BIFURCATIONS AND STABILITY OF LAGRANGIAN RELATIVE EQUILIBRIA FOR A GYROSTAT IN THE THREE BODY PROBLEM

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Abstract. In this paper we consider the non–canonical Hamiltonian dynamics of a gyrostat in the frame of the three body problem. Using geometric/mechanics methods we study the approximate dynamics of the truncated Legendre series representation of the potential of an arbitrary order. Working in the reduced problem, we study the existence of relative equilibria that we call of Lagrange type following the analogy with the standar techniques. We provide necessary and sufficient conditions for the linear stability of Lagrangian relative equilibria if the gyrostat form is close to a sphere. Thus, we generalize the classical results on equilibria of the three–body problem and many results on them obtained by the classic approach for the case of rigid bodies.

1. INTRODUCTION

A new interest in the study of configurations of relative equilibria by the use of differential geometry methods instead of more classical ones has appeared in the last years. See for instance, Wang et al. [8] concerning the problem of a rigid body in a central Newtonian field or Maciejewski [5] on the problem of two rigid bodies in mutual Newtonian attraction.

A gyrostat is a mechanical system $S$ composed by a rigid body $S'$ and other bodies $S''$ deformable or rigid, connected in such way that their relative motion with respect to its rigid part do not change the distribution of masses of the total system (see Leimanis [4]). Results of papers [5] and [8] are generalized in Mondéjar et al. [3] to the case of two gyrostats in mutual Newtonian attraction.

Key words and phrases. Hamiltonian system, three–body problem, gyrostat, Lagrangian equilibria.

2000 Mathematics Subject Classification: 34J15, 34J20, 53D17, 70F07, 70K42, 70H14.

This work was supported partially by MCYT grant numbers MTM2005-03860, MTM2005-06098-C02-01 and BFM2003-02137, by Fundación Séneca, grant numbers 00684-FI-04 and PC-MC/3/00074/FS/02, and JCCM grant numbers PAI06-0114 and PBC05-011-3.
Concerning the problem of the motion of three rigid bodies, Vidiakin [7] and Duboshin [10] prove the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries (for a more modern reference see [9]). See Guirao et al. [2] for a complete study of the Eulerian relative equilibria.

Vera [6] and Vera et al. [6,7] have studied the non–canonical Hamiltonian dynamics of \( n + 1 \) bodies in Newtonian attraction, in which \( n \) of them are rigid bodies with a spherical distribution of masses or material points and the other one is a triaxial gyrostat. Working in the reduced problem, global considerations about the conditions for relative equilibria were made.

In this paper, we consider the problem of analyzing the non–canonical Hamiltonian dynamics of two bodies in Newtonian attraction from a qualitative point of view. Thus, we shall describe the approximate dynamics appeared when we take the truncated Legendre series representation of the potential function at an arbitrary order.

We provide global conditions on the existence of relative equilibria in the case \( S_1 \) and \( S_2 \) are spherical or punctual bodies and \( S_0 \) is a gyrostat. Following the analogy with the classical results we shall call such equilibria of Lagrange type. Necessary and sufficient conditions for their existence of such equilibria are stated and moreover the explicit expressions of them are presented. It allows us to study the stability of them. We develop a complete study of the linear stability of Lagrangian relative equilibria when the gyrostat form is close to a sphere.

As a consequence of this geometric/mechanic study we obtain and generalize some results previously stated by using classical methods in previous works. On the other hand new results not obtained with standard techniques are presented.

Methods introduce in this work can be used in similar problems. A natural extension of this work, that we state as a problem for future, is to study the nonlinear stability of the Lagrangian relative equilibria obtained in this paper.

2. Equations of motion

Let \( S_0 \) be a gyrostat of mass \( m_0 \) and \( S_1, S_2 \) be two spherical rigid bodies of masses \( m_1 \) and \( m_2 \) respectively. We use the following notation

\[
M_2 = m_1 + m_2, \quad M_1 = m_1 + m_2 + m_0, \quad g_1 = \frac{m_1 m_2}{M_2}, \quad g_2 = \frac{m_0 M_2}{M_1}
\]
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For $u, v \in \mathbb{R}^3$, $u \cdot v$ is the dot product, $|u|$ is the Euclidean norm of the vector $u$ and $u \times v$ is the cross product. $I_{\mathbb{R}^3}$ is the identity matrix and $0$ is the zero matrix of order three. We consider $I = \text{diag}(A, A, C)$ the diagonal tensor of inertia of the gyrostat and let $z = (\Pi, \lambda, p_\lambda, \mu, p_\mu) \in \mathbb{R}^{15}$ be a generic element of the twice reduced problem obtained using the symmetries of the system. $\Pi = \mathbb{I} \Omega + l_r$ is the total rotational angular momentum vector of the gyrostat in the body frame, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of $S_0$ and $l_r = (0, 0, l)$ is the constant gyrostatic momentum. The elements $\lambda, \mu, p_\lambda$ and $p_\mu$ be respectively the barycentric coordinates and the linear momenta expressed in the body frame $J$.

