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Analysis of the Gibbs phenomenon in stationary subdivision schemes[☆]

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A B S T R A C T

In this paper sufficient conditions to determine if a stationary subdivision scheme produces Gibbs oscillations close to discontinuities are presented. It consists of the positivity of the partial sums of the values of the mask. We apply the conditions to non-negative masks and analyze (numerically when the sufficient conditions are not satisfied) the Gibbs phenomenon in classical and recent subdivision schemes like B-splines, Deslauriers and Dubuc interpolation subdivision schemes and the schemes proposed in Siddiqi and Ahmad (2008).

1. Introduction and review

Subdivision schemes are powerful tools for generating curves and surfaces. They are used in many applications in different fields such as computer aided geometric design, computer animation or computer graphics. Following the notation used by Dyn and Levin in [1], a univariate stationary subdivision scheme with finitely support mask $\mathbf{a} = \{a_j\}_{j\in\mathbb{Z}}$ is defined as beginning with an initial sequence of finite data $f^0 = \{f_i^0\}_{i\in\mathbb{Z}}$. New values are obtained through refinement at level $k+1$, denoted by $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$. It is the maximal set obtained by applying the rule:

$$(S_{\mathbf{a}} f^k)_i = f_i^{k+1} := \sum_{j\in\mathbb{Z}} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}.$$

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There are two rules to define the points on the level $k+1$:

$$(S_{\mathbf{a}}f^k)_{2i} = f_{2i}^{k+1} = \sum_{\gamma\in\mathbb{Z}} a_{2\gamma} f_{i-\gamma}^k, \qquad i \in \mathbb{Z},$$
$$(S_{\mathbf{a}}f^k)_{2i+1} = f_{2i+1}^{k+1} = \sum_{\gamma\in\mathbb{Z}} a_{2\gamma+1} f_{i-\gamma}^k, \qquad i \in \mathbb{Z}. \tag{1}$$

We can represent these rules using algebraic formalism in terms of z-transforms. The symbol of the mask $\mathbf{a} = \{a_j\}_{j\in\mathbb{Z}}$ is defined as $a(z) = \sum_{j\in\mathbb{Z}} a_j z^j$. If $a^{[k]}$ is denoted as the k iterated symbol (see [1]), then, it is known that $a^{[k]}(z) = \prod_{l=1}^{k} a(z^{2^{l-1}})$. Therefore, if $0 \le l < 2^k$ then

$$f_{2^k i+l}^k = (S_{\mathbf{a}}^k f^0)_{2^k i+l} = (S_{\mathbf{a}^{[k]}} f^0)_{2^k i+l} = \sum_{\gamma\in\mathbb{Z}} a_{2^k\gamma+l}^{[k]} f_{i-\gamma}^0, \tag{2}$$

with

$$a_j^{[k]} = \sum_{i\in\mathbb{Z}} a_i^{[k-1]} a_{j-2i}, \tag{3}$$

being $a_j^{[1]} = a_j$, $\forall j$. In what follows, we review the notion of convergence of the subdivision scheme:

Definition 1. A subdivision scheme $S_{\mathbf{a}}$ is uniformly convergent if for any initial data $f^0 = \{f_i^0\}_{i\in\mathbb{Z}}$, there exists a continuous function $\hat{f}$ such that for any closed interval I, $\hat{f}$ satisfies:

$$\lim_{k\to\infty} \sup_{i\in 2^k I} |(S_{\mathbf{a}}^k f^0)_i - \hat{f}(2^{-k} i)| = 0. \tag{4}$$

Then, we call $S_{\mathbf{a}}^\infty f^0 = \hat{f}$.

In this paper we will impose to the coefficients of the mask the following conditions:

C1. The mask $\mathbf{a} = \{a_j\}_{j\in\mathbb{Z}}$ is defined as:

$$\begin{cases} -1 \le a_j \le 1, & M \le j \le M+N, \\ a_M, a_{M+N} \ne 0, & \\ a_j = 0, & j > M+N \text{ or } j < M, \end{cases} \tag{5}$$

being $M < 0$ and N fixed integers with $N \ge 2$ and $M+N > 0$.

