (a subsequence of) $\overline{K}_n$. Again, from (11) and thanks to the locality of H-convergence (see [1, Prop. 1.4.5] or [6, Th. 13.4 (ii)]) it follows that

$$K (x) = \overline{K} (x) \quad \text{a.e.} \quad x \in \Omega,$$

which completes the proof. ■

Theorem 2.4 gives us a relaxation of the original optimal design problem in which we have replaced the original state equation (1) by the relaxed one (8), this last system being written in terms of the homogenized tensor K^* for which we have the information that comes from Theorem 2.2. In the one-dimensional case, we have an explicit expression for the optimal tensor:

Remark 1 *In the 1-D case, the effective coefficient K^* is explicitly known. Indeed, from Theorem 2.2 it follows that K^* equals the harmonic mean, that is,*

$$K^* (x) = \frac{k_1 k_2}{\theta (x) k_2 + (1 - \theta (x)) k_1}, \quad x \in \Omega.$$

Hence, the relaxed problem (RP) has the simpler form

$$\textit{Minimize in } \theta : \quad J^*(\theta) = \frac{1}{2} \int_0^T \int_\Omega \frac{k_1 k_2}{\theta (x) k_2 + (1 - \theta (x)) k_1} |u_x (t, x)|^2 \, dxdt$$

subject to

$$\begin{cases} (\theta\beta_1 + (1 - \theta) \beta_2) u' - \left(\frac{k_1 k_2}{\theta k_2 + (1-\theta) k_1} u_x\right)_x = f & \textit{in} \quad (0, T) \times \Omega, \\ u = 0 & \textit{in} \quad (0, T) \times \partial\Omega, \\ u (0, x) = u_0 (x) & \textit{in} \quad \Omega, \\ \theta \in L^\infty (\Omega; [0, 1]) , \quad \int_\Omega \theta (x) \, dx = L |\Omega| . \end{cases}$$

Once the existence of optimal relaxed designs has been proved in Theorem 2.4, we stop here our study based on the Homogenization method. We will go back to it in the section devoted to the numerical resolution of the relaxed problem (RP).

3 A Young Measure Approach

We now analyze problem (P_t) from a different perspective. Precisely, we use the so-called div-curl Young measures as a key tool. We refer the reader to [8, 15, 20] for the main properties of this class of measures and some applications to optimal design in conductivity and stabilization in linear elasticity.

3.1 Div-curl Young measure associated with problem (P$_t$)

To begin with, we rewrite the heat equation in system (1) in divergence-free form

$$\mathrm{div}_{(t,x)}\left[\left(-\beta\left(t,x\right)u\left(t,x\right),K\left(t,x\right)\nabla u\left(t,x\right)\right)+F(t,x)\right]=0 \tag{12}$$

where the $\mathrm{div}_{(t,x)}$ operator now includes the time variable t as the first variable and $F(t,x)$ is a vector field such that $div_{(t,x)}F=f$. Since F will not play an important role, we put $F=0$ for simplicity throughout this section. However, all the results that follow hold true for F (and therefore f) different from zero.

For $u_0\in H_0^1\left(\Omega\right)$, an *integral* solution (or solution in the Young measure sense) of (12) exists (see [9, Section 6]). Precisely, we recall that

$$u\in L^{\infty}\left(\left(0,T\right);H_0^1\left(\Omega\right)\right)\quad\text{with } u'\in L^2\left(\left(0,T\right)\times\Omega\right)$$

is said to be an integral solution of (12) if this equation is satisfied in $H^{-1}\left(\left(0,T\right)\times\Omega\right)$ and the initial and boundary conditions also hold.

Now let $\mathcal{X}_n$ be an admissible sequence of designs for (P$_t$) and let u_n be its corresponding sequence of integral solutions. Consider the two sequences of vector fields

$$\begin{cases} G_n\left(t,x\right)=\left(-\left(\mathcal{X}_n\left(t,x\right)\beta_1+\left(1-\mathcal{X}_n\left(t,x\right)\right)\beta_2\right)u_n\left(t,x\right),\ K_n\left(t,x\right)\nabla u_n\left(t,x\right)\right), \\ H_n\left(t,x\right)=\left(u_n'\left(t,x\right),\ \nabla u_n\left(t,x\right)\right). \end{cases} \tag{13}$$

Since both sequences G_n and H_n are uniformly bounded in $\left(L^2\left(\left(0,T\right)\times\Omega\right)\right)^{N+1}$, we may associate with (a subsequence of) the pair $\left(G_n,H_n\right)$ a family of parameterized measures $\nu=\left\{\nu_{(t,x)}\right\}_{(t,x)\in(0,T)\times\Omega}$. Note also that the pair $\left(G_n,H_n\right)$ satisfies

$$\mathrm{div}_{(t,x)}G_n=0\quad\text{and}\quad\mathrm{curl}\ H_n=0.$$

For this reason, the measure ν is called a div-curl Young measure. We know that such class of measures enjoy the commutation property (see [20])

$$\int_{\mathbf{R}^{N+1}\times\mathbf{R}^{N+1}}\rho\cdot\lambda\,d\nu_{(t,x)}(\rho,\lambda)=\int_{\mathbf{R}^{N+1}\times\mathbf{R}^{N+1}}\rho\,d\nu_{(t,x)}(\rho,\lambda)\cdot\int_{\mathbf{R}^{N+1}\times\mathbf{R}^{N+1}}\lambda\,d\nu_{(t,x)}(\rho,\lambda), \tag{14}$$

which is a direct consequence of the div-curl lemma (see [21]). We also notice that by Aubin's lemma,

$$u_n\to u\quad\text{strong in } L^2\left(\left(0,T\right)\times\Omega\right).$$

Due to the particular form of $\left(G_n,H_n\right)$, each individual $\nu_{(t,x)}$ is supported in the union of the two linear manifolds

$$\Lambda_i=\left\{(\rho,\lambda)\in\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}:\rho_1=-\beta_i u,\quad\overline{\rho}=k_i\overline{\lambda}\right\},\quad i=1,2 \tag{15}$$

where

$$\rho=(\rho_1;\overline{\rho})\in\mathbb{R}\times\mathbb{R}^N\quad\text{and}\quad\lambda=\left(\lambda_1;\overline{\lambda}\right)\in\mathbb{R}\times\mathbb{R}^N. \tag{16}$$

Hence, the measure $\nu_{(t,x)}$ may be written as

$$\nu_{(t,x)}=\theta\left(t,x\right)\nu_{1,(t,x)}+\left(1-\theta\left(t,x\right)\right)\nu_{2,(t,x)}, \tag{17}$$

with supp $\nu_{i,(t,x)}\subset\Lambda_i$, $i=1,2$. The meaning of the manifolds Λ_i, $i=1,2,$, and in particular of the dummy variables (ρ,λ), follows from the fact that the measure $\nu_{(t,x)}$ gives the limiting probability distribution as $n \to \infty$ of the values of (G_n, H_n) near the point (t, x). See [17, Ch. 1] for more details.

The importance of having more information on this measure is the following. Suppose that $\mathcal{X}_n$ is a minimizing sequence for (P_t) with the property that $|\nabla u_n|^2$ is equi-integrable (note than only spatial derivatives are involved in the cost functional). Then, by the fundamental property of Young measures (see [17, Th. 6.2]), we may represent the limit of the costs associated with $\mathcal{X}_n$ through the measure ν. Precisely,

$$\lim_{n\to\infty} J_t(\mathcal{X}_n) = \frac{1}{2}\int_0^T\int_\Omega \left[k_1\theta(t,x)\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}^{(2)}_{1,(t,x)} + k_2(1-\theta(t,x))\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}^{(2)}_{2,(t,x)}\right] dxdt \tag{18}$$

where $\overline{\nu}^{(2)}_{i,(t,x)}$, $i = 1, 2$ stands for the projection of $\nu_{i,(t,x)}$ onto the last $N-$components of the second copy of $\mathbb{R}^{N+1}$. Therefore, with each minimizing sequence of the original problem (P_t) we associate an optimal div-curl Young measure. Our goal is to understand the structure of this measure.

