Dynamical Systems encode Concurrent Computing Systems

Guirao, Juan L. G.\textsuperscript{1}, Pelayo, Fernando L.\textsuperscript{2}, Pelayo, Maria L.\textsuperscript{2} and Valverde, Jose C.\textsuperscript{3}

\textsuperscript{1}Department of Applied Mathematics, Polytechnic University of Cartagena
\textsuperscript{2}Department of Computing Systems, University of Castilla - La Mancha
\textsuperscript{3}Department of Applied Mathematics, University of Castilla - La Mancha
Juan.Garcia@upct.es, fpelayo@dsi.uclm.es, mpelayo@dsi.uclm.es, Jose.Valverde@uclm.es

Abstract

This paper presents the very first step in the line of applying Discrete Dynamical Systems, DDSs, theory to the analysis of Concurrent Computing Systems. In order to do that, Petri Nets, PNs, are properly encoded into DDSs, so defining the corresponding Phase Space with its metric structure and the evolution operator of the system. We conclude this study by reflecting on some identifiable problems.

Key words: Formal Computing Science, Applied Mathematics, Petri Nets, Discrete Dynamical Systems.

1 Introduction

Formal Models are used both to describe and to analyze the behaviour of computer systems. Among these models, we have Process Algebras, Event Structures, Markov Chains, Petri Nets and some others. Software designers work happily with process algebras, since they have a very similar syntax to programming languages, but they are not able, in general, to capture true concurrency, and even formal verification is a bit harder than it is in other formalisms like PNs.

Petri Nets were first conceived by Carl Adam Petri \cite{4}. They predate traditional Process Algebras at being able to model concurrent systems.

Petri Nets are widely used for modelling and analyzing concurrent systems, because of their graphical nature and the solid mathematical foundations supporting them. Furthermore, one of the main advantages of PNs is that they easily capture \emph{true concurrency}, i.e., they are able to model the simultaneous execution of actions in the system.

It is also well known that many scientists and technicians try to find out the future and the past state of a process whose present state they are observing. The fact is that
the future and past states of many biological, ecological, physical or even computer processes can be predicted, if their present states and the laws governing their evolution are knowing, provided these laws do not change in time.

A Dynamical System is the mathematical formalization of a deterministic process, created to deal with these kind of challenges. Thus, in this formalization, we have to include the elementary parts cited before, i.e., the set of all possible states and the evolution law in time.

In this sense, we can state a formal definition of a Dynamical System (see [1, 2, 7]), as follows:

**Dynamical System**: A Dynamical System is a triple $(X, \tau, \Phi)$, where $X$ is a set, $\tau$ is a subset of $\mathbb{R}$ which is a monoid, and $\Phi : \tau \times X \to X$ is a function verifying:

1. $\Phi(0, x) = x \quad \forall x \in X$, i.e., $\Phi_0 = id_X$
2. $\Phi(t, \Phi(s, x)) = \Phi(t + s, x) \quad \forall t, s \in \tau, \forall x \in X$

The set $X$ is called the *state space* (or *phase space*) of the system. Very often, the state space can be characterized by $\mathbb{R}^n$ or a submanifold in it. But, as we will show later, it could be a finite set as, for instance, $\mathcal{P}(\{0, 1\}^n)$. Also, it is very common that the state space allows for comparison between two states by means of a distance, making this set a metric space. In fact, it uses to be very interesting to have a complete metric (state) space.

Depending on the monoid $\tau$, it is distinguished between Continuous (time) Dynamical Systems, if $\tau = \mathbb{R}$ ($\mathbb{R}^+ \cup \{0\}$ or $\mathbb{R}^- \cup \{0\}$); and Discrete (time) Dynamical Systems, if $\tau = \mathbb{Z}$ ($\mathbb{N} \cup \{0\}$ or $\mathbb{Z}^- \cup \{0\}$). In this work, we consider a particular case of discrete.

The function $\Phi : \tau \times X \to X$ is called the evolution operator and, generally, it is a continuous function in the state variable and if $\tau = \mathbb{R}$ ($\mathbb{R}^+ \cup \{0\}$ or $\mathbb{R}^- \cup \{0\}$), it is also continuous in the time variable. This continuity is supposed to be with respect to metric in $X$.