C2. The scheme $S_{\mathbf{a}}$ is convergent. Therefore, one necessary condition is

$$\sum_{\gamma\in\mathbb{Z}} a_{2\gamma} = \sum_{\gamma\in\mathbb{Z}} a_{2\gamma+1} = 1. \tag{6}$$

C3. For any $f \in \mathcal{C}^n$, $n \ge 2$ and any $h > 0$, with

$$\begin{cases} f_i^0 = f(ih),\ i \in \mathbb{Z}, & \text{if the scheme has primal parametrization,} \\ f_i^0 = f\big((i - \frac{1}{2})h\big),\ i \in \mathbb{Z}, & \text{if the scheme has dual parametrization,} \end{cases}$$

then

$$\max_{x\in\mathbb{R}} |(S_{\mathbf{a}}^\infty f^0)(x) - f(x)| \le K h^n,$$

being K a constant which depends on f, and it does not depend on h.

The order of approximation determines the precision of the subdivision scheme. Gibbs oscillations could occur in the limit functions close to discontinuity zones. The aim of this paper is to analyze the properties of the masks of the schemes in order to determine when these oscillations will appear. We describe the Gibbs phenomenon using the definition by Gottlieb and Shu and characterize the schemes with Gibbs oscillations in Section 2. We present some classical subdivision schemes and, finally, we apply these conditions to subdivision schemes with non-negative masks.

2. Gibbs phenomenon in stationary subdivision schemes

In this section we use the characterization of Gibbs phenomenon introduced by Gottlieb and Shu in [2]. Given a punctually discontinuous function f and its sampling f_h defined by $f_{i,h} = f(ih)$ ($f_{i,h} = f(ih - \frac{1}{2}h)$ if the scheme has dual parametrization), the Gibbs phenomenon deals with the convergence of $(S^\infty f_h)$ towards f when h goes to 0. It can be delimited by two properties (see [2]).

1. Away from the discontinuity (we denote it as ξ) the convergence is rather slow and for any point x,

$$|f(x) - (S^\infty f_h)(x)| = O(h). \tag{7}$$

2. There is an overshoot, close to the discontinuity, that does not diminish with the reduction of h. Thus,

$$\max_{x\in\mathbb{R}} |f(x) - (S^\infty f_h)(x)| \text{ does not tend to zero with } h. \tag{8}$$

We will prove the following theorem that introduces some conditions to obtain stationary subdivision schemes avoiding oscillations.

Theorem 2.1. *Given $0 \le \xi \le h$, let f be any function defined by*

$$\begin{aligned} \forall x \le \xi, \quad & f(x) = f_-(x), f_- \in \mathcal{C}^n(]-\infty, \xi]), \\ \forall x > \xi, \quad & f(x) = f_+(x), f_+ \in \mathcal{C}^n([\xi, +\infty[), \end{aligned}$$

with $n \ge 2$ and $f_-(\xi) > f_+(\xi)$. Let $S_{\mathbf{a}}$ be a univariate stationary subdivision scheme with:

$$\lambda_l^{[k]}(i) = \begin{cases} \sum_{\tau \le i} a^{[k]}_{2^k\tau+l}, & i < 0, \\ 0, & i = 0, \\ \sum_{\tau \ge i} a^{[k]}_{2^k\tau+l}, & i > 0, \end{cases} \tag{9}$$

$0 \le l < 2^k$, being $a^{[k]}$ defined in Eq. (3). Then, if $\lambda_l^{[k]}(i) \ge 0$, $\forall\, i, k$; and if h is sufficiently small we have:

P1. *If $|x| \ge \max\{|M-1|, |M+N+1|\}h$, then*

$$|f(x) - (S^\infty_{\mathbf{a}} f_h)(x)| = O(h^n),$$

with $n \ge 2$.