3.2 Variational reformulation and relaxation

We now proceed to the analysis of problem (P_t) in a similar fashion as in the stationary case [20]. First step in this process is to put (P_t) into a variational setting. So, we consider the functions

$$W(\rho,\lambda) = \begin{cases} k_1\left|\overline{\lambda}\right|^2 & \text{if } (\rho,\lambda)\in\Lambda_1, \\ k_2\left|\overline{\lambda}\right|^2 & \text{if } (\rho,\lambda)\in\Lambda_2, \\ +\infty & \text{else,} \end{cases} \tag{19}$$

and

$$V(\rho,\lambda) = \begin{cases} 1 & \text{if } (\rho,\lambda)\in\Lambda_1, \\ 0 & \text{if } (\rho,\lambda)\in\Lambda_2, \\ +\infty & \text{else.} \end{cases} \tag{20}$$

Then we associate with problem (P_t) the equivalent variational problem

$$(\mathrm{VP}_t)\quad \text{Minimize in } (G,u):\quad \frac{1}{2}\int_0^T\int_\Omega W\left(G(t,x), \nabla_{(t,x)}u(t,x)\right) dxdt$$

subject to

$$\begin{cases} G\in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1((0,T)\times\Omega;\mathbb{R}), \\ \operatorname{div}_{(t,x)}G = 0 & \text{in } H^{-1}((0,T)\times\Omega), \\ u|_{\partial\Omega} = 0 \quad \text{a. e. } t\in[0,T], & u(0) = u_0 \quad \text{in } \Omega, \\ \int_\Omega V(G(t,x),\nabla u(t,x))\,dx = L|\Omega| & \text{a. e. } t\in[0,T]. \end{cases}$$

The crucial step in this approach is the computation of the constrained quasi-convexification CQW of the density W because it provides us with a relaxation of (VP_t). We remind that as is usual in non-convex vector variational problems, a full relaxation of this type of problems is obtained by replacing the original density W by its constrained quasi-convex envelope (see [18, 20] and the references there in). So, we concentrate on the computation of this new relaxed density.

For fixed $(\theta,\rho,\lambda)\in[0,1]\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$ the constrained quasi-convex density $CQW(\theta,\rho,\lambda)$ is computed by solving the problem in measures

$$(\mathrm{MP}_t)\ \text{ Minimize in } \nu: \quad CQW(\theta,\rho,\lambda)=k_1\theta\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}_1^{(2)}+k_2(1-\theta)\int_{\mathbb{R}^N}\left|\overline{\lambda}\right|^2 d\overline{\nu}_2^{(2)}$$

subject to

$$\begin{cases}\nu=\theta\nu_1+(1-\theta)\nu_2, \text{ with supp } \nu_i\subset\Lambda_i,\ i=1,2,\\ \\ \nu \text{ is a div-curl Young measure verifying the commutation property associated with (14), and}\\ \\ \rho=\int_{\mathbb{R}^{N+1}}yd\nu^{(1)}(y), \quad \lambda=\int_{\mathbb{R}^{N+1}}zd\nu^{(2)}(z), \text{ with } \nu^{(i)} \text{ the two marginals.}\end{cases}$$

We notice that after solving (MP_t) we plan to use the localization principle for div-curl Young measures (see [20]) to analyze the optimal cost given by (18). In fact, for almost everywhere $(t,x)\in(0,T)\times\Omega$, we have the identification $\theta=\theta(t,x)$, $\rho=G(t,x)$ and $\lambda=H(t,x)$, where G and H are the weak limits of G_n and H_n, respectively.

From the expression of the first moment of ν and taking into account (15), (16) and (17), it follows that

$$\begin{cases}\rho_1=-(\theta\beta_1+(1-\theta)\beta_2)u,\\ \\ \overline{\rho}=k_1\theta\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_1^{(2)}+k_2(1-\theta)\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_2^{(2)},\\ \\ \lambda_1=\theta\int_{\Lambda_1}y_1d\nu_1+(1-\theta)\int_{\Lambda_2}y_1d\nu_2,\\ \\ \overline{\lambda}=\theta\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_1^{(2)}+(1-\theta)\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_2^{(2)}.\end{cases}$$

On the other hand, the commutation condition (14) on ν implies that

$$\int_{\Lambda_1\cup\Lambda_2}y\cdot z\ d\nu(y,z)=\rho\cdot\lambda.$$

Developing the left-hand side of this expression,

$$\begin{aligned}\int_{\Lambda_1\cup\Lambda_2}y\cdot z\ d\nu(y,z) &= -\theta\beta_1u\lambda_1^1-(1-\theta)\beta_2u\lambda_1^2\\ &\quad +k_1\theta\int_{\mathbb{R}^N}|\overline{y}|^2d\overline{\nu}_1^{(2)}+k_2(1-\theta)\int_{\mathbb{R}^N}|\overline{y}|^2d\overline{\nu}_2^{(2)}.\end{aligned}$$

Next, we introduce the second moments

$$s_1=\int_{\mathbb{R}^N}|\overline{y}|^2d\overline{\nu}_1^{(2)} \quad\text{and}\quad s_2=\int_{\mathbb{R}^N}|\overline{y}|^2d\overline{\nu}_2^{(2)}.$$

If we put

$$\lambda_1^1=\int_{\Lambda_1}y_1d\nu_1,\quad \lambda_1^2=\int_{\Lambda_2}y_1d\nu_2,\quad \overline{\lambda}_1=\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_1^{(2)}\quad\text{and}\quad \overline{\lambda}_2=\int_{\mathbb{R}^N}\overline{y}d\overline{\nu}_2^{(2)},$$

then we can write the conditions we have in the form

$$\begin{cases}\overline{\rho}=k_1\theta\overline{\lambda}_1+k_2(1-\theta)\overline{\lambda}_2,\\ \\ \overline{\lambda}=\theta\overline{\lambda}_1+(1-\theta)\overline{\lambda}_2, \qquad \lambda_1=\theta\lambda_1^1+(1-\theta)\lambda_1^2,\\ \\ k_1\theta s_1+k_2(1-\theta)s_2-\rho\cdot\lambda=\theta\beta_1u\lambda_1^1+(1-\theta)\beta_2u\lambda_1^2.\end{cases}$$

The first two equations can be used to solve for $\overline{\lambda}_1$ and $\overline{\lambda}_2$, namely,

$$\overline{\lambda}_1 = \frac{1}{\theta\left(k_1-k_2\right)}\left(\overline{\rho}-k_2\overline{\lambda}\right), \qquad \overline{\lambda}_2 = \frac{1}{\left(1-\theta\right)\left(k_2-k_1\right)}\left(\overline{\rho}-k_1\overline{\lambda}\right).$$

The other two equations can also be used to solve for λ_1^1 and λ_1^2. If $u \neq 0$, then

$$\begin{cases} \lambda_1^1 = -\frac{1}{u\theta(\beta_1-\beta_2)}\left(\theta\left(\beta_2-\beta_1\right)u\lambda_1 + \overline{\rho}\cdot\overline{\lambda} - \left[k_1\theta s_1 + k_2(1-\theta)s_2\right]\right), \\ \lambda_1^2 = -\frac{1}{u(1-\theta)(\beta_1-\beta_2)}\left(\left(1-\theta\right)\left(\beta_2-\beta_1\right)u\lambda_1 + \overline{\rho}\cdot\overline{\lambda} - \left[k_1\theta s_1 + k_2(1-\theta)s_2\right]\right). \end{cases}$$

For $u = 0$ there is an infinity of possibilities for λ_1^1 and λ_1^2, namely

$$\lambda_1^1 = \gamma, \qquad \lambda_1^2 = \frac{1}{(1-\theta)}\left(\lambda_1 - \theta\gamma\right)$$

with any $\gamma \in \mathbb{R}$. With all of these notations, (MP$_t$) reads in the simpler form:

$$\text{Minimize in } (s_1, s_2): \quad k_1\theta s_1 + k_2\left(1-\theta\right)s_2$$

subject to

$$s_1 \geq \frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta^2\left(k_1-k_2\right)^2}, \qquad s_2 \geq \frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{\left(1-\theta\right)^2\left(k_2-k_1\right)^2}$$

where the two inequalities appearing in the constraints are a consequence of Jensen's inequality.

It is elementary to realize that the minimum of this problem is attained for

$$s_1 = \frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta^2\left(k_1-k_2\right)^2} \quad \text{and} \quad s_2 = \frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{\left(1-\theta\right)^2\left(k_2-k_1\right)^2},$$

and thus,

$$CQW\left(\theta,\rho,\lambda\right) \geq k_1\frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta\left(k_1-k_2\right)^2} + k_2\frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{\left(1-\theta\right)\left(k_2-k_1\right)^2}$$

Here $\rho = (\rho_1, \overline{\rho})$ and $\lambda = (\lambda_1, \overline{\lambda})$.

Our next task is to see if this lower bound can be attained by a first-order div-curl laminate (see [20] for the definition and main properties of this subclass of div-curl Young measures). This would give us more information on the minimizing sequences of (VP$_t$).

