A Note on a Family of Newton Type Iterative Processes

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Abstract

In this paper, we study the convergence of a family of iteration methods to solve nonlinear equations in the complex plane. Two analysis of convergence are provided. We give a Kantorovich-type convergence theorem under mild differentiability conditions with error analysis.

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1 Introduction

Hernández and Salanova [5] define a new family of iterative processes of second order depending on a real parameter $\alpha \geq 0$ by

$$x_{\alpha,n+1} = x_{\alpha,n} - \frac{h(x_{\alpha,n})}{h'(x_{\alpha,n})}\left(1 + \alpha h(x_{\alpha,n})\right), \quad n \geq 0,$$

to solve a nonlinear scalar equation $h(x) = 0$. A thorough analysis is realized in [5], it is shown that an iterative processes of above family can always be applied to solve $h(x) = 0$ and this process is faster than Newton's method. They also give a Kantorovich theorem to prove the convergence in the complex plane.

We continue with the analysis of the convergence in the complex plane. We consider the problem of solving the equation

$$f(z) = 0 \tag{1}$$

where $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is an holomorfic function on some open convex domain D. Let $z_0 = z_{\alpha,0} \in D$ and be the family of iterative processes defined in [5] for all $n \geq 0$ by

$$z_{\alpha,n+1} = F_\alpha(z_{\alpha,n}) = z_{\alpha,n} - \frac{f(z_{\alpha,n})}{f'(z_{\alpha,n})} \left(1 + \alpha f(z_{\alpha,n})\right), \tag{2}$$

where $\alpha \geq 0$, to solve equation (1). This family of iterations includes the Newton's method as a specific choose of the parameter ($\alpha = 0$).

On the one hand, we study the Kantorovich convergence of family (2) by means of majorizing sequences ([7],[9]) where function f satisfy a Lipschitz-type condition. We also give error bound expressions depending on the real parameter α.

Let us denote

$$\overline{B(z,r)} = \{w \in \mathbb{C}; |w - z| \leq r\} \quad \text{and} \quad B(z,r) = \{w \in \mathbb{C}; |w - z| < r\}.$$

2 The Newton-Kantorovich convergence

Hernández and Salanova [5] study the convergence of the family of methods (2) under standard original Kantorovich conditions [7]. Here we analyse the convergence of family (2) under milder differentiability conditions. The basic assumption made is that the first derivative f' of f is Lipschitz continuous in D. Let us assume throughout this section that

($\mathbf{c}_1$) $|f(z_0)| = a$,

($\mathbf{c}_2$) $|f'(z_0)| = b$,

($\mathbf{c}_3$) $\left|\dfrac{f'(z) - f'(w)}{f'(z_0)}\right| \leq k|z - w|,\ z, w \in D,\ k > 0,$

(c$_4$) $b - 2ak \geq 0$.

To establish the convergence of (2) and uniqueness of solution, we will need the following two results. The proof of the first one follows inmediately.

Lemma 2.1 *Let α be a fixed real number that satisfies $0 \leq \alpha < \dfrac{b-2ak}{8ab}$. Then we have:*

(i) $\left[b + \dfrac{4b^2\alpha}{k}, \dfrac{b^2}{2ak}\right] \neq \emptyset$.

(ii) *If $N \leq \dfrac{b^2}{2ak}$, the equation*

$$p(t) \equiv \frac{kN}{2}t^2 - bt + a = 0 \tag{3}$$

has two positive roots r_1 and r_2 ($r_1 \leq r_2$). Besides $N = \dfrac{b^2}{2ak}$ if and only if $r_1 = r_2$.

Lemma 2.2 *Let p be the polynomial defined in (3). Then the sequence*

$$t_0 = t_{\alpha,0} = 0,$$

$$t_{\alpha,n+1} = P_\alpha(t_{\alpha,n}) = t_{\alpha,n} - \frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})}(1 + \alpha p(t_{\alpha,n})), \quad n \geq 0, \tag{4}$$

is increasing and converges quadratically to r_1 for all $0 \leq \alpha < \dfrac{b-2ak}{8ab}$.