P2. *If $|x| \le \max\{|M-1|, |M+N+1|\}h$, there exists $\alpha_h = O(h)$ such that*

$$f_{1,h} - \alpha_h \le f_+(h) - \alpha_h \le (S^\infty_{\mathbf{a}} f_h)(x) \le f_-(0) + \alpha_h = f_{0,h} + \alpha_h. \tag{10}$$

Proof. We follow the sketch of the proof presented by Amat et al. in [3].

For any iteration k, there exist p_k^-, p_k^+ such that, for all $i \notin [p_k^-, p_k^+]$ the evaluation $(S^{k+1}_{\mathbf{a}} f_h)_{2i+l}, l = 0, 1$ is applied starting only from regular data.

Let us consider $k = 0$. As $0 \le \xi \le h$, the discontinuity is between the values $f_{0,h}$ and $f_{1,h}$, then by Eq. (1) when $\gamma = i$ and $\gamma = i - 1$, these values are used. By definition of the mask $\mathbf{a}$, see Eq. (5), we only have to determine when the values $a_{2i-2}, a_{2i-1}, a_{2i}, a_{2i+1}$ are different from zero. Therefore, we can define:

$$p_0^- = \frac{M-1}{2}, \quad p_0^+ = \frac{M+N}{2} + 1. \tag{11}$$

Let us consider $k = 1$. The points calculated using $f_{0,h}$ and $f_{1,h}$ at level $k = 1$ are $i \in [p_0^-, p_0^+]$, then we have that $p_1^- = \frac{M-1}{2} + 2p_0^- = 3p_0^-$, $p_1^+ = \frac{M+N}{2} + 2p_0^+ = 3(\frac{M+N}{2}) + 2$. By induction, $p_k^- = (2^{k+1} - 1)p_0^- = (2^{k+1} - 1)(\frac{M-1}{2})$, $p_k^+ = (2^{k+1} - 1)(\frac{M+N}{2}) + 2^k$. Then for the condition (C3) for $|x| \geq \max\{|M-1|, |M+N+1|\}h$, (P1) is satisfied.

In order to prove the second part of the theorem, we consider the initial data and iterate the scheme.

By Eqs. (1) and (6) we have that

$$\begin{aligned}
(S_{\mathbf{a}}f_h)_{2i} &= \sum a_{2\gamma} f_{i-\gamma,h} = f_{i,h} + \sum_{\gamma\leq -1}\Big(\sum_{\tau\leq\gamma} a_{2\tau}\Big)(f_{i-\gamma,h} - f_{i-\gamma-1,h}) + \sum_{\gamma\geq 1}\Big(\sum_{\tau\geq\gamma} a_{2\tau}\Big)(f_{i-\gamma,h} - f_{i-\gamma+1,h}) \\
&= f_{i,h} + \sum_{\gamma\leq -1}\lambda_0(\gamma)(f_{i-\gamma,h} - f_{i-\gamma-1,h}) + \sum_{\gamma\geq 1}\lambda_0(\gamma)(f_{i-\gamma,h} - f_{i-\gamma+1,h}), \\
(S_{\mathbf{a}}f_h)_{2i+1} &= \sum a_{2\gamma+1} f_{i-\gamma,h} = f_{i,h} + \sum_{\gamma\leq -1}\Big(\sum_{\tau\leq\gamma} a_{2\tau+1}\Big)(f_{i-\gamma,h} - f_{i-\gamma-1,h}) \\
&\quad + \sum_{\gamma\geq 1}\Big(\sum_{\tau\geq\gamma} a_{2\tau+1}\Big)(f_{i-\gamma,h} - f_{i-\gamma+1,h}) \\
&= f_{i,h} + \sum_{\gamma\leq -1}\lambda_1(\gamma)(f_{i-\gamma,h} - f_{i-\gamma-1,h}) + \sum_{\gamma\geq 1}\lambda_1(\gamma)(f_{i-\gamma,h} - f_{i-\gamma+1,h}),
\end{aligned} \tag{12}$$