Note that due to the strict convexity of $|\cdot|^2$, the equality in Jensen's inequality holds if and only if the associated measure is a Dirac mass in the corresponding components, that is,

$$\overline{\nu}_1^{(2)} = \delta_{\frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}} \quad \text{and} \quad \overline{\nu}_2^{(2)} = \delta_{\frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}}.$$

Moreover, since supp $\nu_i \subset \Lambda_i$, $i = 1, 2$, the projection of ν_i onto the last $N-$components of the first copy of $\mathbb{R}^{N+1}$ has the form

$$\overline{\nu}_1^{(1)} = \delta_{k_1\frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}} \quad \text{and} \quad \overline{\nu}_2^{(1)} = \delta_{k_2\frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}}.$$

So, the optimal first-order laminate we are looking for looks like

$$\nu = \theta\delta_{\left(-\beta_1 u, k_1\frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}; \lambda_1^1, \frac{\overline{\rho}-k_2\overline{\lambda}}{\theta(k_1-k_2)}\right)} + \left(1-\theta\right)\delta_{\left(-\beta_2 u, k_2\frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}; \lambda_1^2, \frac{\overline{\rho}-k_1\overline{\lambda}}{(1-\theta)(k_2-k_1)}\right)}. \tag{21}$$

Notice that each one of these two mass points belongs to one of the two manifolds Λ_i, so that the fine, one-dimensional oscillations recorded in this measure truly correspond to the same fine, one-dimensional oscillations for a sequence of admissible $\mathcal{X}$'s. Moreover, the information about the direction of oscillations is coming from the difference of the two mass-points corresponding to the gradient components (including time)

$$\left(\lambda_1^1, \frac{\overline{\rho} - k_2\overline{\lambda}}{\theta\,(k_1 - k_2)}\right) - \left(\lambda_1^2, \frac{\overline{\rho} - k_1\overline{\lambda}}{(1-\theta)\,(k_2 - k_1)}\right).$$

Except for a multiplicative constant, this direction is given by

$$((k_1 - k_2)\theta(1-\theta)(\lambda_1^1 - \lambda_1^2), \overline{\rho} - (\theta k_1 + (1-\theta)k_2)\overline{\lambda}). \tag{22}$$

Because this difference has, in general, a non-vanishing component in the time variable (the first component), these oscillations will take place also with respect to time.

According to our previous formulae, we must choose λ_1^1 and λ_1^2 such that

$$\begin{cases} \lambda_1 = \theta\lambda_1^1 + (1-\theta)\,\lambda_1^2, \\ \\ -\theta\beta_1 u\lambda_1^1 - (1-\theta)\,\beta_2 u\lambda_1^2 = \rho\cdot\lambda - \left[k_1\frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{(1-\theta)(k_2-k_1)^2}\right], \end{cases}$$

that is, if $u \neq 0$, then

$$\begin{cases} \lambda_1^1 = -\frac{1}{u\theta(\beta_1-\beta_2)}\left(\theta\,(\beta_2-\beta_1)\,u\lambda_1 + \overline{\rho}\cdot\overline{\lambda} - \left[k_1\frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{(1-\theta)(k_2-k_1)^2}\right]\right), \\ \\ \lambda_1^2 = -\frac{1}{u(1-\theta)(\beta_1-\beta_2)}\left((1-\theta)\,(\beta_2-\beta_1)\,u\lambda_1 + \overline{\rho}\cdot\overline{\lambda} - \left[k_1\frac{\left|\overline{\rho}-k_2\overline{\lambda}\right|^2}{\theta(k_1-k_2)^2} + k_2\frac{\left|\overline{\rho}-k_1\overline{\lambda}\right|^2}{(1-\theta)(k_2-k_1)^2}\right]\right), \end{cases} \tag{23}$$

and for $u = 0$ there is an infinity of possibilities for λ_1^1 and λ_1^2, namely

$$\lambda_1^1 = \gamma, \quad \lambda_1^2 = \frac{1}{(1-\theta)}\,(\lambda_1 - \theta\gamma)$$

with $\gamma \in \mathbb{R}$. In this last case, the div-curl compatibility condition reduces to

$$\rho\cdot\lambda = \left[k_1\frac{\left|\overline{\rho} - k_2\overline{\lambda}\right|^2}{\theta\,(k_1-k_2)^2} + k_2\frac{\left|\overline{\rho} - k_1\overline{\lambda}\right|^2}{(1-\theta)\,(k_2-k_1)^2}\right].$$

The above means that optimal measures leading to the exact value for $CQW\,(\theta,\rho,\lambda)$ may be found in the form of first-order laminates of the kind (21). Note also that thanks to the particular form of this measure, the first component of the vector field G, say G_1, is equal to $-\,(\theta\beta_1 + (1-\theta)\,\beta_2)\,u$. This, together with the divergence-free character of G leads to the equation

$$-\,((\theta\beta_1 + (1-\theta)\,\beta_2)\,u)' + \operatorname{div}\overline{G} = 0$$

where we have put $G = \left(G_1, \overline{G}\right)$. Our conclusion is then that

$$CQW(\theta,\rho,\lambda) = k_1\frac{\left|\overline{\rho} - k_2\overline{\lambda}\right|^2}{\theta\,(k_1-k_2)^2} + k_2\frac{\left|\overline{\rho} - k_1\overline{\lambda}\right|^2}{(1-\theta)\,(k_2-k_1)^2}.$$

We then find a relaxation of (VP_t) in the following form:

THEOREM 3.1 *Assume that the solution of system (3) has the regularity*

$$u \in L^2\left(0,T;H^2\left(\Omega\right)\right) \tag{24}$$

and depends continuously on the initial datum in the corresponding norms. Then the variational problem

$$(RP_t) \quad \text{Minimize in } \left(\theta,\overline{G},u\right): \quad \overline{J_t}(\theta,\overline{G},u)=\frac{1}{2}\int_0^T\int_\Omega\left[k_1\frac{\left|\overline{G}-k_2\nabla u\right|^2}{\theta\left(k_1-k_2\right)^2}+k_2\frac{\left|\overline{G}-k_1\nabla u\right|^2}{\left(1-\theta\right)\left(k_2-k_1\right)^2}\right]dxdt$$

subject to

$$\begin{cases} G\in L^2\left((0,T)\times\Omega;\mathbb{R}^{N+1}\right), & u\in H^1\left((0,T)\times\Omega;\mathbb{R}\right),\\ \left(\left(\theta\beta_1+\left(1-\theta\right)\beta_2\right)u\right)'-div\,\overline{G}=0 & in\ H^{-1}\left((0,T)\times\Omega\right),\\ u|_{\partial\Omega}=0 \quad a.\ e.\ t\in[0,T], & u\left(0\right)=u_0 \quad in\ \Omega,\\ \theta\in L^\infty\left((0,T)\times\Omega;[0,1]\right), \quad \int_\Omega\theta\left(t,x\right)dx=L|\Omega| \quad a.e.\ t\in(0,T). & \end{cases}$$

is a relaxation of (VP_t) in the sense that

(i) *there exists at least one minimizer for (RP_t),*

(ii) *the infimum of (VP_t) equals the minimum of (RP_t), and*

(iii) *the underlying Young measure associated with (RP_t) (and therefore the optimal microstructure of (VP_t)) can be found in the form of a first-order laminate whose direction of lamination can be given explicitly in terms of optimal solutions for (RP_t).*

Proof. Once the constrained quasi-convex density CQW has been computed, the proof is standard in non-convex, vector, variational problems, but is included here for the sake of completeness.

A first technical point we must deal with concerns the equi-integrability property of $|\nabla u_n|^2$ that is needed to represent the limit cost associated with a minimizing sequence of designs through its corresponding Young measure. This problem may be easily overcome if we assume the regularity of the solutions u_n as stated above. By using the Sobolev embedding theorem this implies that $|\nabla u_n|^2\in L^{p/2}\left(\Omega\right)$ for some $p>2$ and a.e. $t\in[0,T]$. From this and Hölder inequality one deduces that $|\nabla u_n|^2$ is equi-integrable.