The twice reduced Hamiltonian of the system, see [9] for more details, has the following expression

\begin{equation}
H(z) = \frac{|p_\lambda|^2}{2g_1} + \frac{|p_\mu|^2}{2g_2} + \frac{1}{2} \Pi I^{-1} \Pi - l_r \cdot I^{-1} \Pi + V(\lambda, \mu)
\end{equation}

Let $M = \mathbb{R}^{15}$, and we consider the manifold $(M, \{ , \}, H)$, with Poisson brackets $\{ , \}$ defined by using the the Poisson tensor

\begin{equation}
B(z) = \begin{pmatrix}
\hat{\Pi} & \hat{\lambda} & \hat{p}_\lambda & \hat{\mu} & \hat{p}_\mu \\
\hat{\lambda} & 0 & I_{\mathbb{R}^3} & 0 & 0 \\
\hat{p}_\lambda & -I_{\mathbb{R}^3} & 0 & 0 & 0 \\
\hat{\mu} & 0 & 0 & 0 & I_{\mathbb{R}^3} \\
\hat{p}_\mu & 0 & 0 & -I_{\mathbb{R}^3} & 0
\end{pmatrix}
\end{equation}

In $B(z)$, $\hat{v}$ is considered to be the image of the vector $v \in \mathbb{R}^3$ by the standard isomorphism between the Lie Algebras $\mathbb{R}^3$ and $\mathfrak{so}(3)$, i.e.

$$
\hat{v} = \begin{pmatrix}
0 & -v_3 & v_2 \\
v_3 & 0 & -v_1 \\
-v_2 & v_1 & 0
\end{pmatrix}
$$

The equations of the motion is given by the following expression

$$
\frac{dz}{dt} = \{z, H(z)\} = B(z) \nabla_z H(z)
$$

with $\nabla_u V$ is the gradient of $V$ with respect to an arbitrary vector $u$. 

Developing \( \{z, \mathcal{H}(z)\} \), we obtain the following group of vectorial equations of the motion

\[
\frac{d\Pi}{dt} = \Pi \times \Omega + \lambda \times \nabla_\lambda \mathcal{V} + \mu \times \nabla_\mu \mathcal{V}
\]

(3)

\[
\frac{d\lambda}{dt} = \frac{p_\lambda}{g_1} + \lambda \times \Omega, \quad \frac{dp_\lambda}{dt} = p_\lambda \times \Omega - \nabla_\lambda \mathcal{V}
\]

\[
\frac{d\mu}{dt} = \frac{p_\mu}{g_2} + \mu \times \Omega, \quad \frac{dp_\mu}{dt} = p_\mu \times \Omega - \nabla_\mu \mathcal{V}
\]

We denote by \( z_e = (\Pi_e, \lambda_e, \mu_e, p_\lambda^e, p_\mu^e) \) a generic relative equilibrium of

\[
\frac{dz}{dt} = \{z, \mathcal{H}(z)\} = B(z) \nabla_z \mathcal{H}(z)
\]

The potential function \( \mathcal{V}(\lambda, \mu) \) is given by the expression,

\[
- \left( \frac{G m_1 m_2}{|\lambda|} + G m_1 \int_{S_0} \frac{d m(Q)}{|Q + \mu + \frac{m_1}{M_2} \lambda|} + G m_2 \int_{S_0} \frac{d m(Q)}{|Q - \mu + \frac{m_1}{M_2} \lambda|} \right)
\]

3. APPROXIMATE HAMILTONIAN DYNAMICS

To simplify the problem we assume that the gyrostat \( S_0 \) is symmetrical around the third axis of inertia \( Oz \) of the body frame \( \hat{J} \) and with respect to the plane \( Oxy \) of the same one. If the mutual distances are bigger than the individual dimensions of the bodies, then we can develop the potential using a convergent series of high speed. Under these hypotheses, we will be able to carry out a study of certain relative equilibria in different approximate dynamics.

Applying the Legendre development of the potential, we have that \( \mathcal{V}(\lambda, \mu) \) has the form

\[
- \left( \frac{G m_1 m_2}{|\lambda|} + G m_1 \sum_{i=0}^{\infty} \frac{A_{2i}}{|\mu + \frac{m_1}{M_2} \lambda|^{2i+1}} + G m_2 \sum_{i=0}^{\infty} \frac{A_{2i}}{|\mu - \frac{m_1}{M_2} \lambda|^{2i+1}} \right)
\]

where \( A_0 = m_0, \ A_2 = (C - A)/2 \) and \( A_{2i} \) are certain coefficients related with the geometry of the gyrostat, see [9] for details.