Proof. Note that $P'_\alpha(t) \geq 0$ in $[0, r_1]$ where

$$P'_\alpha(t) = L_p(t) - \alpha p(t)(2 - L_p(t))$$

and $L_p(t) = \dfrac{p(t)p''(t)}{p'(t)^2}$ [4]. Then by mathematical induction on n, it follows that $t_{\alpha,n} \leq r_1, \ n \geq 0$.

On the other hand, it is easy to prove that $t_{\alpha,n} \leq t_{\alpha,n+1}$ for all $n \in \mathbb{N}$ and consequently the proof is completed. ■

Now we can state an existence-uniqueness theorem.

Theorem 2.3 *Assume that conditions* $(\mathbf{c}_1)$–$(\mathbf{c}_4)$ *are satisfied and* $0 \le \alpha < \dfrac{b-2ak}{8ab}$. *Then the sequence* $\{z_{\alpha,n}\}$ *defined by (2) converges to a solution* z^* *of equation (1) in* $\overline{B(z_0, r_1)} \cap D$ *for* $N \in \left[b + \dfrac{4b^2\alpha}{k}, \dfrac{b^2}{2ak}\right]$. *The limit* z^* *is the unique solution of (1) in* $B(z_0, r) \cap D$ *where* $r = r_2 + \dfrac{2(N-b)}{kN}$. *Moreover* $|z^* - z_{\alpha,n}| \le r_1 - t_{\alpha,n}$, $n \ge 0$.

So as to show the previous theorem we need the following lemma.

Lemma 2.4 *The sequence* $\{t_{\alpha,n}\}$ *defined by (4) is a majorizing sequence of the sequence* $\{z_{\alpha,n}\}$ *given by (2), i.e.*

$$|z_{\alpha,n+1} - z_{\alpha,n}| \le t_{\alpha,n+1} - t_{\alpha,n}, \quad n \ge 0. \tag{5}$$

Proof. By mathematical induction, it suffices to show that the following statements are true for all $n \ge 0$:

$[\mathbf{I}_n]$ $f'(z_{\alpha,n}) \ne 0$,

$[\mathbf{II}_n]$ $\left|\dfrac{f'(z_0)}{f'(z_{\alpha,n})}\right| \le \dfrac{p'(t_0)}{p'(t_{\alpha,n})}$,

$[\mathbf{III}_n]$ $|f(z_{\alpha,n})| \le p(t_{\alpha,n})$,

$[\mathbf{IV}_n]$ $\left|\dfrac{f(z_{\alpha,n})}{f'(z_0)}\right| \le -\dfrac{p(t_{\alpha,n})}{p'(t_0)}$,

$[\mathbf{V}_n]$ $|z_0 - z_{\alpha,n+1}| \le t_{\alpha,n+1}$.

All the above statements are true for $n = 0$ by initial hypotheses $(\mathbf{c}_1)$–$(\mathbf{c}_4)$. Then we assume that $[\mathbf{I}_k]$–$[\mathbf{V}_k]$ are true for $k = 1, 2, \ldots, n$. From general hypotheses and

$$\left|\frac{f'(z_0) - f'(z_{\alpha,n+1})}{f'(z_0)}\right| \le k|z_0 - z_{\alpha,n+1}| \le \frac{kN}{b} t_{\alpha,n+1},$$

we obtain

$$\left|1 - \frac{f'(z_{\alpha,n+1})}{f'(z_0)}\right| \le 1 + \frac{p'(t_{\alpha,n+1})}{p'(t_0)} < 1.$$

Then

$$\left|\frac{f'(z_0)}{f'(z_{\alpha,n+1})}\right| \le \frac{p'(t_0)}{p'(t_{\alpha,n+1})}.$$

Therefore $[\mathbf{I}_{n+1}]$ and $[\mathbf{II}_{n+1}]$ are true.

