Third order iterative methods without using second Fréchet derivative

Great quantity of general problems may be reduced to finding zeros. The roots of a nonlinear equation cannot in general be expressed in closed form. Thus, in order to solve nonlinear equations, we have to use approximate methods. One of the most important techniques to study these equations is the use of iterative processes, starting from an initial approximation x_0, called pivot, successive approaches (until some predetermined convergence criterion is satisfied) x_i are computed , $i = 1, 2, \dots$, with the help of certain iteration function $\Phi : X \to X$,

$$x_{i+1} := \Phi(x_i), \quad i = 0, 1, 2 \dots \tag{1}$$

Certainly Newton's method is the most useful iteration for this purpose. The advance of computational techniques has allowed the development of some more complicated iterative methods in order to obtain greater order of convergence as Chebyshev and Halley methods. In these methods we have to evaluate first and overall second derivatives. These difficulties are usually harder than the advantage because of the order of these methods. So, two order iterative methods are widely used.

In this paper, we present a modification of classical third order iterative methods. The main advantage of these methods is they do not need evaluate any second derivative, but having the same properties of convergence than the classical third order methods. The methods will depend, in each iteration, of a parameter α_n. These parameters will be a control of the good approximation to the second derivatives. We will use second order divided differences. We will study their convergence by recurrence relations and we will test their competitively with respect the classical methods. They seemed to work very well in our preliminary numerical results.

Let $F : B \subset X \to X$ a nonlinear operator, X a Banach space and B an open convex set.

If we are interesting to approximate a solution of the nonlinear equation

$$F(x) = 0, \tag{2}$$

Chebyshev method can be written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_F(x_n)\right) \left[F'(x_n)\right]^{-1} F(x_n), \tag{3}$$

where

$$L_F(x_n) = \left[F'(x_n)\right]^{-1} F''(x_n) \left[F'(x_n)\right]^{-1} F(x_n).$$

Our modification will be

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} \mathcal{L}_F(x_n)\right) \left[F'(x_n)\right]^{-1} F(x_n), \tag{4}$$

where

$$\mathcal{L}_F(x_n) = \left[F'(x_n)\right]^{-1} \mathcal{D}_F(x_n) \left[F'(x_n)\right]^{-1} F(x_n),$$

$$\mathcal{D}_F(x_n) = [x_n - \alpha_n F(x_n), x_n, x_n + \alpha_n F(x_n); F],$$

and $[\cdot, \cdot, \cdot; F]$ denotes the second divided difference of the operator F.

The method will depend, in each iteration, of a parameter α_n. This parameter will be a control of the good approximation to the second derivative. In practice, $\{\alpha_n\}$ will be a increasing sequence in $(0, 1]$, and $||\alpha_n F(x_n)||$ will be small enough.

Remark 1 In order to control the stability in practice, the α_n can be computed such that

$$tol_c << ||\alpha_n F(x_n)|| \leq tol_u$$

where tol_c is related with the computer precision and tol_u is a free parameter for the user.

Taylor series expansions show that with these approximations the method (in the scalar case) can be written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_f(x_n) + O(L_f^2(x_n))\right) \left[f'(x_n)\right]^{-1} f(x_n), \tag{5}$$

thus, if the method converges, it has order three.

We are interesting to obtain sufficient conditions of convergence. We establish a convergence theorem using recurrence relations in a similar way that Gutiérrez and Hernández.

We have tested it on some nonlinear equations. We present a comparison with the classical Chebyshev method.