**Definition 1.** The approximate potential of order \( k \) \( \mathcal{V}^{(k)}(\lambda, \mu) \) is defined as the following expression

\[
- \left( \frac{G m_1 m_2}{|\lambda|} + G m_1 \sum_{i=0}^{k} \frac{A_{2i}}{|\mu + \frac{m_1}{M_2} \lambda|^{2i+1}} + G m_2 \sum_{i=0}^{k} \frac{A_{2i}}{|\mu - \frac{m_1}{M_2} \lambda|^{2i+1}} \right)
\]
Definition 2. Let $M = \mathbb{R}^{15}$ and let the manifold $(M, \{,\}, \mathcal{H}^k)$, with Poisson brackets $\{,\}$ be defined by using the Poisson tensor

$$
\mathbf{B}(z) = \begin{pmatrix}
\hat{\Pi} & \hat{\lambda} & \hat{p}_\lambda & \hat{\mu} & \hat{p}_\mu \\
\hat{\lambda}_1 & 0 & I_{\mathbb{R}^3} & 0 & 0 \\
\hat{p}_\lambda & -I_{\mathbb{R}^3} & 0 & 0 & 0 \\
\hat{\mu} & 0 & 0 & I_{\mathbb{R}^3} & 0 \\
\hat{p}_\mu & 0 & 0 & -I_{\mathbb{R}^3} & 0
\end{pmatrix}
$$

**Figure 1.** Gyrostat in the three body problem
We defined the approximate dynamics of order $k$ to the differential equations of motion given by the following expression

$$\frac{dz}{dt} = \{z, H^k(z)\} = B(z)\nabla_z H^k(z)$$

being

$$H^k(z) = \left| p_\lambda \right|^2 \frac{2}{2g_1} + \left| p_\mu \right|^2 \frac{1}{2g_2} \Pi^{-1} \Pi - 1 \cdot \Pi^{-1} \Pi + V^{(k)}(\lambda, \mu)$$

In this setting we have the following result.

**Theorem 3.** In the approximate dynamics of order 0, $|\Pi|^2$ is an integral of motion and also when the gyrostat is of revolution $\pi_3$ is another integral of motion for all approximate dynamics.

**Proof.** The proof is consequence of two facts. On the one hand, by calculation is easy to verify that

$$\nabla_z (|\Pi|^2)B(z)\nabla_z H^0(z) = 0$$

and on the other hand, obtain in a similar way, when the gyrostat is of revolution is

$$\nabla_z (\pi_3)B(z)\nabla_z H^k(z) = 0$$

where $\pi_3$ is the third component of the rotational angular momentum of the gyrostat. $\square$

4. **Relative Equilibria**

The relative equilibria are the equilibria of the twice reduced problem whose Hamiltonian function is obtained in [9] for the case $n = 2$. If we denote by $z_e = (\Pi_e, \lambda^e, p_\lambda^e, \mu^e, p_\mu^e)$ a generic relative equilibrium of an approximate dynamics of order $k$, then this verifies the equations

$$\Pi_e \times \Omega_e + \lambda^e \times (\nabla_\lambda V^{(k)})_e + \mu^e \times (\nabla_\mu V^{(k)})_e = 0$$

$$\frac{p_\lambda^e}{g_1} + \lambda^e \times \Omega_e = 0, \quad p_\lambda^e \times \Omega_e = (\nabla_\lambda V^{(k)})_e$$

$$\frac{p_\mu^e}{g_2} + \mu^e \times \Omega_e = 0, \quad p_\mu^e \times \Omega_e = (\nabla_\mu V^{(k)})_e$$

with $(\nabla_\lambda V^{(k)})_e$ and $(\nabla_\mu V^{(k)})_e$ are the value of $\nabla_\lambda V^{(k)}$ and $\nabla_\mu V^{(k)}$ in $z_e$.

[9] provides us the following result which will play a key role in this work because it will be used to obtain necessary conditions for the existence of relative equilibria in approximate dynamics.
Lemma 4. If \( z_e = (\Pi_e, \lambda^e, p^\lambda_e, \mu^e, p^\mu_e) \) is a relative equilibrium of an approximate dynamics of order \( k \) the following relationships are verified

\[
|\Omega_e|^2 |\lambda^e|^2 - (\lambda^e \cdot \Omega_e)^2 = \frac{1}{g_1} (\lambda^e \cdot (\nabla_\lambda V^{(k)})_e)
\]

\[
|\Omega_e|^2 |\mu^e|^2 - (\mu^e \cdot \Omega_e)^2 = \frac{1}{g_2} (\mu^e \cdot (\nabla_\mu V^{(k)})_e)
\]

We will study the relative equilibria in the approximate dynamics for which their vectors \( \Omega_e, \lambda^e, \mu^e \) satisfy some special geometric properties.