Using Altman technique ([1],[10]) and taking into account (2), we deduce by Taylor's formula that

$$f(z_{\alpha,n+1}) = f(z_{\alpha,n}) + f'(z_{\alpha,n})(z_{\alpha,n+1} - z_{\alpha,n}) + \int_{z_{\alpha,n}}^{z_{\alpha,n+1}} (f'(z) - f'(z_{\alpha,n}))\, dz$$

$$= -\alpha f(z_{\alpha,n})^2 + \int_{z_{\alpha,n}}^{z_{\alpha,n+1}} (f'(z) - f'(z_{\alpha,n}))\, dz.$$

Taking norms, we have

$$|f(z_{\alpha,n+1})| \le \alpha p(t_{\alpha,n})^2 + \frac{kb}{2}(t_{\alpha,n+1} - t_{\alpha,n})^2.$$

Repeating the same process for the polynomial p, we get

$$p(t_{\alpha,n+1}) \le -\alpha p(t_{\alpha,n})^2 + \frac{kN}{2}(t_{\alpha,n+1} - t_{\alpha,n})^2.$$

As $p'(t_{\alpha,n}) \le p(t_0) = b^2$ and $1 + \alpha p(t_{\alpha,n}) \ge 1$ we infer that

$$p(t_{\alpha,n+1}) - |f(z_{\alpha,n+1})| \ge \left(\frac{k}{2b^2}(N - b) - 2\alpha\right) p(t_{\alpha,n})^2.$$

Hence

$$|f(z_{\alpha,n+1})| \le p(t_{\alpha,n+1}), \tag{6}$$

since $N \ge b + \dfrac{4b^2\alpha}{k}$.

Consequently $[\mathbf{III}_{n+1}]$ is true and $[\mathbf{IV}_{n+1}]$ follows from an analogous way. Finally,

$$|z_{\alpha,n+1} - z_{\alpha,n}| = \left|\frac{f(z_{\alpha,n})}{f'(z_{\alpha,n})}(1 + \alpha f(z_{\alpha,n})\right| \le -\frac{p(t_{\alpha,n})}{p'(t_{\alpha,n})}(1 + \alpha p(t_{\alpha,n}))$$

$$= t_{\alpha,n+1} - t_{\alpha,n},$$

then (5) holds and $\{t_{\alpha,n}\}$ majorizes $\{z_{\alpha,n}\}$. Now $[\mathbf{V}_{n+1}]$ is deduced inmediately. ■

Proof of theorem 2.3. The fact that the sequence $\{t_{\alpha,n}\}$ defined by (4) majorizes the sequence $\{z_{\alpha,n}\}$ given by (2) is a consequence of lemma 2.4. So the convergence of $\{t_{\alpha,n}\}$ implies the convergence of $\{z_{\alpha,n}\}$ to a limit z^*. When $n \to \infty$ in (6), we deduce that $F(z^*) = 0$.

Moreover, for $q \geq 0$, it follows from (5) that $|z_{\alpha,n+q} - z_{\alpha,n}| \leq t_{\alpha,n+q} - t_{\alpha,n}$, and making $q \to \infty$ we obtain $|z^* - z_{\alpha,n}| \leq r_1 - t_{\alpha,n}$, $n \geq 0$. Besides $|z^* - z_0| \leq r_1 - t_0 = r_1$.

To show the uniqueness of the solution z^*. Assume that there exists another solution w^* of equation (1) in $B(z_0, r)$ where $r = r_2 + \dfrac{2(N-b)}{kN}$. Following Argyros and Chen ([2],[3]), we have

$$f(w^*) - f(z^*) = (w^* - z^*) \int_0^1 f'(z^* + t(w^* - z^*))\, dt = 0$$

and

$$\left|1 - f'(z_0)^{-1} \int_0^1 f'(z^* + t(w^* - z^*))\, dt\right|$$
$$\leq k \left[|z_0 - z^*| \int_0^1 (1-t)\, dt + |z_0 - w^*| \int_0^1 t\, dt \right] < k \left(\frac{r_1 + r}{2} \right) = 1.$$

Therefore $w^* = z^*$ follows from $\displaystyle\int_0^1 f'(z^* + t(w^* - z^*))\, dt \neq 0$. ∎

Notice that Hernández and Salanova [5] give uniqueness of solution of equation (1) in the ball $\overline{B(z_0, \frac{2a}{b}(2 - \sqrt{2}))}$ for the family (2).