We are now in position to describe the main steps of the proof. To begin with, we notice that (RP_t) may be written in an equivalent form as

$$\left(\widetilde{RP_t}\right) \qquad \text{Minimize in } \nu: \qquad \widetilde{J}_t\left(\nu\right)=\frac{1}{2}\int_0^T\int_\Omega\left(\int_{\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}}\overline{\rho}\cdot\overline{\lambda}d\nu_{(t,x)}\left(\rho,\lambda\right)\right)dxdt$$

where the competing measures are of the form (21) and

$$\left(G\left(t,x\right),\nabla_{(t,x)}u\left(t,x\right)\right)=\left(\int_{\mathbb{R}^{N+1}}yd\nu^{(1)}_{(t,x)}\left(y\right),\ \int_{\mathbb{R}^{N+1}}zd\nu^{(2)}_{(t,x)}\left(z\right)\right)$$

with

$$\mathrm{div}_{(t,x)}\ G=0, \qquad u|_{\partial\Omega}=0, \qquad u\left(0\right)=u_0.$$

In particular,

$$\inf_{\nu} \widetilde{J}_t(\nu) = \inf_{(\theta,\overline{G},u)} \overline{J}\left(\theta,\overline{G},u\right).$$

Now let $\mathcal{X}_n$ be a minimizing sequence for (P_t) and let (G_n, H_n) be the pair associated with $\mathcal{X}_n$ as in (13). Then (see [17, Th. 6.2]), up to a subsequence still denoted by $\mathcal{X}_n$,

$$\lim_{n\to\infty} J_t(\mathcal{X}_n) = \widetilde{J}_t(\nu^\star), \tag{25}$$

where $\nu^\star$ is the measure associated with (G_n, H_n). This proves that

$$\inf_{\nu} \widetilde{J}_t(\nu) \le \inf_{\mathcal{X}} J_t(\mathcal{X}).$$

Conversely, if ν is admissible for $\left(\widetilde{RP_t}\right)$, then there exists a pair (G_n, H_n) associated with some sequence of characteristic functions $\mathcal{X}_n$ such that its corresponding $|\nabla u_n|^2$ is equi-integrable (see [8, 20] and [17, Th. 8.7] for its equivalent in the context of gradient Young measures). We notice that, although the volume constraint is satisfied at the limit, that is,

$$\lim_{n\to\infty} \int_\Omega \mathcal{X}_n(t,x)\, dx = L\,|\Omega| \qquad \text{a.e. } t \in (0,T)$$

in principle, each individual $\mathcal{X}_n$ does not satisfy this volume constraint. But this a technical difficulty that may be overcome as in the homogenization approach. So, we may assume that $\mathcal{X}_n$ is admissible for (P_t). Thanks again to the equi-integrability of the gradients, up to a subsequence,

$$\lim_{n\to\infty} J_t(\mathcal{X}_n) = \widetilde{J}_t(\nu)$$

and so

$$\inf_{\nu} \widetilde{J}_t(\nu) \ge \inf_{\mathcal{X}} J_t(\mathcal{X}).$$

Combining this with (25), we obtain that $\nu^\star$ is a minimizer for $\left(\widetilde{RP_t}\right)$ and therefore so is its associated $\left(\theta,\overline{G},u\right)$ for (RP_t). This proves (i), (ii) and the fact that the underlying Young measure is a first-order laminate. Concerning the direction of lamination, it has also been indicated above in (22). All of these expressions depend on the optimal solution $\left(\theta,\overline{G},u\right)$ of (RP_t) through the identification $\theta, \overline{\rho} = \overline{G}, \overline{\lambda} = \nabla u, \lambda_1 = u'$. The proof is now complete. ■

Remark 2 *For the case $\beta_1 = \beta_2$, the regularity (24) is satisfied if Ω is of class C^1 and $u_0 \in H_0^1(\Omega)$ (see [7, p. 360]).*

3.3 Analysis of (RP_t) and Conjecture

The relaxed formulation (RP_t) is rather complicated to deal with as it depends on too many fields. We conjecture that the problem

$$(\underline{\mathrm{RP}_t}) \quad \text{Minimize in } \theta: \quad \underline{J_t}(\theta) = \frac{1}{2}\int_0^T\!\!\int_\Omega \frac{k_1 k_2}{\theta k_2 + (1-\theta)\,k_1} \left|\nabla u\right|^2 dxdt$$

subject to

$$\begin{cases} \left((\theta\beta_1 + (1-\theta)\,\beta_2)\,u\right)' - \operatorname{div}\left(\frac{k_1k_2}{\theta k_2 + (1-\theta)k_1}\nabla u\right) = 0 & \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 \quad \text{a. e. } t \in [0,T], & u(0) = u_0 \quad \text{in } \Omega, \\ \theta \in L^\infty\left((0,T)\times\Omega;[0,1]\right), \quad \int_\Omega \theta(t,x)\,dx = L|\Omega| \quad a.e. \quad t\in(0,T) & \end{cases} \tag{26}$$

is also a relaxation for our original problem. Our intuition here is rooted in the fact that if in the expression for CQW, we find the minimum in $\overline{\rho}$ for $\overline{\lambda}$ fixed, then we arrive at a linear relationship, given by the harmonic mean, between $\overline{\lambda}$ and $\overline{\rho}$

$$\overline{\rho} = \frac{k_1 k_2}{(1-\theta)k_1 + \theta k_2}\overline{\lambda}. \tag{27}$$

In this case, some elementary algebra leads to the fact that

$$\overline{\rho}\cdot\overline{\lambda} = \left[k_1 \frac{\left|\overline{\rho} - k_2\overline{\lambda}\right|^2}{\theta\,(k_1-k_2)^2} + k_2 \frac{\left|\overline{\rho} - k_1\overline{\lambda}\right|^2}{(1-\theta)\,(k_2-k_1)^2}\right],$$

so that $\lambda_1^1 - \lambda_1^2 = 0$ in (22), and, in addition, $\overline{\rho} - (\theta k_1 + (1-\theta)k_2)\overline{\lambda}$ is just a multiple of $\overline{\lambda}$. This means that the direction of lamination is orthogonal to $\nabla u(x,t)$ with no component in the time axis. But if this is the case, then the optimal local volume fraction θ should be independent of time $\theta = \theta(x)$, though the direction of lamination is changing with time.

Conjecture 1 *Minimizing sequences for the time-dependent initial optimal design problem (P_t) can be recovered through optimal solutions of problem $(\underline{RP_t})$, where the harmonic mean plays a fundamental role. Such optimal sequences of characteristic functions correspond to first-order laminates with local volume fraction $\theta(x)$ (***independent of time***) and direction of lamination orthogonal to $\nabla u(x,t)$.*

See [19] for more on these ideas for the elliptic case. It is also important to notice that because we have put no source term in the equation from the beginning of this section (for the sake of simplicity), our problem is no longer a typical compliance situation (for which in general the arithmetic mean plays an important role. See for instance [21]). In our case, we conjecture that

$$\overline{G} = \lambda_\theta^- \nabla u. \tag{28}$$

In the next section, we support this surprising and non trivial conjecture with some numerical experiments.

4 Numerical Applications

In this section, we solve numerically in the two dimensional case ($N = 2$) the relaxed formulations (RP) and (RP$_t$) obtained from the Homogenization and Young measure theory respectively.

4.1 Numerical resolution of the relaxed problems

We first explain the numerical resolution of the relaxed problem (RP) derived from the homogenization method (see section 2.2).

A convenient way to minimize J^* consists first in using a parametrization of the homogenized tensor $K^* \in G_\theta$ in terms of its Y-transform (we refer to [1, p. 122]): the Y-transform is the map on the set of symmetric matrices defined by

$$Y(K^*) = (\lambda_\theta^+ I_N - K^*)((\lambda_\theta^-)^{-1}K^* - I_N)^{-1}. \tag{29}$$

For $N = 2$, denoting by y_1, y_2 the eigenvalues of $Y(K^*)$, K^* belongs to G_θ if and only if

$$\min(k_1,k_2)^2 \le y_1 y_2 \le \max(k_1,k_2)^2, \quad y_1, y_2 \ge 0. \tag{30}$$

The advantage is that the set $Y(G_\theta)$ does not depend on θ. Its inverse mapping is

$$K^*(Y) = (\lambda_\theta^+ I_N + Y)((\lambda_\theta^-)^{-1}Y + I_N)^{-1}. \tag{31}$$

We then parameterize a composite design by (θ, Y^*) with $Y^* = Y(A^*)$ for some $A^* \in G_\theta$. The interest is that the constraints on θ and Y are now uncoupled making easier the implementation of gradient algorithm. Consequently, $A^* \in G_\theta$ is parameterized by the density θ, the two eigenvalues y_1 and y_2 and the angle of rotation ϕ such that

$$K^*(\theta, y_1, y_2, \phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\lambda_\theta^+ + y_1}{y_1/\lambda_\theta^- + 1} & 0 \\ 0 & \frac{\lambda_\theta^+ + y_2}{y_2/\lambda_\theta^- + 1} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{32}$$

Finally, we compute the first derivative of the resulting function (still denoted by J^*) with respect to θ, Y^* and ϕ and apply a gradient algorithm. The first derivative in any direction $(\delta\theta, \delta Y^*, \delta\phi)$ takes the following expression

$$\begin{aligned} \frac{\partial J^*(\theta, Y^*, \phi)}{\partial(\theta, Y^*, \phi)} \cdot (\delta\theta, \delta Y^*, \delta\phi) &= \int_\Omega \int_0^T \left(\frac{1}{2} K_\phi^* \nabla u \cdot \nabla u + K_\phi^* \nabla u \cdot \nabla p \right) dt \; \delta\phi dx \\ &+ \int_\Omega \int_0^T \left(\frac{1}{2} K_{Y^*}^* \nabla u \cdot \nabla u + K_{Y^*}^* \nabla u \cdot \nabla p \right) dt \;\; \cdot \delta Y^* dx \\ &+ \int_\Omega \int_0^T \left(\frac{1}{2} K_\theta^* \nabla u \cdot \nabla u + K_\theta^* \nabla u \cdot \nabla p + (\beta_1 - \beta_2) u' p \right) dt \; \delta\theta dx \end{aligned} \tag{33}$$

where p designates the adjoint solution of the backward system

$$\begin{cases} -\beta(\theta)p' - \text{div } (K^*(\theta, Y^*, \phi)\nabla p) = \text{div } (K^*(\theta, Y^*, \phi)\nabla u) & \text{in} \quad (0,T) \times \Omega, \\ p = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ p\,(T, x) = 0 & \text{in} \quad \Omega \end{cases} \tag{34}$$

and $K_\theta^*, K_{Y^*}^*, K_\phi^*$ the derivatives of K^* with respect to θ, Y^* and ϕ respectively. At last, we use lagrangian multipliers to enforce the constraints $\theta \in L^\infty(\Omega, [0,1])$, $\int_\Omega \theta(x)dx = L|\Omega|$ and (30).