Definition 5. We say that \( z_e \) is a Lagrangian relative equilibrium in an approximate dynamics of order \( k \), if \( \lambda^e, \mu^e \) are not proportional and \( \Omega_e \) is perpendicular to the plane that these generate.

In this setting we have the following result.

Proposition 6. In a Lagrangian relative equilibrium for any approximate dynamics of arbitrary order, moments are not exercised on the gyrostat.

Proof. The proof follows from the equations of motion and the potential relations in the equilibrium.

In the next section we endeavour to obtain necessary and sufficient conditions for the existence of Lagrangian relative equilibria.

5. Lagrangian relative equilibria

5.1. Necessary condition of existence.

Proposition 7. Let \( z_e = (\Pi_e, \lambda^e, p^\lambda_e, \mu^e, p^\mu_e) \) be Lagrangian relative equilibria. Then we have

\[
g_2(\tilde{A}_{11})_e = g_1(\tilde{A}_{22})_e
\]

\[
(\tilde{A}_{12})_e = 0
\]

with

\[
|\Omega_e|^2 = \frac{(\tilde{A}_{11})_e}{g_1} = \frac{(\tilde{A}_{22})_e}{g_2}
\]

Proof. If \( z_e = (\Pi_e, \lambda^e, p^\lambda_e, \mu^e, p^\mu_e) \) is a Lagrangian relative equilibrium, in an approximate dynamics of order \( k \), the following identities are verified

\[
\lambda^e \times (\nabla_\lambda V^{(k)})_e = 0, \quad g_1 |\Omega_e|^2 (\lambda^e \times \mu^e) = (\nabla_\lambda V^{(k)})_e \times \mu^e
\]

\[
\mu^e \times (\nabla_\mu V^{(k)})_e = 0, \quad g_2 |\Omega_e|^2 (\lambda^e \times \mu^e) = \lambda^e \times (\nabla_\mu V^{(k)})_e
\]
In the relative equilibria, from the equation (12) of the Appendix A, we deduce
\[(\tilde{A}_{12})_e (\lambda^e \times \mu^e) = 0, \quad g_1 | \Omega_e |^2 (\lambda^e \times \mu^e) = (\tilde{A}_{11})_e (\lambda^e \times \mu^e)\]
\[(\tilde{A}_{21})_e (\lambda^e \times \mu^e) = 0, \quad g_2 | \Omega_e |^2 (\lambda^e \times \mu^e) = (\tilde{A}_{22})_e (\lambda^e \times \mu^e)\]
being \((\tilde{A}_{ij})_e\) the evaluation in the equilibria of \(\tilde{A}_{ij}\).

Concluding, we have the following relations
\[(\tilde{A}_{12})_e = 0, \quad | \Omega_e |^2 = \frac{(\tilde{A}_{11})_e}{g_1} = \frac{(\tilde{A}_{22})_e}{g_2}\]
and the proof is over.

**Proposition 8.** If \(z_e = (\Pi^e, \lambda^e, p^e_0, \mu^e, p^e_0)\) is Lagrangian relative equilibria in an approximate dynamics of order \(k\), then denoting by \(| \lambda^e | = Z, | \mu^e + \frac{m_1}{M_2} \lambda^e | = X, | \mu^e - \frac{m_2}{M_2} \lambda^e | = Y\), the system of equations
\[
\begin{align*}
X^{2k+3} &= \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)} \\
Y^{2k+3} &= \sum_{i=0}^{k} \beta_i Z^3 Y^{2(k-i)}
\end{align*}
\]
has positive real solutions.

**Proof.** The proof follows by using the expressions of \(\tilde{A}_{ij}\) given in (13).

**Remark 9.** The parameters that have influence in the study of the number of the different configurations of Lagrangian relative equilibria will be \(Z\) and \(\beta_i\) \((i = 1, 2, \ldots, k)\).

5.2. **Sufficient condition of existence.** If we fix \(Z\) and there exist \(X\) and \(Y\) verifying the system of equations
\[
\begin{align*}
X^{2k+3} &= \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)} \\
Y^{2k+3} &= \sum_{i=0}^{k} \beta_i Z^3 Y^{2(k-i)}
\end{align*}
\]
with respect to an appropriate reference system, we can build Lagrangian relative equilibria. If \(X = Y \neq Z\) is a solution of the previous system, then the \(S_i\) \((i = 0, 1, 2)\) form an isosceles triangle. If \(X \neq Y \neq Z\) then the \(S_i\) form a scalene triangle.
Proposition 10 shows the form of the Lagrangian relative equilibria when \( S_0, S_1, S_2 \) forms an isosceles triangle. In a similar way Proposition 11 describes the Lagrangian relative equilibria expressions when \( S_0, S_1, S_2 \) form a scalene triangle. Thus, we obtain the following results.