being:

$$\lambda_l(i) = \begin{cases} \sum_{\tau\leq i} a_{2\tau+l}, & i < 0, \\ 0, & i = 0, \\ \sum_{\tau\geq i} a_{2\tau+l}, & i > 0, \end{cases} \tag{13}$$

with $l = 0, 1$. If $i \leq 0$, then

$$\begin{aligned}
(S_{\mathbf{a}}f_h)_{2i} &= f_{i,h} + \lambda_0(i-1)(f_{1,h} - f_{0,h}) + O(h) = f_{0,h} + \lambda_0(i-1)(f_{1,h} - f_{0,h}) + (f_{i,h} - f_{0,h}) + O(h), \\
(S_{\mathbf{a}}f_h)_{2i+1} &= f_{i,h} + \lambda_1(i-1)(f_{1,h} - f_{0,h}) + O(h) \\
&= f_{0,h} + \lambda_1(i-1)(f_{1,h} - f_{0,h}) + (f_{i,h} - f_{0,h}) + O(h).
\end{aligned} \tag{14}$$

If $i > 0$, then

$$\begin{aligned}
(S_{\mathbf{a}}f_h)_{2i} &= f_{i,h} + \lambda_0(i)(f_{0,h} - f_{1,h}) + O(h) = f_{1,h} + \lambda_0(i)(f_{0,h} - f_{1,h}) + (f_{i,h} - f_{1,h}) + O(h), \\
(S_{\mathbf{a}}f_h)_{2i+1} &= f_{i,h} + \lambda_1(i)(f_{0,h} - f_{1,h}) + O(h) = f_{1,h} + \lambda_1(i)(f_{0,h} - f_{1,h}) + (f_{i,h} - f_{1,h}) + O(h).
\end{aligned} \tag{15}$$

Since $0 \leq \lambda_0(i), \lambda_1(i)$, $\forall i$, and $f_{0,h} - f_{1,h} > 0$, we obtain that if $i \in [p_0^-, p_0^+]$

$$f_{1,h} - O(h) \leq (S_{\mathbf{a}}f_h)_{2i}, (S_{\mathbf{a}}f_h)_{2i+1} \leq f_{0,h} + O(h).$$

If $k = 2$ we get by Eq. (2),

$$\begin{aligned}
(S_{\mathbf{a}}^2 f_h)_{4i} &= S_{\mathbf{a}}(S_{\mathbf{a}}f_h)_{4i} = (S_{\mathbf{a}^{[2]}}f_h)_{4i} = \sum_{\gamma\in\mathbb{Z}} a_{4\gamma}^{[2]} f_{i-\gamma,h} \\
&= f_{i,h} + \sum_{\gamma\leq -1}\Big(\sum_{\tau\leq\gamma} a_{4\tau}^{[2]}\Big)(f_{i-\gamma,h} - f_{i-\gamma-1,h}) + \sum_{\gamma\geq 1}\Big(\sum_{\tau\geq\gamma} a_{4\tau}^{[2]}\Big)(f_{i-\gamma,h} - f_{i-\gamma+1,h}) \\
&= f_{i,h} + \sum_{\gamma\leq -1}\lambda_0^{[2]}(\gamma)(f_{i-\gamma,h} - f_{i-\gamma-1,h}) + \sum_{\gamma\geq 1}\lambda_0^{[2]}(\gamma)(f_{i-\gamma,h} - f_{i-\gamma+1,h}).
\end{aligned} \tag{16}$$

By Eq. (3) with,

$$\lambda_0^{[2]}(i) = \begin{cases} \sum_{\tau\leq i}\Big(\sum_{l\in\mathbb{Z}} a_l a_{4\tau-2l}\Big), & i < 0, \\ 0, & i = 0, \\ \sum_{\tau\geq i}\Big(\sum_{l\in\mathbb{Z}} a_l a_{4\tau-2l}\Big), & i > 0. \end{cases} \tag{17}$$