## 2 Petri Nets

The representation of a Petri Net is a graph which has two kinds of nodes: places and transitions. Places are usually related to conditions or states, whereas transitions are associated with events or actions, which cause the changes of state in a system. The arcs in the net represent the conditions that must be fulfilled for executing an action (firing a transition), and the new conditions or states obtained after firing that transition. The behaviour of a PN can easily be represented by means of a linear equation, the so called state equation, which is the most formal tool for the analysis of PNs.

**Petri Net**: An Ordinary Petri Net (OPN) is a triple $N = (P, T, F)$ consisting of two sets $P$ and $T$, and a relation $F$ defined over $P \cup T$, such that:

1. $P \cap T = \emptyset$
2. $F \subseteq (P \times T) \cup (T \times P)$
3. $\text{dom}(F) \cup \text{cod}(F) = P \cup T$

$P$ is said to be the set of places, $T$ is called the set of transitions and $F$ is named the flow relation. $F$ relates places and transitions by arcs connecting them. In the classical representation of PNs, places are circles and transitions are rectangles.

Let $X$ be the set $X = P \cup T$. Then, for all $x \in X$ two sets are defined:

- $\bullet x = \{y \in X \mid (y, x) \in F\}$ (precondition set of $x$),
- $\bullet x^* = \{y \in X \mid (x, y) \in F\}$ (postcondition set of $x$).

**Example 1:** Let $N = (P, T, F)$ be an Ordinary Petri Net, where:

- $P = \{p_1, p_2, p_3\}$
- $T = \{t_1, t_2\}$
- $F = \{(p_1, t_1), (p_2, t_1), (t_1, p_3), (p_3, t_2)\}$

This Petri Net is graphically represented in Fig. 1.

![Figure 1: Example of Petri Net](image)

The state of a system described by a PN is captured by means of the so called Markings. They are defined as follows.

**Markings of Ordinary Petri Nets:** Let $N = (P, T, F)$ be an Ordinary Petri Net. A function $M : P \rightarrow \mathbb{N}$ is a Marking of $N$. Thus, $(P, T, F, M)$ is a Marked Ordinary Petri Net, MOPN.

Markings of Petri Nets are graphically represented by including in the places as many points as tokens.
**Example 2:** The Petri Net of Fig. 1 can be marked as shown in Fig. 2:

\[ M(p_1) = 1, \ M(p_2) = 1, \ M(p_3) = 0 \]

![Petri Net Diagram](image)

Figure 2: Example of Marked Petri Net

This Marked Ordinary Petri Net will be codified in our particular DDS as the vector state \((1, 1, 0)\).

Given a MOPN \((P, T, F, M)\) with \(P = \{p_1, \ldots, p_n\}\), a Marking, \(M\) of it which has tokens \((m)\) in places \(p_{i_1}, \ldots, p_{i_m}\) with \(m \leq n\), will be codified by a binary \(n\)-tuple containing \(1\)'s in \(p_{i_1}, \ldots, p_{i_m}\) positions and the remainder \(n - m\) positions contain \(0\)'s.

The semantics of a MOPN is defined by the following firing rule which establishes when we can fire a transition and by the marking obtained after firing.

**Firing rule:** Let \(N = (P, T, F, M)\) be a MOPN. A transition \(t \in T\) is enabled at marking \(M\), denoted by \(M[t]\), if for all place \(p \in P\) such that \((p, t) \in F\), we have \(M(p) > 0\) (\(M(p) = 1\), in the particular case we are dealing with).

An enabled transition \(M\) can be fired, thus producing a new marking, \(M'\):

\[ M'(p) = M(p) - W_f(p, t) + W_f(t, p) \quad \forall p \in P \]

where

\[
\begin{align*}
W_f(x) &= 1 \quad \text{if } x \in F \\
W_f(x) &= 0 \quad \text{if } x \notin F,
\end{align*}
\]

for all \(x \in (T \times P) \cup (P \times T)\). It is denoted by \(M[t]M'\).