**Proposition 10.** With respect to an appropriate reference system we have that \( z_e = (\Pi_e, \lambda_e, \mu_e, \lambda_e^p, \mu_e^p) \) given by
\[
\lambda_e = (x_1, y_1, 0), \quad \mu_e = (x_2, y_2, 0),
\]
\[
\lambda_e^p = g_1 \omega_e (-y_1, x_1, 0), \quad \mu_e^p = g_2 \omega_e (-y_2, x_2, 0)
\]
\[
\Omega_e = (0, 0, \omega_e), \quad \Pi_e = (0, 0, C \omega_e + l)
\]
with
\[
x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{Z(m_1 - m_2)}{2(m_1 + m_2)}, \quad y_2 = \pm \frac{\sqrt{4X^2 - Z^2}}{2}
\]
and
\[
\omega_e^2 = \sum_{i=0}^{k} G(m_0 + m_1 + m_2) \beta_i X^{2i+3}
\]
being \( \beta_0 = 1 \), \( \beta_i = \frac{\alpha_i}{m_i} \) for \( i \geq 1 \), are isosceles Lagrangian relative equilibria. The total angular momentum vector of the system is given by
\[
L = (0, 0, C \omega_e + l + \omega_e^2 \sum_{i=1}^{2} g_i (x_i^2 + y_i^2))
\]

**Proposition 11.** With respect to an appropriate reference system we have that \( z_e = (\Pi_e, \lambda_e, \mu_e, \lambda_e^p, \mu_e^p) \) given by
\[
\lambda_e = (x_1, y_1, 0), \quad \mu_e = (x_2, y_2, 0),
\]
\[
\lambda_e^p = g_1 \omega_e (-y_1, x_1, 0), \quad \mu_e^p = g_2 \omega_e (-y_2, x_2, 0)
\]
\[
\Omega_e = (0, 0, \omega_e), \quad \Pi_e = (0, 0, C \omega_e + l)
\]
with
\[
x_1 = Z, \quad y_1 = 0
\]
\[
x_2 = \frac{m_1(X^2 + Z^2 - Y^2) - m_2(Y^2 + Z^2 - X^2)}{2(m_1 + m_2)Z}
\]
\[
y_2 = \pm \frac{\sqrt{(Z + X + Y)(Z + X - Y)(Z + Y - X)(X + Y - Z)}}{2Z}
\]
and

\[ \omega_e^2 = \sum_{i=0}^{k} \frac{Gm_1(m_0 + m_1 + m_2)\beta_i}{(m_1 + m_2)X^{2i+3}} + \sum_{i=0}^{k} \frac{Gm_2(m_0 + m_1 + m_2)\beta_i}{(m_1 + m_2)Y^{2i+3}} \]

being \( \beta_0 = 1, \beta_i = \frac{\alpha_i}{m_0} \) for \( i \geq 1 \), are scalene Lagrangian relative equilibria. The total angular momentum vector of the system is given by

\[ \mathbf{L} = (0, 0, C\omega_e + l + \sum_{i=1}^{2} g_i(x_i^2 + y_i^2)) \]

In the sequel we study the Lagrangian relative equilibria in the approximate dynamics of orders zero and one respectively.

5.3. Lagrangian relative equilibria in an approximate dynamics of order zero. When \( k = 0 \), the equations (6) are

\[ \begin{cases} 
  X^3 = Z^3 \\
  Y^3 = Z^3 
\end{cases} \]

then easily we deduce that \( X = Y = Z \). This means that \( S_0, S_1 \) and \( S_2 \) form an equilateral triangle. Moreover, is

\[ |\Omega_e|^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3} \]

On the other hand a parametrization of \( \mathbf{z}_e = (\Pi_e, \lambda_e, \mathbf{p}_\lambda, \mu_e, \mathbf{p}_\mu) \) is given by

\[ \lambda_e = (x_1, y_1, 0), \quad \mathbf{p}_\lambda = g_1\omega_e(-y_1, x_1, 0) \]
\[ \mu_e = (x_2, y_2, 0), \quad \mathbf{p}_\mu = g_2\omega_e(-y_2, x_2, 0) \]
\[ \Omega_e = (0, 0, \omega_e), \quad \Pi_e = (0, 0, C\omega_e + l) \]

being

\[ x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{Z(m_1 - m_2)}{2(m_1 + m_2)}, \quad y_2 = \pm \frac{\sqrt{3}Z}{2} \]

This parametrization of the relative equilibria will play a key role in the study of their stability properties.
5.4. Lagrangian relative equilibria in an approximate dynamics of order one. For $k = 1$, the equations (6) are

$$\begin{align*}
X^5 - Z^3X^2 - \beta_1 Z^3 &= 0 \\
Y^5 - Z^3Y^2 - \beta_1 Z^3 &= 0
\end{align*}$$

(7)

being $Z$ and $\beta_1$ parameters. We study the number of positive real roots of the polynomial

$$p(X) = X^5 - Z^3X^2 - \beta_1 Z^3$$

according to the values of the parameters $Z$ and $\beta_1$.