Analogously for $4i+1$, $4i+2$, $4i+3$. Therefore, if $\lambda_l^{[2]}(i) \geq 0$, $\forall\ i$, $0 \leq l < 4$, then

$$f_{1,h} - O(h) \leq (S_{\mathbf{a}^{[2]}} f_h)_{4i}, (S_{\mathbf{a}^{[2]}} f_h)_{4i+1}, (S_{\mathbf{a}^{[2]}} f_h)_{4i+2}, (S_{\mathbf{a}^{[2]}} f_h)_{4i+3} \leq f_{0,h} + O(h).$$

Using the same reasoning, through an induction process, we obtain the result for all k. ■

2.1. *Gibbs phenomenon in classical subdivision schemes*

In this section we analyze the Gibbs phenomenon using Theorem 2.1 in some typical examples of subdivision schemes (see [1]). We will present the limit functions obtained by the different schemes when the initial data is a sampling of the function,

$$g(x) = \begin{cases} \sin(\pi x), x \in [0, 0.5], \\ -\sin(\pi x), x \in]0.5, 1]. \end{cases} \tag{18}$$

The limit function will be represented by a solid line while the initial data points will be represented with • markers. The results can be seen in Table 2.

2.1.1. *B-splines*

A B-spline scheme of degree $m > 0$ (see [1]) is defined by the following mask:

$$(a_m)_j = \begin{cases} \frac{1}{2^m} \binom{m+1}{j+1+[\frac{m}{2}]}, & -[\frac{m}{2}] - 1 \leq j \leq m - [\frac{m}{2}], \\ 0, & j > m - [\frac{m}{2}] \text{ or } j < -[\frac{m}{2}] - 1. \end{cases} \tag{19}$$

Then, the B-spline schemes do not produce Gibbs phenomenon at the discontinuity zones because of the positivity of the mask (Theorem 2.1). Since the limiting curves generated by the B-spline scheme of degree m are $\mathcal{C}^{m-1}$, then we can construct a subdivision scheme with high continuity that does not produce Gibbs phenomenon. A known example is Chaikin's algorithm (see [4]). This scheme is used, for example, to design non-linear subdivision schemes avoiding Gibbs oscillations (see [3]). An example of the results obtained by this scheme is presented in Table 2(a) for 1D subdivision and Table 2(g) for a 2D curve generation example.

2.1.2. *Deslauriers and Dubuc interpolatory schemes [5]*

The symmetric interpolatory schemes designed by Deslauriers and Dubuc in [5] consist of replicating the even values, i.e. $(S_{\mathbf{dd}} f^k)_{2i} = f_i^k$ and calculating the odd values f_{2i+1}^{k+1} as the evaluation of the polynomial interpolation at $2^{-k-1}(2i+1)$ obtaining the $2m$ values $f_{i-m+1}^k, \ldots, f_{i+m}^k$. Then, the subdivision schemes are defined as:

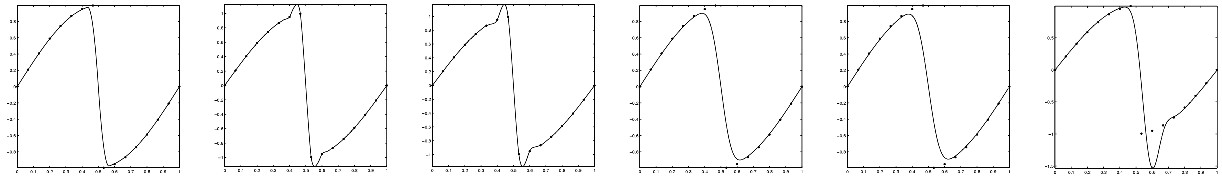
$$\begin{aligned} (S_{\mathbf{dd1}} f^k)_{2i+1} &= \frac{1}{2} f_i^k + \frac{1}{2} f_{i+1}^k, \\ (S_{\mathbf{dd2}} f^k)_{2i+1} &= -\frac{1}{16} f_{i-1}^k + \frac{9}{16} f_i^k + \frac{9}{16} f_{i+1}^k - \frac{1}{16} f_{i+2}^k, \\ (S_{\mathbf{dd3}} f^k)_{2i+1} &= \frac{3}{256} f_{i-2}^k - \frac{25}{256} f_{i-1}^k + \frac{150}{256} f_i^k + \frac{150}{256} f_{i+1}^k - \frac{25}{256} f_{i+2}^k + \frac{3}{256} f_{i+3}^k. \end{aligned} \tag{20}$$

