EQUILIBRIA, STABILITY AND HAMILTONIAN HOPF
BIFURCATION OF A GYROSTAT IN AN
INCOMPRESSIBLE IDEAL FLUID

JUAN L. G. GUIRAO¹ AND JUAN A. VERA²

ABSTRACT. For a gyrostat in an incompressible ideal fluid, by writ-
ing the Kirchhoff’s equations as a Lie–Poisson system and using a
non–canonical Hamiltonian formulation, we provide the expressions
of the equilibria when the gyrostatic momentum is constant with the
form \( \mathbf{l} = (0, 0, l) \) and present necessary and sufficient conditions for
the stability of some of them via the energy–Casimir method and the
study of the linearized equations of the motion. Finally, using a re-
cent geometric method introduced by Hanssmann and Van der Me-
er, we give a sufficient condition for the existence of a non–degen-
erate Hamiltonian Hopf bifurcation at those equilibria when the gyrostat
is symmetric.

1. INTRODUCTION

In the classic literature the general study of the dynamics of rigid solids
and gyrostats has been extensively presented. The Eulerian, Lagrangian
and Hamiltonian formulations of such dynamics have been the main tools
used in these problems (see for instance [4, 5, 3] or [15]).

A gyrostat is a mechanical system \( S \) composed by a rigid solid \( S_1 \) to
which other bodies \( S_2 \) are connected; these other bodies may be variable
or rigid, but the key property is that they must not be rigidly connected
to \( S_1 \), so that the movements of \( S_2 \) with respect to \( S_1 \) do not modify the
distribution of mass within the compound system \( S \).

For instance, we can consider a rigid main body \( S_1 \), designated as the
platform, supporting additional bodies \( S_2 \), which possess axial symmetry

¹Departamento de Matemática Aplicada y Estadística. Universidad Politéc-
nica de Cartagena, Hospital de Marina, 30203 Cartagena, Región de Murcia,
Spain, e-mail: juan.garcia@upct.es, Telf/Fax: +0034968338913—Corresponding Author
Information—.
and are designated as rotors. These rotors may rotate with respect to the platform in such a way that the mass distribution within the system as a whole is not altered; this will produce an internal angular momentum, designated as gyrostat momentum, which will be normally regarded as a constant. Note that when this constant vector is zero, the motion of the system is reduced to the motion of a rigid solid, see for instance Figure 1 where a gyrostat in the frame of the three body problem is presented.

Volterra was the first to introduce the concept of a gyrostat in [16], in order to study the motion of the Earth’s polar axis and explaining variations in the Earth’s latitude by means of internal movements that do not alter the planets’s distribution of mass.

The most common problems find in the literature on these systems are the following:

1. Equilibria and stabilities in rigid bodies and gyrostats, either with fixed point or in orbit.
2. Periodic solutions, bifurcations, or chaos, in various gyrostat motion problems.
3. Integrability and first integrals for the problem.
These problems are undoubtedly appealing, not only in the field of astronautics because of the need for placing satellites (whether rigid or gyrostatic) in stable orbits with stable orientations, but also in hydrodynamics, specifically in the study of equilibria and their stabilities for underwater vehicles with symmetric rotors. This will be our object of work.

We consider the problem of the motion of a gyrostat in an incompressible ideal fluid, see Figure 2. A classical study of the problem of a rigid body in an incompressible ideal fluid can be found, for instance, at the monographs [12] and [10].

Our first step in section 2 is to obtain the Lie–Poisson equations of the problem by using a non–canonical Hamiltonian formulation. In section 3 we determine the equilibrium solutions of the problem when the gyrostatic momentum (i.e. the relative angular momentum of $S_2$ respect to $S_1$) is constant and it has the form $l = (0, 0, l)$.

![Figure 2. Gyrostat in a incompressible ideal fluid. $l$ is the added gyrostatic momentum produced by a rotor wheel.](image-url)

In section 4 and for the case of the equilibrium solutions $E_{I_3}$, i.e. solutions which physically correspond to uniform translations and stationary rotations around the third axis of inertia of the system, by the energy–Casimir method, in which we use the invariants given by the energy and the Casimirs to determine stability of the equilibria, we obtain sufficient conditions for their stability, see [2] for more details. Moreover, by studying the linearized equations of motion we are able to present necessary conditions for the stability as well of these equilibria. We remark that
we have obtained necessary and sufficient conditions for the stability in the triaxial and symmetric cases. The obtained results generalize previous ones stated in [8], [9] and [11] for rigid bodies. Finally in last section applying the geometric method introduced by Hanssmann and Van der Meer [7] we provide sufficient conditions for the existence of a non–degenerate Hamiltonian Hopf bifurcation at the studied equilibria points when the gyrostat is symmetric, see [14] for more details on this notion. This technique allows us to obtain these results with an easier and cleaner approach to the classical one which implies in general difficult calculations which will make it to cumbersome to complete the study like was the case in this problem. These results complete [6] where sufficient conditions for the existence of a non–degenerate Hopf bifurcation at the equilibrium points corresponding to the vertical rotations of a gyrostat with a fixed point suffering the effects of an axially symmetric potential where stated. For future work one could try to study necessary and sufficient conditions for the stability of the equilibrium solutions $E_\omega$ and try to obtain the regions of stability as functions of two or three parameters of the possible non null components of the gyrostatic momentum.