Applying the Descartes rule of signs, if $\beta_1 \geq 0$ then this polynomial can only have a positive real root.

If $\beta_1 < 0$ then we can have two positive real roots, a real root (positive) or none. The discriminant of the polynomial, denoted by $\text{discrim}(p, X)$, is given by

$$\text{discrim}(p, X) = \beta_1 Z^{12}(3125\beta_1^3 + 108Z^6)$$

Then if $\text{discrim}(p, X) < 0$, the polynomial $p$ has two real roots, if $\text{discrim}(p, x) = 0$, it has a positive double root, while if $\text{discrim}(p, x) > 0$, it has not positive root.

The discriminant is zero when the following relation is verified

$$\beta_1 = -\frac{3\sqrt{20}}{25}Z^2$$

By the previous results we can make a complete study of the bifurcations of the equilibria in an approximate dynamics of order 1.

**Proposition 12.** Let $z_e = (\Pi_e, \lambda^e, p^e_\lambda, \mu^e, p^e_\mu)$ be a Lagrangian relative equilibria, in an approximate dynamics of order one. Then:

1. If $\beta_1 \geq 0$ (gyrostat oblate) an only 2-parametric family exists forming $S_0$, $S_1$, $S_2$ an isosceles triangle.
2. If $\beta_1 < 0$ (gyrostat prolate) then
   
   a1): If $-\frac{7Z^2}{32} < \beta_1 < 0$, there are two types of relative equilibria:
   
   - One 2-parametric family of relative equilibria forming $S_0$, $S_1$, $S_2$ an isosceles triangle with $X = Y \neq Z$.
   - Two 2-parametric families of relative equilibria forming $S_0$, $S_1$, $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   a2): If $-\frac{3\sqrt{20}}{25}Z^2 < \beta_1 < -\frac{7Z^2}{32}$, there are two types of relative equilibria:
• Two 2-parametric families of relative equilibria forming $S_0, S_1, S_2$ an isosceles triangle with $X = Y \neq Z$.
• Four 2-parametric families of relative equilibria forming $S_0, S_1, S_2$ an scalene triangle with $X \neq Y \neq Z$.

b): If $\beta_1 = -\frac{3\sqrt{20}}{25}Z^2$ an only 2-parametric family exists forming $S_0, S_1, S_2$ an isosceles triangle, with $X = Y \neq Z$.

c): If $\beta_1 < -\frac{3\sqrt{20}}{25}Z^2$ relative equilibria don’t exist.

Remark 13. It is easy to see that when the gyrostat is oblate, in the previous equilibria, it rotates quicker around the principal axis of inertia $C$ that when the gyrostat is prolate.

Remark 14. To study the Lagrangian relative equilibria in an approximate dynamics of order $k$ anyone, we should study the positive real solutions of the equation

$$X^{2k+3} - \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)} = 0$$

If we know the number of positive roots in the approximate dynamics of order $k$, we can know the number of positive roots of the polynomial equation that arises in the approximate dynamics of order $k+1$. This study reduces to calculate the number of positive roots of the equation

$$\beta_{k+1} = \frac{X^2 [X^{2k+3} - \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)}]}{Z^3}$$

Figure 2. Bifurcations of the equilibria in the plane $\beta_1 Z$
5.5. \textbf{Lagrangian relative equilibria in an approximate dynamics of order one when $S_0$ is close to a sphere.} If $S_0$ is close to a sphere then $\beta_1 \approx 0$. To first order in $\beta_1$ the parametrization of $z_e = (\Pi_e, \lambda^e, \mu^e, \Omega^e, \Pi^e)$ is given by

$$\lambda^e = (x_1, y_1, 0), \quad \mu^e = g_1\omega_e(-y_1, x_1, 0)$$

$$\Omega^e = (0, 0, \omega_e), \quad \Pi^e = (0, 0, C\omega_e + l)$$

being

$$x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{Z(m_1 - m_2)}{2(m_1 + m_2)},$$

$$y_2 = \pm \left( \frac{\sqrt{3}Z}{2} + \frac{2\sqrt{3}}{9Z} \beta_1 + o(\beta_1) \right).$$

6. \textbf{Linear stability of the Lagrangian relative equilibrium}

The tangent flow of the equations (3) at the equilibrium $z_e$ is

$$\frac{d\delta z}{dt} = \mathfrak{U}(z_e) \delta z$$

with $\delta z = z - z_e$ and $\mathfrak{U}(z_e)$ the Jacobian matrix of (3) in $z_e$.