When $m = 1$ (**dd1**), it is easy to see that this scheme does not produce Gibbs oscillations. When $m = 2$ the mask is $\mathbf{dd2} = (-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16})$. Since $\lambda_1(-2) = -\frac{1}{16}$, the scheme does not satisfy the sufficient conditions of Theorem 2.1. In fact, in the numerical experiments we can clearly observe Gibbs oscillations. See Table 2(b) for a 1D subdivision experiment and Table 2(h) for 2D subdivision. Finally, when $m = 3$ we present the values of λ_1 in Table 1. As $\lambda_1(-2) = -\frac{22}{256}$, analogously to case $m = 2$, this scheme can produce Gibbs oscillations. The results for this scheme are shown in Table 2(c) for a 1D subdivision experiment and Table 2(i) for 2D subdivision.

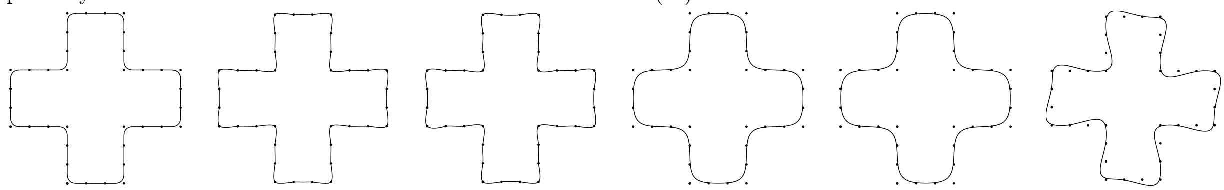
Table 1
Values of λ_1 of the Deslauriers and Dubuc's scheme with $m = 3$.

i	-3	-2	-1	0	1	2
$\lambda_1(i)$	$\frac{3}{256}$	$-\frac{22}{256}$	$\frac{1}{2}$	0	$-\frac{22}{256}$	$\frac{3}{256}$

Table 2
Figures (a), (b), (c), (d), (e), (f) present 1D subdivision experiments for the schemes Chaikin, Deslauriers and Dubuc with $m = 2, 3$, (21) and (22) with $\omega = 0$ and $\omega = 4$. In this figures, the original sampling of function (18) is represented with • markers and the limit function with a solid line. Figures (g), (h), (i), (j), (k), (l) present an application to $2D$ curve generation for the same schemes.



(a) Scheme proposed by Chaikin. (b) DD's scheme with $m = 2$. (c) DD's scheme with $m = 3$. (d) Siddiqi's sch. in (21). (e) Wang-Li's sch. $\omega = 0$. (f) Wang-Li's scheme with $\omega = 4$.



(g) Scheme proposed by Chaikin. (h) DD's scheme with $m = 2$. (i) DD's scheme with $m = 3$. (j) Siddiqi's sch. in (21). (k) Wang-Li's sch. $\omega = 0$. (l) Wang-Li's scheme with $\omega = 4$.