We would like to note that since a place can belongs to the precondition set of more than one different transitions, a token in it could potentially enable more than one transition and, after firing one of them (transitions), more than one different marking can be reached. This fact has lead us to consider as Phase Space not the set of binary \(n\)-tuples but the set of all its subsets, in order to properly capture these cases.
In the Example 2, the fire of $t_1$ generates the marking $M'$ given by:

$$M'(p_1) = 0, \; M'(p_2) = 0, \; M'(p_3) = 1$$

These definitions can be extended in order to consider the evolution by executing an arbitrary number of transitions simultaneously.

**Concurrent activation of transitions:** Let $N = (P, T, F, M)$ be a MOPN. Let $R \subseteq T$ be a subset of transitions. It is said that all transitions in $R$ are enabled at marking $M$, denoted by $M[R]$, if and only if (iff)

$$M(p) \geq \sum_{t \in R} W_f(p, t), \; \forall p \in P,$$

where $W_f(p, t)$ is defined as in previous definition.

Moreover, we say that a multiset of transitions $R$ is enabled at marking $M$ iff

$$M(p) \geq \sum_{t \in T} W_f(p, t) \cdot R(t),$$

for all $p \in P$.

The firing of a multiset of transitions $R$ at the marking $M$ generates a new marking $M'$, defined by:

$$M'(p) = M(p) - \sum_{t \in T} (W_f(p, t) - W_f(t, p)) \cdot R(t)$$

This evolution of the PN in a single step is denoted by $M[R]M'$.

This is the way in which a PN evolves and it is assumed the best in terms of accuracy to the real behaviour of concurrent computing systems.

### 3 Petri Nets into Discrete Dynamical Systems

The DDS which encodes the MOPN $N = (P, T, F, M)$ is the triple $(X, \tau, \Phi)$, where:

- $X = \mathcal{P}(\{0, 1\}^n)$ is the set of all subsets of $\{0, 1\}^n$, being $n$ the number of places of the MOPN. This is a finite set of $2^n$ elements.
- $\tau$ is the monoid $\mathbb{N} \cup \{0\}$
- $\Phi : \tau \times X \rightarrow X$ is the evolution operator $\Phi$ verifying:
  1. $\Phi(0, A) = A \; \forall A \in X$, i.e., $\Phi_0 = id_X$
  2. $\Phi(1, A) = B \; A, B \in X$ where:
     - $A = \{x_1, \ldots, x_k\}$ where $x_i \in \{0, 1\}^n$ encodes Markings of the MOPN $N$
     - $B = \bigcup_{i=1}^k B_i$
DSs encode CCS

- \( B_i = \bigcup_{j=1}^{t_i} \{ y_j^i \} \), i.e. the union of all \((t_i)\) possible reachable markings from \( x_i \), defined by \( x_i[R_i]y_j^i \) being \( R_i \) the set of transitions of the net enabled at marking \( x_i \)

3. \( \Phi(t, \Phi(s, A)) = \Phi(t + s, A) \quad \forall t, s \in \tau, \forall A \in X \)

As we commented above, for our pretensions to formalize PNs as DDSs, we consider as Phase Space the set \( \mathcal{P}([0,1]^n) \). Now, we have to determine a metric \( d \) on this set, such that the pair \( (\mathcal{P}([0,1]^n), d) \) is a complete metric (state) space. In order to do that, following the reference [5], we begin considering on \( \{0,1\}^n \) the metric induced from the Bayre metric given by

\[
d(x, y) = \frac{1}{2^{l(x \cap y)}} - \frac{1}{2^n}, \quad x, y \in \{0,1\}^n
\]

where \( l(x \cap y) \) is the length of the longest common initial part of the vectors \( x \) and \( y \). The function so defined is a metric. Effectively, from the definition it is obvious that \( d \) is symmetric and

\[
d(x, y) = 0 \iff x = y
\]

On the other hand, for all \( x, y, z \in \{0,1\}^n \) it is true that

\[
d(x, y) \leq d(x, z) + d(z, y)
\]

To check this, it is sufficient to observe that for all \( x, y \in \{0,1\}^n \) the function \( d(x, y) \) show the coincidence grade of the initial part of \( x \) and \( y \). So, if by reduction to the absurd, we suppose that for some \( x, y, z \in \{0,1\}^n \) is

\[
d(x, y) > d(x, z) + d(z, y)
\]

and we call \( k, l, m \) the length of the longest common initial part of the pairs of vectors \((x, y), (x, z)\) and \((z, y)\), then \( k < l, m \). Note that if \( k \geq l \) (or \( k \geq m \)) then

\[
\frac{1}{2^k} - \frac{1}{2^n} \leq \frac{1}{2^l} - \frac{1}{2^n} \leq \frac{1}{2^m} - \frac{1}{2^n} + \frac{1}{2^n} - \frac{1}{2^n}
\]

what is inconsistent with the supposition made before.