6.1. \textbf{Order zero approximate dynamics.} The characteristic polynomial of $\mathfrak{U}(z_e)$ has the following expression

$$P(\lambda) = \lambda^3(\lambda^2 + \Phi^2)(\lambda^2 + \omega_e^2)^3(\lambda^4 + \omega_e^2\lambda^2 + q)$$

where

$$\omega_e^2 = \frac{G(m_0 + m_1 + m_2)}{Z^4}, \quad q = \frac{27G^2(m_1m_0 + m_2m_0 + m_1m_2)}{4Z^6}$$

and $\Phi = \frac{(C - A)\omega_e + l}{A}$.

Then the following results are verified.

\textbf{Proposition 15.} $z_e$ is spectral stable if

$$(m_0 + m_2 + m_1)^2 \geq 27(m_1m_0 + m_2m_0 + m_1m_2)$$

If

$$(m_0 + m_2 + m_1)^2 < 27(m_1m_0 + m_2m_0 + m_1m_2)$$

then $z_e$ is unstable.
Proof. The proof follows from the form of the minimum polynomial of $\mathcal{U}(z_e)$ which has the following expression

$$Q(\lambda) = \lambda^2(\lambda^2 + \Phi^2)(\lambda^2 + \omega_e^2)(\lambda^4 + \omega_e^2 \lambda^2 + q).$$

□

Proposition 16. The linear system

$$\frac{d\delta z}{dt} = \mathcal{U}(z_e)\delta z$$

is unstable.

Proof. In this case the minimum polynomial of $\mathcal{U}(z_e)$ has the zero 0 as double root, that is why the matrix $\mathcal{U}(z_e)$ is not diagonalizable and the proof is over. □

6.2. Order one approximate dynamics. Similar results show that the characteristic polynomial in an order one approximate dynamics has the following expression

$$P(\lambda) = \lambda(\lambda^2 + \Phi^2)(\lambda^2 + m)(\lambda^2 + n)h(\lambda)$$

with

$$h(\lambda) = \lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s$$

Thus we have the following result

Proposition 17. The Lagrangian relative equilibria in order one approximate dynamics is spectral stable (lineally stable) if the following conditions are verified

$$p^2q^2 - 3rp^3 - 6p^2s - 4q^3 + 14pq r + 16qs - 18r^2 \geq 0 \ (> 0)$$

$$p^2qr - 48sr - 9sp^3 + 32pq s - 4q^2 r + 3pr^2 \geq 0 \ (> 0)$$

$$r, s \geq 0 \ (> 0), \ 3p^2 - 8q \geq 0 \ (> 0), \ pr - 16s \geq 0 \ (> 0)$$

$$m, n \geq 0 \ (> 0)$$

$$\text{discrim}(h) \geq 0 \ (> 0)$$

where

$$\text{discrim}(h) = 18p^3 rqs - 4p^3 r^3 - 128q^2 s^2 + 16q^4 s - 4q^3 r^2 - 27p^4 s^2$$

$$- 80prq^2 s + 256s^3 - 27 r^4 - 6p^2 r^2 s - 192 pr s^2 + 18p^3 r q + 144q p^2 s^2$$

$$+ q^2 p^2 r^2 - 4q^3 p^2 s + 144sr^2 q$$

Proof. The coefficients of the characteristic polynomial are expressed in function of the parameters of our problem, i.e, the masses and the coefficient $\beta_1$ and the proof follows from the application of the Sturm Theorem. □
**Remark 18.** If $z_e$ is an arbitrary relative equilibrium, the conditions of the statement of Proposition 17 have very complicated expressions in the parameters of the problem, only can be studied via numerical analysis.

If $S_0$ is close to a sphere, the coefficients of $P$, to first order in the parameter $\beta_1$, have the following relationships

$$
m = \frac{G(m_0 + m_1 + m_2)}{Z^3} + o_1(\beta_1), \quad n = \frac{G(m_0 + m_1 + m_2)}{Z^3} + o_2(\beta_1),$$

$$s = o_3(\beta_1)$$

$$r = \frac{27G^3(m_0 + m_1 + m_2)(m_1m_0 + m_2m_0 + m_1m_2)}{4Z^6} + o_4(\beta_1)$$

$$q = \frac{G^2(4m_0^2 + 4m_1^2 + 4m_2^2 + 35m_0m_1 + 35m_0m_2 + 35m_1m_2)}{4Z^6} + o_5(\beta_1)$$

$$p = \frac{2G(m_0 + m_1 + m_2)}{Z^3} + o_6(\beta_1)$$

If the function

$$o_3(\beta_1) = \frac{81G^4m_0(m_1 + m_2)(m_0 + m_1 + m_2)^2}{4}\beta_1 + o(\beta_1^2)$$

is positive and

$$(m_0 + m_2 + m_1)^2 > 27(m_1m_0 + m_2m_0 + m_1m_2)$$

then $z_e$ is linearly stable in order one approximate dynamics. Then if $S_0$ is close to a sphere and $C > A$ then $z_e$ is linearly stable if (10) is verified.