2.2. Subdivision schemes with non-negative masks

Using Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $S_{\mathbf{a}}$ be a convergent subdivision scheme, with mask $\mathbf{a} = \{a_M, \ldots, a_{M+N}\}$. If $a_j > 0$, $M \le j \le M + N$ then $S_{\mathbf{a}}$ does not produce Gibbs oscillations close to a discontinuity.*

Examples of this type of schemes are B-spline subdivision schemes. We apply the corollary in the following scheme designed by Siddiqi and Ahmad in [6]:

$$\begin{aligned}(S_{\mathbf{sa}}f^k)_{2i} &= \frac{1}{122880}f^k_{i-2} + \frac{3119}{122880}f^k_{i-1} + \frac{6719}{20480}f^k_i + \frac{31927}{61440}f^k_{i+1} + \frac{15349}{122880}f^k_{i+2} + \frac{81}{40960}f^k_{i+3},\\ (S_{\mathbf{sa}}f^k)_{2i+1} &= \frac{81}{40960}f^k_{i-2} + \frac{15349}{122880}f^k_{i-1} + \frac{31927}{61440}f^k_i + \frac{6719}{20480}f^k_{i+1} + \frac{3119}{122880}f^k_{i+2} + \frac{1}{122880}f^k_{i+3}.\end{aligned} \tag{21}$$

A one dimensional example for this scheme is presented in Table 2(d). In Table 2(j) we present an experiment for 2D curve generation for the same scheme.

Theorem 2.3 (*Siddiqi and Ahmad [6]*)**.** *The scheme $S_{\mathbf{sa}}$ defined in Eq. (21), converges and has smoothness $\mathcal{C}^6$.*

Also, Corollary 2.2 can be used in the schemes proposed in [7,8].

Finally, we show the following example designed by Wang and Li in [9]:

$$\begin{aligned}(S_{\mathbf{wl}}f^k)_{2i} &= \frac{1}{256}\big((9-7\omega)f^k_{i-2} + (84-28\omega)f^k_{i-1} + (126+14\omega)f^k_i + (36+20\omega)f^k_{i+1} + (1+\omega)f^k_{i+2}\big),\\ (S_{\mathbf{wl}}f^k)_{2i+1} &= \frac{1}{256}\big((1-\omega)f^k_{i-2} + (36-20\omega)f^k_{i-1} + (126-14\omega)f^k_i + (84+28\omega)f^k_{i+1} + (9+7\omega)f^k_{i+2}\big).\end{aligned} \tag{22}$$

Wang and Li calculate the conditions of the convergence proving the following theorem:

Theorem 2.4 (*Wang and Li* [9]). *The limiting curves generated by the subdivision scheme $S_{\mathbf{wl}}$ are $\mathcal{C}^0$ continuous in the range $-6.2 < \omega < 6.2$, $\mathcal{C}^1$ continuous in the range $-5.4 < \omega < 5.4$, $\mathcal{C}^2$ continuous in the range $-5.3 < \omega < 5.3$, $\mathcal{C}^3$ continuous in the range $-\frac{13}{3} < \omega < \frac{13}{3}$, $\mathcal{C}^4$ continuous in the range $-4 < \omega < 4$, $\mathcal{C}^5$ continuous in the range $-3 < \omega < 3$, $\mathcal{C}^6$ continuous in the range $-2 < \omega < 2$, and $\mathcal{C}^7$ when $\omega = 0$.*

If the parameter $\omega \in [-1, 1]$ then the masks are positive since $\lambda_0(-2) = \frac{1}{256}(1+\omega)$ and $\lambda_1(2) = \frac{1}{256}(1-\omega)$. Thus, by Corollary 2.2, these schemes do not produce Gibbs oscillations. Some examples are presented in Table 2(e) and (f) when $\omega = 0$ and when $\omega = 4$ respectively. In Table 2(k) and (l) we present two experiments for 2D curve generation for the same scheme and the same parameters.

3. Conclusions

In this paper the behavior of stationary subdivision schemes in the presence of strongly varying data, and in particular the possible apparition of Gibbs phenomenon, has been analyzed. A way to construct subdivision schemes without Gibbs oscillations has been showed. Using some sufficient conditions, we have proved that the schemes with non-negative masks do not produce Gibbs oscillations. Therefore, this result has been used for schemes with high continuity.

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