Now we get error expressions for the sequence $\{t_{\alpha,n}\}$ defined by (4). Following Ostrowski [8], we can deduce following error estimates for $r_1 - t_{\alpha,n}$, $n \geq 0$.

Theorem 2.5 *Let p be the polynomial given in (3). Assume that p has two positive roots r_1 and r_2 ($r_1 \leq r_2$). Let $\{t_{\alpha,n}\}$ be the sequence given by (4).*

(a) *If $r_1 < r_2$, let $\theta_\alpha = \dfrac{r_1}{r_2}\sqrt{\rho_\alpha}$ and $\Delta_\alpha = \dfrac{r_1}{r_2}\sqrt{\sigma_\alpha}$. Then*

$$\frac{(r_2 - r_1)\Delta_\alpha^{2^n}}{\sqrt{\sigma_\alpha} - \Delta_\alpha^{2^n}} \leq r_1 - t_{\alpha,n} \leq \frac{(r_2 - r_1)\theta_\alpha^{2^n}}{\sqrt{\rho_\alpha} - \theta_\alpha^{2^n}}, \quad n \geq 0,$$

where $\rho_\alpha = \dfrac{1}{2}\left[2-\alpha kN(r_2-r_1)^2\right]$, $\sigma_\alpha = \dfrac{2-\alpha kNr_2^2}{2-\alpha kNr_1^2}$, $\theta_\alpha < 1$ *and* $\Delta_\alpha < 1$.

(b) *If* $r_1 = r_2$, *let* $\tau_\alpha = \dfrac{1}{4}(2-\alpha kNr_1^2)$. *Then*

$$r_1\tau_\alpha^n \le r_1 - t_{\alpha,n} \le \frac{r_1}{2^n}, \quad n \ge 0.$$

where $\tau_\alpha < 1$.

Proof. Let us write $a_{\alpha,n} = r_1 - t_{\alpha,n}$ and $b_{\alpha,n} = r_2 - t_{\alpha,n}$. Hence

$$p(t_{\alpha,n}) = \frac{kN}{2}a_{\alpha,n}b_{\alpha,n} \quad \text{and} \quad p'(t_{\alpha,n}) = -\frac{kN}{2}(a_{\alpha,n}+b_{\alpha,n}).$$

By (4) we obtain

$$a_{\alpha,n} = a_{\alpha,n-1}^2\frac{2-\alpha kNb_{\alpha,n-1}^2}{2(a_{\alpha,n-1}+b_{\alpha,n-1})} \tag{7}$$

and

$$b_{\alpha,n} = b_{\alpha,n-1}^2\frac{2-\alpha kNa_{\alpha,n-1}^2}{2(a_{\alpha,n-1}+b_{\alpha,n-1})}.$$

If $r_1 < r_2$, denote $\delta_{\alpha,n} = \dfrac{a_{\alpha,n}}{b_{\alpha,n}}$ to get

$$\delta_{\alpha,n} = \delta_{\alpha,n-1}^2\frac{2-\alpha kN(r_2-t_{\alpha,n-1})^2}{2-\alpha kN(r_1-t_{\alpha,n-1})^2} = \delta_{\alpha,n-1}^2\phi_\alpha(t_{\alpha,n-1}).$$

Taking into account that the function

$$\phi_\alpha(t) = \frac{2-\alpha kN(r_2-t)^2}{2-\alpha kN(r_1-t)^2}$$

is nondecreasing in $[0,r_1]$ for all $\alpha \ge 0$, we have

$$\sigma_\alpha = \phi_\alpha(0) \le \phi_\alpha(t) \le \phi_\alpha(r_1) = \rho_\alpha. \tag{8}$$

Therefore

$$\delta_{\alpha,n} \le \rho_\alpha\delta_{\alpha,n-1} \le \ldots \le \rho_\alpha^{\frac{2^n-1}{2}}\delta_{\alpha,0}^{2^n},$$

$$\delta_{\alpha,n} \geq \sigma_\alpha \delta_{\alpha,n-1} \leq \ldots \leq \sigma_\alpha^{\frac{2^n-1}{2}} \delta_{\alpha,0}^{2^n},$$

and so the first part holds.