But, if \( k < l, m \) the coincidence grade of the initial part of \( x \) and \( z \) and also \( z \) and \( y \) is greater than \( k \). Thus, \( x, y, z \) have a initial coincident part whose length is the minimum of \( l \) and \( m \), which is greater than \( k \), what contradicts that \( k \) is the longest common initial part of the pair of vectors \( x \) and \( y \).

At this point, taking into account this metric \( d \), we can define the distance between a vector \( x \in \{0,1\}^n \) and a subset \( B \) of vectors of \( \{0,1\}^n \), i.e., an element of \( \mathcal{P}([0,1]^n) \) in this manner

\[
d(x, B) = \min \{d(x, y) : y \in B\}
\]

and, consequently, we can establish the distance between two elements \( A, B \) of \( \mathcal{P}([0,1]^n) \) as

\[
d(A, B) = \max \{d(x, B), d(A, y) : x \in A, y \in B\}
\]
Now, in view of the reasoning before, it is easy to check that this application determines a metric on $\mathcal{P}(\{0, 1\}^n)$.

Observe that it is necessary to consider the distance from vectors of the first subset to the second subset and vice versa in order to get the symmetry of the function $d$.

Besides, note that, if two subsets $A, B$ are different, then the minimum distance between them could be

$$\frac{1}{2^n - 1} \cdot \frac{1}{2^n} = \frac{2 - 1}{2^n} = \frac{1}{2^n}$$

what allows to conclude that every Cauchy’s succession in this set is convergent, due to from a particular index, being the distance between every two terms lower or equal than $\frac{1}{2^n}$, all the terms are equal. Therefore, $(\mathcal{P}(\{0, 1\}^n), d)$ is a complete metric space.

4 Identifiable problems

As it is said before, scientists and technicians have to analyze the future and the past state of a process whose present state they are observing and, hence, they know. For this reason, it is necessary to formalize this set of evolutionary states in the context of DSs Theory.

**Orbits of a Dynamical System:** The ordered subset of the state space $X$

$$\text{Orb}(x_0) = \{ x \in X : x = \Phi(t, x_0), t \in \tau \}$$

is named the orbit of the present (or initial) state $x_0$.

Note that orbits of Continuous Dynamical Systems are curves in the state space, while orbits of DDSs are sequences of points in the state space.

Following [7], the main goals in the study of a DS are both giving a complete characterization of the geometry of its orbit structure and analyzing whether or not this structure remain when the system is perturbed slightly.

Since in our particular case of Discrete Dynamical System, we have a finite state space, it is easy to know that every orbit is either periodic or eventually periodic. Therefore, every orbit is an invariant set of the system. However, it is not so easy to determine a priori the different coexistent periods of its orbits.

On the other hand, not every result relative to orbit structure for the well known DDSs given by continuous map of the interval, works here. For instance, the famous Sharkovskii theorem (see [6]) is not true for a system with a period three, because we can only have a finite number of different periodic orbits in the system (one for each initial state).

Also, those questions which can be studied by means of the differentiability of the evolution operator, as attraction of certain orbits, are now very difficult to state.

Obviously, one could count all the diverse orbits, but, for a state space big enough, it could be very hard.

Other question is to analyze the perturbations of these kind of DSs. Often, this problem is formalized mathematically by adding a parameter in the expression of the evolution operator (see [3]). But in our case, the evolution operator is not given by a formula and even to formalize a perturbation of a system is a problem.
Acknowledgements

This work has been partially supported by projects MTM2008-03679/MTM, 00684-FI-04, PAI06-0114, PAC06-0008-6995, PAC08-0173-4838, PEII09-0184-7802 & CGL07-66440-C04-03

References