The relaxed problem (RP_t) (see Theorem 3.1) derived from the second approach, although less standard, may be solved in a similar way using a descent algorithm. Precisely, the minimization of $\overline{J}$ is done over θ and $\overline{G}$ while u is determined via the constraint $((\theta\beta_1 + (1-\theta)\beta_2)u)' - div\overline{G} = 0$. The first variation of $\overline{J_t}$ with respect to $(\theta, \overline{G})$ in any direction $(\delta\theta, \delta\overline{G})$ is given by

$$\begin{aligned} \frac{\partial \overline{J_t}(\theta, \overline{G}, u)}{\partial(\theta, \overline{G})} \cdot (\delta\theta, \delta\overline{G}) &= -\int_0^T \int_\Omega (\beta_1 - \beta_2) u p' \delta\theta \, dx dt + \int_\Omega \Big[(\beta_1 - \beta_2) u p \delta\theta \Big]_0^T dx \\ &+ \frac{1}{2} \int_0^T \int_\Omega \left[-\frac{k_1}{\theta^2} \frac{|\overline{G} - k_2 \nabla u|^2}{(k_1 - k_2)^2} + \frac{k_2}{(1-\theta)^2} \frac{|\overline{G} - k_1 \nabla u|^2}{(k_2 - k_1)^2} \right] \delta\theta \, dx dt \\ &+ \int_0^T \int_\Omega \left[\frac{k_1}{\theta} \frac{(\overline{G} - k_2 \nabla u)}{(k_1 - k_2)^2} + \frac{k_2}{1-\theta} \frac{(\overline{G} - k_1 \nabla u)}{(k_2 - k_1)^2} + \nabla p \right] \cdot \delta\overline{G} \, dx dt \end{aligned} \tag{35}$$

where p is solution of the following problem :

$$\begin{cases} (\theta\beta_1 + (1-\theta)\beta_2)p' = \frac{k_1 k_2}{(k_1 - k_2)^2} div \left(\frac{(\overline{G} - k_2 \nabla u)}{\theta} + \frac{(\overline{G} - k_1 \nabla u)}{1-\theta} \right) & \text{in} \quad (0,T) \times \Omega, \\ p = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ p(T, x) = 0 & \text{in} \quad \Omega. \end{cases} \tag{36}$$

Once again, a multiplier is necessary to deal with the constraints on θ. Finally, the resolution of problem $(\underline{\mathrm{RP}_t})$ from Section 3.3 is standard and we refer to [14] for the details in the context of the wave equation.

For all the variables, we use a continuous finite element approximation of second order with respect to x on a uniform mesh and a finite difference approximation of first order with respect to t. In the resolution of problem (RP_t), since $\overline{G}$ and θ are time-space variables, a regularization of the variable p via a viscosity term in (36) is applied (see [13] for a similar phenomenon where the density is time-space dependent).

4.2 Numerical experiments

4.2.1 Examples with $u_0 \neq 0$ and $f = 0$

We consider the following simple initial data on the unit square : $\Omega = (0,1)^2$:

$$u_0(x) = \sin(\pi x_1)\sin(\pi x_2), \quad x = (x_1, x_2) \in \Omega \tag{37}$$

and take $T = 0.5$, $L = 1/2$, $(\beta_1, k_1) = (10, 0.1)$ and $(\beta_2, k_2) = (20, 1)$. At last, the numerical results presented in this section are obtained with the spatial discretisation parameter $h = 1/50$ and with the temporal discretisation parameter $dt = h/4$.

We first give the results obtained for problem (RP) derived from the Homogenization approach. The algorithm is initialized with constant functions: we take $\theta \equiv L|\Omega|$, $y_i \equiv (k_1+k_2)/2, i = 1, 2$, and $\phi \equiv 0$ on Ω. Figure 1 depicts the functions θ and ϕ, local minima of J^*. Figure 2 depicts the corresponding function y_1 and y_2. We obtain $J^*(\theta, y_1, y_2, \phi) \approx 0.202$ and we observe that θ is a characteristic function in $L^\infty(\Omega, \{0, 1\})$. The corresponding gradient part of the energy with respect to the time is given in Figure 3 highlighting the diffusion of the heat. We also observe - this is the main drawback of gradient method - that the result depends on the initialization. Figure 4 depicts the iso-values of θ and ϕ obtained at convergence of the algorithm initialized still with $\theta = L|\Omega|$, $\phi = 0$ but now with $y_1 = \min(k_1, k_2)$ and $y_2 = \max(k_1, k_2)$. Figure 5 depicts the corresponding function y_1 and y_2. The value of the cost function is however similar highlighting the existence of local minima and a low dependence of J^* with respect to the variables.

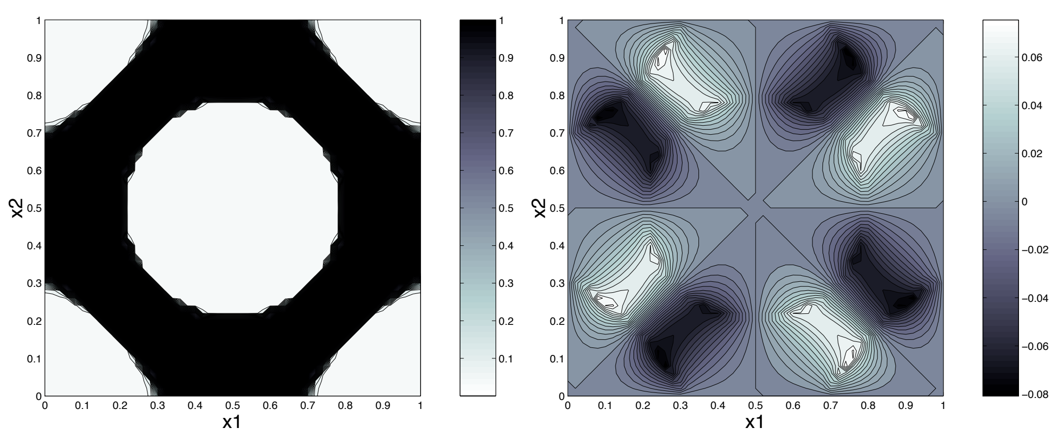

Figure 1: Resolution of (RP) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ (**Left**) and ϕ (**Right**)- $J^*(\theta, y_1, y_2, \phi) \approx \mathbf{0.202}$.

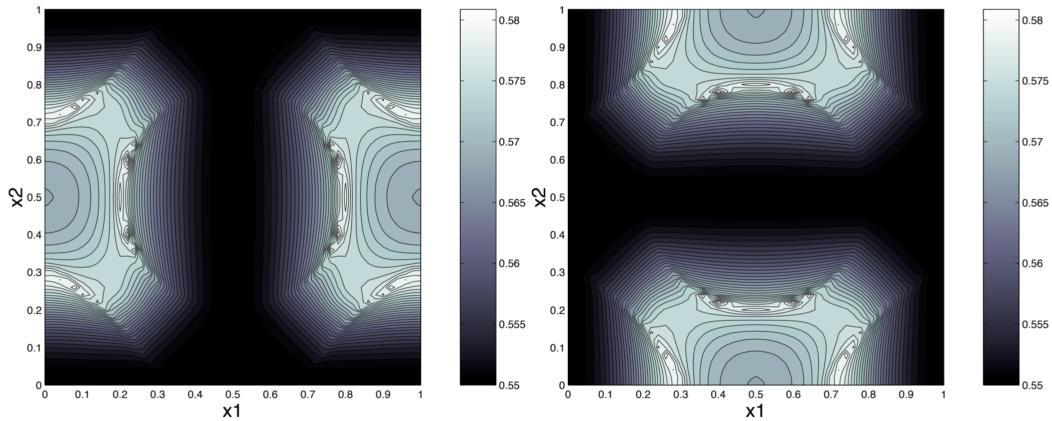

Figure 2: Resolution of (RP) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of y_1 (**Left**) and y_2 (**Right**) corresponding to Figure 1- $J^*(\theta, y_1, y_2, \phi) \approx \mathbf{0.202}$.