If $r_1 = r_2$, then $a_{\alpha,n} = b_{\alpha,n}$. By (7) we deduce

$$a_{\alpha,n} = \frac{a_{\alpha,n-1}}{4}(2 - \alpha k N a_{\alpha,n-1}^2).$$

Repeating an analogous process to the first part we get

$$a_{\alpha,n} \leq \frac{a_{\alpha,n-1}}{2} \leq \ldots \leq \frac{a_{\alpha,0}}{2^n}$$

and

$$a_{\alpha,n} \geq \tau_\alpha a_{\alpha,n-1} \geq \ldots \geq \tau_\alpha^n a_{\alpha,0}.$$

Thus the second part also holds.

From $\sigma_\alpha \geq 0$, (8) and $\rho_\alpha < 1$, it follows that $\Delta_\alpha < \theta_\alpha < 1$. Besides it is obvious that $\tau_\alpha < 1$. So the proof is completed. ∎

Remark. We give now an optimization result by means of asymptotic error constant [6]. Let us denote the assumptotic error constant of sequence (4) by $C_\alpha = \left|\frac{P''_\alpha(r_1)}{2}\right|$, where P_α is defined in (4). Then, from $N \geq b + \frac{4b^2\alpha}{k}$ it follows that

$$C_\alpha = -\frac{kN - 2\alpha(kNr_1 - b)^2}{kNr_1 - b}.$$

It is easy to check that function

$$h_\alpha(N) = -\frac{kN - 2\alpha(kNr_1 - b)^2}{kNr_1 - b}$$

is nondecreasing. Then the optime value of N in $\left[b + \frac{4b^2\alpha}{k}, \frac{b^2}{2ak}\right]$ is obtained for $N = b + \frac{4b^2\alpha}{k}$. Therefore we will consider $N = b + \frac{4b^2\alpha}{k}$ in practical situations.

Numerical result. To illustrate theorem 2.3, let us consider the equation $f(z) = 0$ where $f(z) = e^z - 1$ is an holomorfic function in $\mathbb{C}$. If we choose $D = B(0, 0.5)$ and $z_0 = 0.2(1 + i)$, then

$$a = |f(z_0)| = 0.31259, \quad b = |f'(z_0)| = 1.2214,$$

$$k = 1.34986 \quad \text{and} \quad 0 \leq \alpha < 0.123592.$$

Taking into account $\alpha = 0.1$, we have $N = 1.66347$. Therefore, from the definition (3),

$$p(t) = 1.12273t^2 - 1.2214t + 0.31259.$$

This polynomial has two real roots: $r_1 = 0.411825$ and $r_2 = 0.676065$. Hence, by theorem 2.3, the sequence of iterates $\{z_{0.1,n}\}$ given by (2) converges to the solution $z^* = 0$ of $f(z) = 0$ in $\overline{B(z_0, 0.411825)} \cap D$, see Table 1. Moreover the solution $z^* = 0$ is unique in $B(z_0, 1.06981) \cap D$.

Notice that Hernández and Salanova [5] would obtain uniqueness of the solution $z^* = 0$ in $\overline{B(z_0, 0.299837)} \cap D$. Consequently, the uniqueness domain has been increased considerablely.

Finally, observe that the sequence $\{z_{0.1,n}\}$ converges to $z^* = 0$ faster than the sequences of Newton's method $\{z_{0,n}\}$, see Tables 1 and 2.

n	$z_{0.1,n}$
0	0.200000000000000+0.200000000000000i
1	0.002463980472679+0.029343451406702i
2	-0.000340904885976+0.000619548395 67i
3	0.000000044957044-0.000000016900126i
4	0.000000000000000+0.000000000000000i

Table 1: Process iterative (2)

n	$z_{0,n}$
0	0.200000000000000+0.200000000000000i
1	0.002410647342520+0.037343309184660i
2	-0.000692598436544+0.000098570978707i
3	0.000000235040194-0.000000068293593i
4	0.000000000000025-0.000000000000016i

Table 2: Newton's method

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