7. **Conclusions and future works**

The approximate dynamics of a gyrostat (or rigid body) in Newtonian interaction with two spherical or punctual rigid bodies has been considered. For orders zero and one of the approximate dynamics we provide a complete study of Lagrangian relative equilibria. The bifurcations of the Lagrangian relative equilibria is completely described for an approximate dynamics of order one. Necessary and sufficient conditions are given for the linear stability of the Lagrangian relative equilibria in zero order and one order approximate dynamics if the gyrostat $S_0$ has form close to a sphere.

Several results obtained in previous works by classical methods have been obtained and generalized in a different way. And other results, not previously considered, have been studied.

The methods employed in this work are susceptible of being used in similar problems. Numerous problems are open, and among them it is necessary to consider the study of the "inclined" relative equilibria, in
which $\Omega_e$ form an angle $\alpha \neq 0$ and $\pi/2$ with the vector $\lambda^e \times \mu^e$. The study of the nonlinear stability of the relative equilibria obtained here is the natural continuation of this work.

8. Appendix A

The following expressions are obtained of the potential $\mathcal{V}^{(k)}$.

$$(\nabla_\lambda \mathcal{V}^{(k)})_e = \left( \frac{Gm_1 m_2}{\lambda^e} \right) + \frac{Gm_1 m_2}{M_2} \sum_{i=0}^{k} \frac{\alpha_i (\mu^e + \frac{m_1}{M_2} \lambda^e)}{\mu^e + \frac{m_1}{M_2} \lambda^e |2i+3|}
$$

$$- \frac{Gm_1 m_2}{M_2} \sum_{i=0}^{k} \frac{\alpha_i (\mu^e - \frac{m_2}{M_2} \lambda^e)}{\mu^e - \frac{m_2}{M_2} \lambda^e |2i+3|}$$

$$(\nabla_\mu \mathcal{V}^{(k)})_e = Gm_1 \sum_{i=0}^{k} \frac{\alpha_i (\mu^e + \frac{m_1}{M_2} \lambda^e)}{\mu^e + \frac{m_1}{M_2} \lambda^e |2i+3|} + Gm_2 \sum_{i=0}^{k} \frac{\alpha_i (\mu^e - \frac{m_2}{M_2} \lambda^e)}{\mu^e - \frac{m_2}{M_2} \lambda^e |2i+3|}$$

Also, the following identities are verified

$$(\nabla_\lambda \mathcal{V}^{(k)})_e = (\tilde{A}_{11})_e \lambda^e + (\tilde{A}_{12})_e \mu^e, \quad (\nabla_\mu \mathcal{V}^{(k)})_e = (\tilde{A}_{21})_e \lambda^e + (\tilde{A}_{22})_e \mu^e$$

being

$$\tilde{A}_{11}(\lambda^e, \mu^e) = \frac{Gm_1 m_2}{\lambda^e} \left( \frac{\sum_{i=0}^{k} \alpha_i (\mu^e + \frac{m_1}{M_2} \lambda^e)}{\mu^e + \frac{m_1}{M_2} \lambda^e |2i+3|} \right)$$

$$+ \frac{Gm_1 m_2}{M_2} \left( \frac{\sum_{i=0}^{k} \alpha_i (\mu^e - \frac{m_2}{M_2} \lambda^e)}{\mu^e - \frac{m_2}{M_2} \lambda^e |2i+3|} \right)$$

$$\tilde{A}_{12}(\lambda^e, \mu^e) = \frac{Gm_1 m_2}{M_2} \left( \sum_{i=0}^{k} \frac{\alpha_i}{\mu^e + \frac{m_1}{M_2} \lambda^e |2i+3|} - \sum_{i=0}^{k} \frac{\alpha_i}{\mu^e - \frac{m_2}{M_2} \lambda^e |2i+3|} \right)$$

$$\tilde{A}_{21}(\lambda^e, \mu^e) = \tilde{A}_{12}(\lambda^e, \mu^e)$$

$$\tilde{A}_{22}(\lambda^e, \mu^e) = Gm_1 \left( \sum_{i=0}^{k} \frac{\alpha_i}{\mu^e + \frac{m_1}{M_2} \lambda^e |2i+3|} \right) + Gm_2 \left( \sum_{i=0}^{k} \frac{\alpha_i}{\mu^e - \frac{m_2}{M_2} \lambda^e |2i+3|} \right)$$

with $\alpha_0 = m_0$, $\alpha_1 = 3(C-A)/2$ and $\alpha_i = (2i+1)A_{2i}$ for $i \geq 2$. 

9. ACKNOWLEDGEMENTS

This work has been partially supported by MCI (Ministerio de Ciencia e Innovación) and FEDER (Fondo Europeo Desarrollo Regional), grant number MTM2008–03679/MTM, Fundación Séneca de la Región de Murcia, grant number 08667/PI/08 and JCCM (Junta de Comunidades de Castilla-La Mancha), grant number PEII09-0220-0222.

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