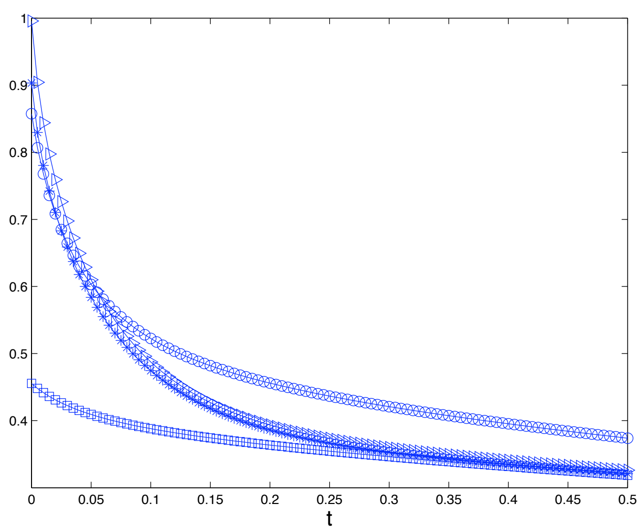

Figure 3: $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Gradient part of the energy vs. t: Homogenization approach corresponding to Figure 1 ($\circ$); Homogenization approach corresponding to Figure 4 ($\star$); Young measure approach corresponding to Figure 7 ($\square$); Arithmetic mean corresponding to Figure 9 ($>$).

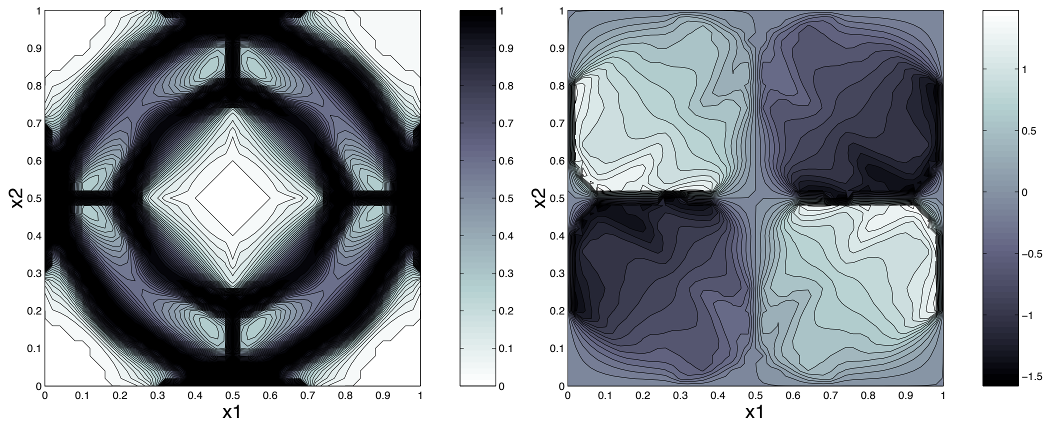

Figure 4: Resolution of (RP) with a different initialization- $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ (**Left**) and ϕ (**Right**)- $J^*(\theta, y_1, y_2, \phi) \approx \mathbf{0.224}$.

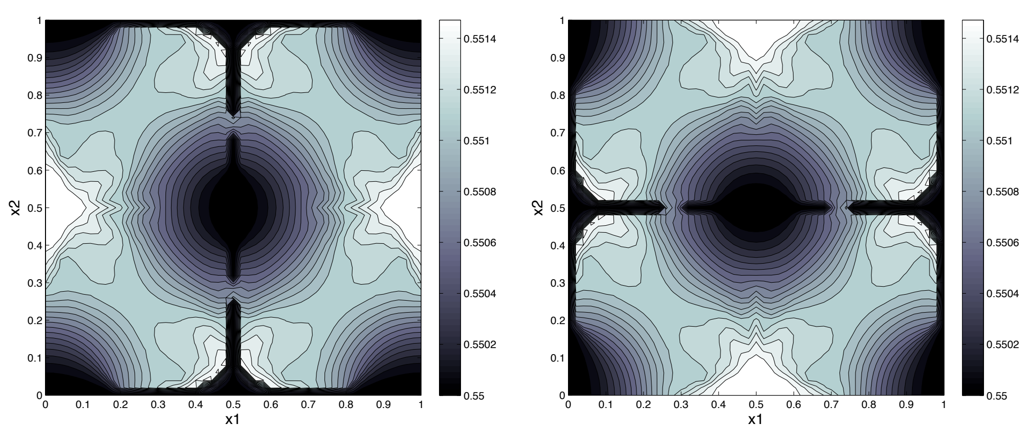

Figure 5: Resolution of (RP) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of y_1 (**Left**) and y_2 (**Right**) corresponding to Figure 4- $J^*(\theta, y_1, y_2, \phi) \approx \mathbf{0.224}$.

We now solve the relaxed problem (RP_t) with a special attention to the time dependence of the optimal density θ. Expected from the theoretical part but a bit surprising, we have obtained in all our simulations that the density θ is almost (up to the numerical approximation) time independent. Let us first give the result in the 1-D version of (37) (easier to represent). Precisely, we take $u_0(x_1) = \sin(\pi x_1)$. Initialized with the density $\theta(x,t) = L$ and $\overline{G} = \lambda_\theta^- \nabla u$ in $(0,T) \times (0,1)$, the descent algorithm provides the density depicted in Figure 6-left. The cost is $\overline{J_t}(\theta, \overline{G}, u) \approx 0.1403$. Up to the numerical approximation and boundary phenomena, we may conclude that this optimal density does not depend on time. We highlight that we obtain the same result for a different initialization: for instance $\theta(x,t) = x \otimes t$ and/or $\overline{G} = \lambda_\theta^+ \nabla u$. We then check that the triplet solution $(\theta, \overline{G}, u)$ satisfy the relation (28): we obtain

$$R_{\theta,\overline{G},u} \equiv \frac{||\overline{G} - \lambda_\theta^- \nabla u||_{L^2((0,T)\times\Omega)}}{||\overline{G}||_{L^2((0,T)\times\Omega)}} \approx 1.62 \times 10^{-4} \tag{38}$$

Finally, the resolution of problem $(\underline{\mathrm{RP}_t})$ leads to the density θ depicted in Figure 6-right, and provides a very similar value of the cost : $\underline{J_t}(\theta) \approx 0.1409$. These results support our Conjecture 1. Remark that we still observe this time independence when we replace the volume constraint (4) by the weaker one

$$\int_0^T \int_\Omega \theta(t,x)\,dxdt = TL|\Omega|, \tag{39}$$

the Young measure analysis being independent of such a constraint.

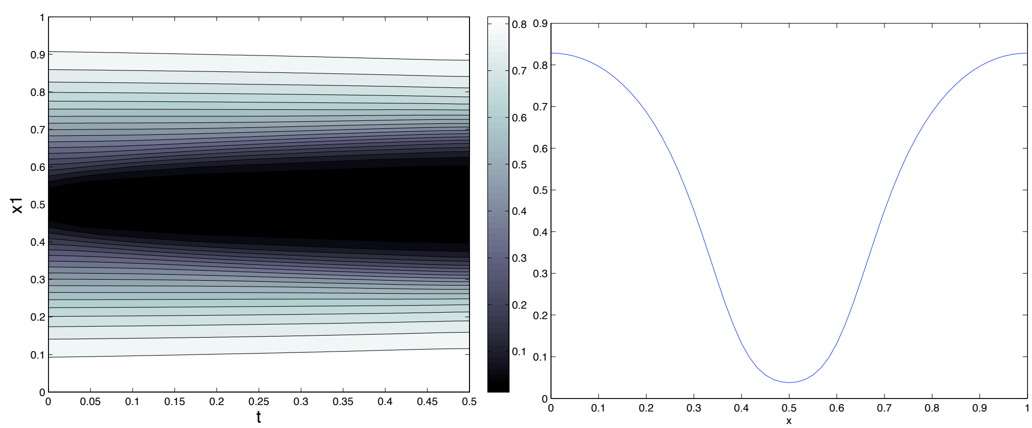

Figure 6: Resolution of (RP_t) and $(\underline{\mathrm{RP}_t})$ in 1D- $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - **Left** : Density $\theta(x,t)$ solution of (RP_t); **Right** : density $\theta(x)$ solution of $(\underline{\mathrm{RP}_t})$.

