On a generalization of Steffensen method

One of the most important techniques to study nonlinear equations is the use of iterative processes, starting from an initial approximation x_0, called pivot, successive approaches (until some predetermined convergence criterion is satisfied) x_i are computed , $i = 1, 2, \ldots$, with the help of certain iteration function $\Phi : X \to X$,

$$x_{i+1} := \Phi(x_i), \quad i = 0, 1, 2 \ldots \tag{1}$$

Certainly Newton method (second order) is the most useful iteration for this purpose. In this case, we need to evaluate a derivative in each step, it is the main difficulty. Steffensen method (second order) can be considered as a simplification of original Newton method where $F^{\prime}(x_k)$ is replaced by a special approximation. If we are interesting to approximate a solution of the nonlinear equation

$$F(x) = x, \tag{2}$$

Steffensen method can be written as

$$x_{k+1} = x_k + (I - [F(x_k), x_k; F])^{-1}(F(x_k) - x_k). \tag{3}$$

where $[\cdot, \cdot; F]$ denotes a divided difference of first order for the operator $F : X \to X$ (X a Banach space).

The main limitation of this method is the first iterations. In this case, $[F(x_k), x_k; F] = [x_k + (F(x_k) - x_k), x_k; F]$ is not a good approximation of $F^{\prime}(x_k)$, since $||F(x_k) - x_k||$ is not small enough. We will consider the following generalization of Steffensen method

$$x_{k+1} = x_k + (I - [\alpha_k(F(x_k) - x_k) + x_k, x_k; F])^{-1}(F(x_k) - x_k), \tag{4}$$

where the parameters α_k will be a control of the good approximation to the first derivative. In practice, $\{\alpha_k\}$ will be a increasing sequence in $(0, 1]$, and $||\alpha_k(F(x_k) - x_k)||$ will be small enough. In order to control the stability, the α_k can be computed such that

$$tol_c << ||\alpha_k(F(x_k) - x_k)|| \leq tol_{user}$$

where tol_{c} is related with the computer precision and tol_{user} is a free parameter for the user.

The structure of the paper is the following: In section two we assert two convergence and uniqueness theorems. These theorems establish sufficient conditions on the operator and the pivot in order to ensure that the sequence of iterates converges to a solution of the equation. In both cases, the original problem in Banach spaces is replaced by a simple one with real sequences. The first theorem is related with the classical theory for Steffensen method. We impose similar hypothesis than Jhonson and Scholz. In some recent works (for Secant method), Hernández and Rubio relax this requirement and they only assume that the divided difference satisfies

$$||[x, y; G] - [v, w; G]|| \leq \omega(||x - v||, ||y - w||), \qquad x, y, v, w \in B$$

where $\omega : IR_+ \times IR_+ \rightarrow IR_+$ is a continuous function nondecreasing in both components. In the second theorem we extend this theory for our method

$$x_{\mathsf{k+1}} = x_{\mathsf{k}} - ([x_{\mathsf{k}}, x_{\mathsf{k}} + \alpha_{\mathsf{k}} G(x_{\mathsf{k}}); G])^{-1} G(x_{\mathsf{k}}) \tag{5}$$

for the equation $G(x) = 0$. In section three some numerical experiments are presented. We compare the introduced method with classical Steffensen and Newton methods. In our experiments, we consider $tol_{\mathsf{user}} = 10^{-4}$. This number is small enough but without numerical instability, since $tol_{\mathsf{user}} >> tol_{\mathsf{c}}$.