In 2D, our numerical results lead to $R_{\theta,\overline{G},u} \approx 4.35 \times 10^{-3}$ highlighting once again the role of the harmonic mean and the validity of Conjecture 1. The results obtained from the relaxed problem (RP_t) derived from the variational approach are qualitatively different from the ones obtained from the first approach. Once again, the density θ is initialized with $\theta \equiv L$ on $(0,T) \times \Omega$ which does not privilege any location for the set of the first material (β_1, k_1). On the other hand, the field $\overline{G}$ is initialized by $\overline{G} = \lambda_\theta^- \nabla u$ where u is solution of (26). Figure 7 displays the iso-values of the (time-independent) function θ. The results seem here independent of the initialization of the algorithm: for instance, we get a similar result if we take $\overline{G} = \lambda_\theta^+ \nabla u$. This suggests that the function θ of Figure 7 is the global minimum: we obtain $\overline{J}(\theta, \overline{G}, u) \approx 0.1806$ which is lower than in the previous cases (see Figures 1 and 4). The corresponding evolution with respect to time of the gradient part of the energy is depicted on Figure 3. Moreover, we observe that θ is no more a characteristic function which suggests that for these data, the initial design problem (P_t) is not well-posed, and therefore justifies the whole relaxation procedure. Furthermore, if we naively consider the arithmetic mean $\lambda_\theta^+ = k_1\theta + k_2(1-\theta)$, then we obtain the distribution of Figure 9 leading to a greater cost equal to 0.213 (see Figure 3 for the corresponding evolution of the integrand of the cost). We have also represented in Figure 8 one of the components of the gradient $\nabla u(x,t)$ on the slice $x_2 = 1/2$ to emphasize the dependence of the optimal direction of lamination with respect to time (which would be constant in the time independent version of the problem).

Moreover, similarly to the hyperbolic case (see [13]), we observe that when the gap $k_2 - k_1$ and $\beta_2 - \beta_1$ between the coefficients is small enough (depending on the data of the problem), the density θ is a characteristic function (see Figure 10 obtained for $(\beta_1, k_1) = (10, 0.1)$ and $(\beta_2, k_2) = (10.2, 0.102)$): this suggests that in this case the problem (P_t) is well-posed.

At last, on a physical point of view, the initial data being fixed, the distribution of the two materials seems to depend mainly on the value of the ratio k_2/k_1 with respect to one. Precisely, the material which have the greater diffusion coefficient (here k_2) is distributed on the center and on the corners of the unit square. The value of the ratio β_2/β_1 and of T seems less preponderant. These observations are related to the exponential diffusion in time of the heat solution u.

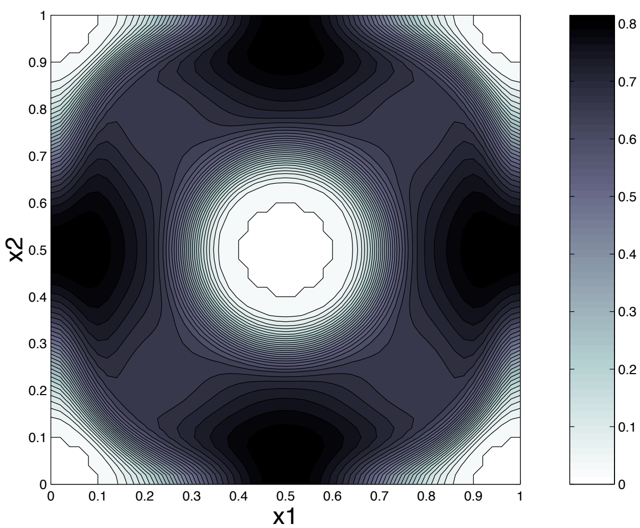

Figure 7: Resolution of (RP_t) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.1806}$.

Obviously, the optimal distribution of the two material depend on the initial condition u_0: we give the result obtained with the initial condition $u_0(x) = e^{-50(x_1-0.3)^2 - 50(x_2-0.3)^2}$ concentrated on $(0.3, 0.3)$. We take the same values for $(\alpha_i, \beta_i), i = 1, 2$ and $L = 1/5$. Here again, the numerical experiments are in agreement with the theoretical part: for $T = 0.5$, we obtain $\overline{J_t}(\theta, \overline{G}, u) \approx 0.0465$ and $\underline{J_t}(\theta) \approx 0.0469$; in particular the optimal density from

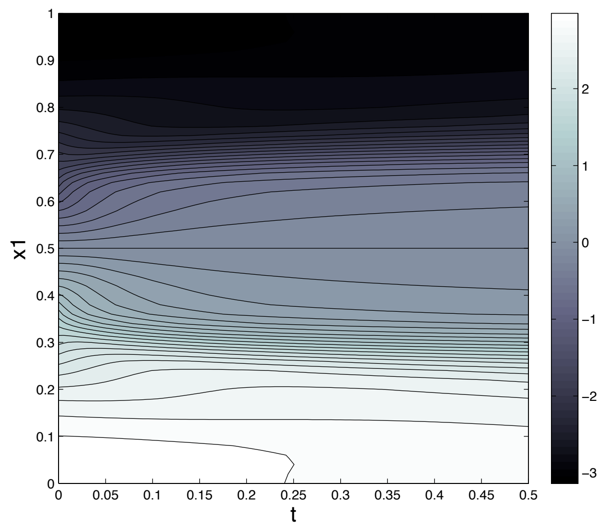

Figure 8: Resolution of (RP_t) - Iso-values of $\partial u/\partial x_1$ for $(t, x_1) \in (0, T) \times (0, 1)$ and $x_2 = 1/2$ associated with Figure 7.

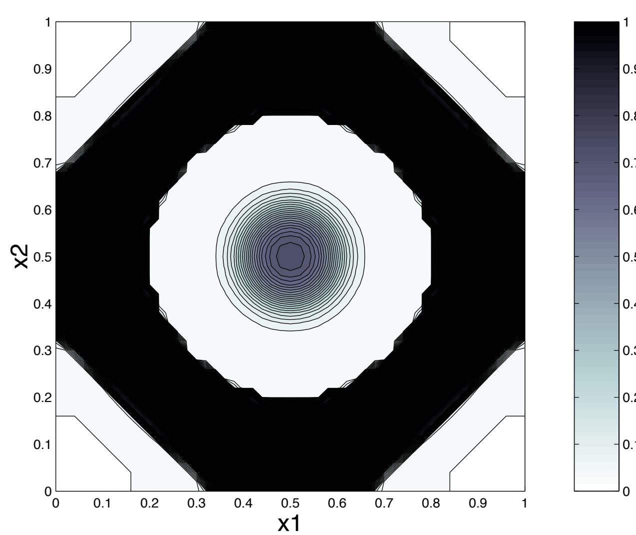

Figure 9: $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ when $k_1\mathcal{X}_\omega + k_2(1 - \mathcal{X}_\omega)$ is directly replaced by the arithmetic mean λ_θ^+- Cost function $\approx$ **0.213**.

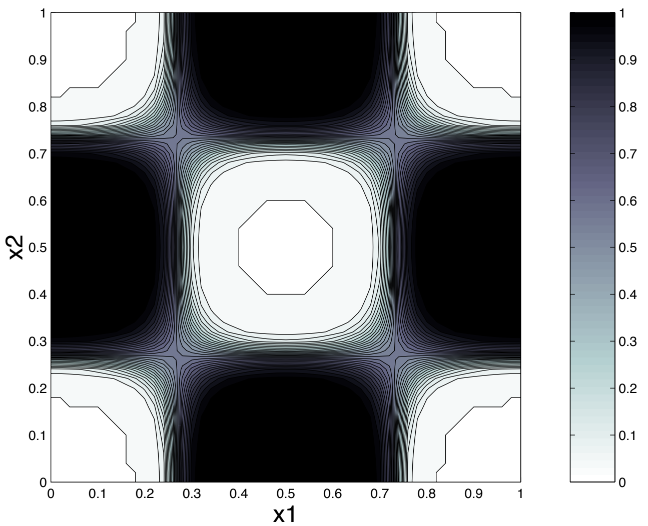

Figure 10: Resolution of (RP$_t$) - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 10.2)$, $(k_1, k_2) = (0.1, 0.102)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.1126}$.

(RP$_t$) is time independent and $R_{\theta,\overline{G},u} \approx 2.19 \times 10^{-3}$. The optimal densities for $T = 0.5$ and $T = 5$ are reported on Figure 11. As expected, the (k_1, β_1)-material is concentrated around the point $(0.3, 0.3)$. Although the variation with respect to T is low, we remark that the optimal density for $T = 5$ remains strictly positive. Results from the Homogenization approach are similar.

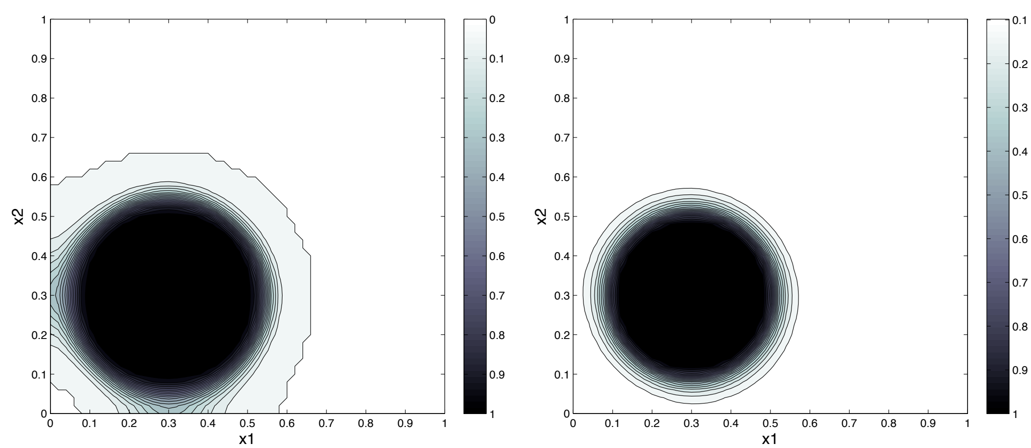

Figure 11: Resolution of (RP$_t$) - $L = 1/5$ - Iso-values of the optimal density: **Left**: $T = 0.5$ leading to $\overline{J_t}(\theta, \overline{G}, u) \approx 0.0465$; **Right**: $T = 5$ leading to $\overline{J_t}(\theta, \overline{G}, u) \approx 0.0812$.

4.2.2 $u_0 = 0$ and $f \neq 0$

For sake of simplicity, we have assumed in Section 3 that $f \equiv 0$ in $(0, T) \times \Omega$. We consider here $f(x, t) = 10$ in $(0, T) \times \Omega$, $u_0 \equiv 0$ in Ω and evaluate whether or not Conjecture 1 still holds in this case. The resolution of (RP_t) for $T = 0.5$ and $T = 5$ are depicted in Figure 12 and 13 respectively. Once again, in both cases, we obtain that the cost $J_t(\theta, \overline{G}, u)$ is very similar to the value of $\underline{J_t}(\theta)$ corresponding to the resolution of $(\underline{\mathrm{RP}_t})$. The resolution of (RP) for $T = 0.5$ leads to Figure 14 and to a cost slightly greater. Lastly, further simulations display the same phenomenon when f is constant in time but not in space. When f depends on time, we obtain time dependent functions θ from (RP_t) and that $\min\,(\mathrm{RP}_t) < \min\,(\underline{\mathrm{RP}_t})$.

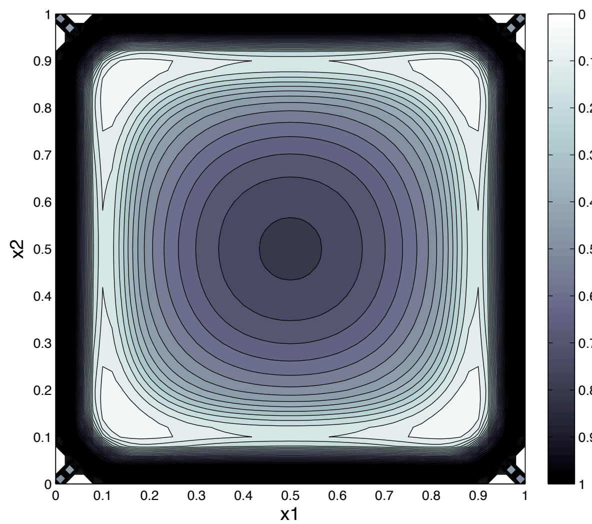

Figure 12: Resolution of (RP_t) for $f = 10$ and $u_0 = 0$ - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{0.0446}$.

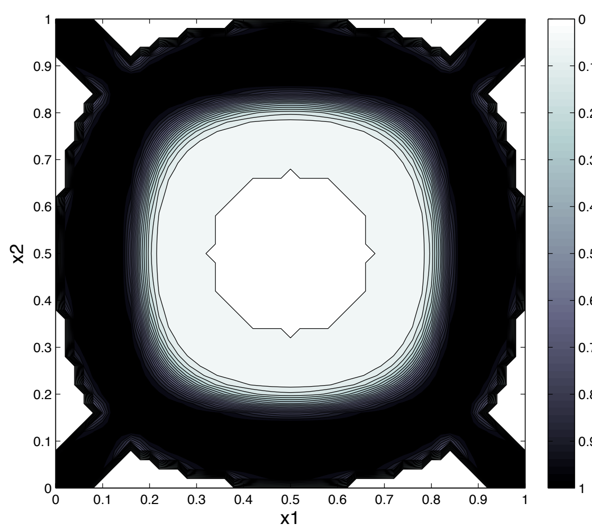

Figure 13: Resolution of (RP_t) for $f = 10$ and $u_0 = 0$ - $L = 1/2$ - $T = 5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ - $\overline{J}(\theta, \overline{G}, u) \approx \mathbf{9.053}$.

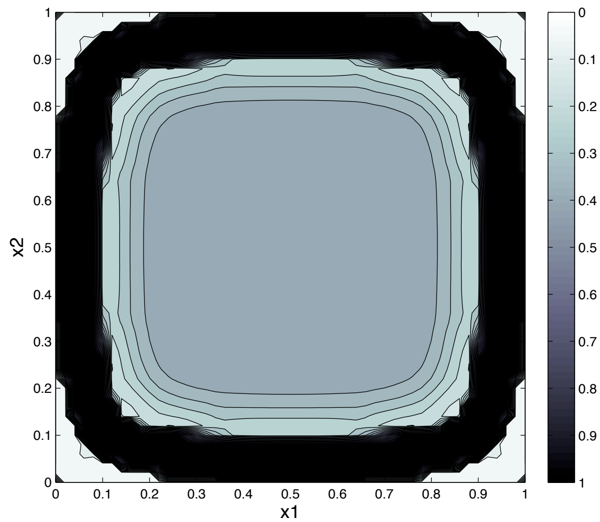

Figure 14: Resolution of (RP) for $f = 10$ and $u_0 = 0$ - $L = 1/2$ - $T = 0.5$ - $(\beta_1, \beta_2) = (10, 20)$, $(k_1, k_2) = (0.1, 1)$ - Iso-values of θ - $J^\star(\theta, Y^\star, \phi) \approx \mathbf{0.0492}$.

5 Concluding remarks and open problems

In this work, we have analyzed theoretically and numerically a typical nonlinear optimal design problem for the heat equation in two different cases: (1st) time-independent designs, and (2nd) time-dependent designs.

A full relaxation for the first case has been obtained by using the homogenization method. The relaxed problem has been solved by using a gradient algorithm as is usual in this context. Writing down the necessary optimality conditions, it is not hard to show that if $(\theta, K^\star)$ is a minimizer for problem (RP), then $K^\star$ is a solution of the point-wise optimization problem

$$\int_0^T \left(\frac{1}{2}K^\star\nabla u \cdot \nabla u - K^\star\nabla u \cdot \nabla p\right) dt = \min_{K^0 \in G_{\theta(x)}} \int_0^T \left(\frac{1}{2}K^0\nabla u \cdot \nabla u - K^0\nabla u \cdot \nabla p\right) dt,$$

where p solves the system

$$\begin{cases} (\theta\beta_1 + (1-\theta)\,\beta_2)\,p' + \operatorname{div}(K^\star\nabla p) = \operatorname{div}(K^\star\nabla u) \\ p|_{\partial\Omega} = 0, \qquad p\,(T) = 0. \end{cases}$$

A deeper analysis of these optimality conditions would be required to characterize the microgeometry of the optimal composite. This is a very interesting analytical problem that it has not been addressed here. A preliminary situation could be considered where the source term and the cost functional, in a steady-state framework, depend on a parameter t as well.

In the second - time-dependent - case, the relaxation procedure has been derived by using the classical tools of non-convex, vector, variational problems: quasi-convexification and div-curl Young measures. The proposed method directly provides the behaviour of (some) minimizing sequences of the original problem. Precisely, this information is codified in the Young measure associated with such a minimizing sequence. We have obtained that this measure is a convex combination of two Dirac masses. In the context of Young measures, we refer to this as a (time-dependent) first-order laminate. In addition, we conjecture that the weights of these Dirac masses (which represent the local volume fraction of the two materials) are time-independent, i.e.. $\theta = \theta(x)$. In other words, the volume fraction $\theta = \theta(x)$ only depends on the spatial variable, but the normal vector to the direction of lamination changes with time according to $\nabla u(t,x)$. This is the way in which a minimizing sequence for the original problem is recovered from a minimizer of the relaxed one. This is a surprising result, but our numerical experiments seem to validate this conjecture. The pursuit of a rigorous proof of such a conjecture is a main open issue. Of particular interest is also the numerical resolution of the relaxed problem obtained from the Young measure approach. This is less standard in this context, but seems to be very robust at least in our experiments.

Finally, it could be interesting to analyze the time-dependent case by using the homogenization method. The concepts of time-dependent $G-$closure and time-dependent laminates would have to be better understood ([11, 12]).

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