2. Lie-Poisson equations of a gyrostat in an incompressible ideal fluid

In this section Kirchhoff’s equations of a gyrostat immersed in an incompressible ideal fluid, in the absence of gravitational forces and torques, is declared as a Poisson system with a certain Hamiltonian function $h$.

2.1. Kirchhoff’s equations of motion of a gyrostat in an ideal fluid. Kirchhoff’s equations of a rigid body in an incompressible ideal fluid are as follows (see [12]):

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial v} \right) + \Omega \times \frac{\partial T}{\partial v} = F
$$

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial \Omega} \right) + \Omega \times \frac{\partial T}{\partial \Omega} + v \times \frac{\partial T}{\partial v} = L
$$

where $T$ is the kinetic energy of the solid–liquid system, $F$ and $L$ are the forces and moments that act on the system. The kinetic energy is given by a quadratic form in the variables $(\Omega, v)$, where $\Omega$ is the angular velocity and $v$ is the translation velocity of the body. We remark that $\mathbf{\Pi} = (\pi_1, \pi_2, \pi_3)$, and $\mathbf{p} = (p_1, p_2, p_3)$ are the angular and linear momenta of the body $S$, measured in a body frame of axes. The relations between
the variables \((\Pi, p)\) and \((\Omega, v)\) are given by the following formula:

\[
\begin{pmatrix}
\Pi \\
p
\end{pmatrix} =
\begin{pmatrix}
J & D \\
D^t & M
\end{pmatrix}
\begin{pmatrix}
\Omega \\
v
\end{pmatrix}
\]

where the previous matrix contains the added inertia tensor \(J\) equal to \(\text{diag}(I_1, I_2, I_3)\), the added mass tensor \(M = \text{diag}(m_1, m_2, m_3)\) and the coupling terms between linear and angular momenta \(D = (d_{ij})_{i,j\in\{1,2\}}\).

Following [8], we focus on the case of a neutrally buoyant body, in which the center of mass coincides with the center of volume \((D = 0)\), there are no forces or torques acting on the body and the body frame coincides with the principal axes of inertia; then the kinetic energy coincides with the Hamiltonian function \(H\) and it is given by

\[
T = \frac{1}{2}(\Pi \cdot \Omega + p \cdot v)
\]

Kirchhoff’s equations in the absence of gravitational forces and torques can be expressed by the following formulas:

\begin{equation}
\frac{d\Pi}{dt} = \Pi \times \Omega + p \times v \quad \frac{dp}{dt} = p \times \Omega
\end{equation}

A generalization of Kirchhoff’s equations for a rigid body consists in assuming that the body submerged in the fluid is a gyrostat, whose gyrostatic momentum is constant \(l = (l_1, l_2, l_3)\); then, Kirchhoff’s equations for a gyrostat can be expressed as follows:

\begin{equation}
\frac{d\Pi}{dt} = (\Pi + l) \times \Omega + p \times v \quad \frac{dp}{dt} = p \times \Omega
\end{equation}

And the problem has the following two first integrals:

\begin{equation}
(\Pi + l) \cdot p = k_0 \quad \|p\|^2 = p_0^2 \quad (k_0, p_0 \in \mathbb{R} \text{ are constant})
\end{equation}

2.2. Kirchhoff’s equations as a Lie–Poisson system. We will use the energy–Casimir method as a tool to study sufficient conditions of (Lyapunov) stability for the equilibrium solutions of a mechanical systems with symmetry. The following theorem describes the method which will play a key role in the sequel.

**Theorem 1** (energy–Casimir method). Let \((M, \{\cdot, \cdot\}, \mathcal{H})\) be a Poisson system, \(m \in M\) an equilibrium solution of the Hamiltonian vector field \(X_\mathcal{H}\) and \(C_1, C_2, ..., C_n \in C^\infty(M)\) integrals of system, verifying

\begin{equation}
d(\mathcal{H} + C_1 + C_2 + ... + C_n)(m) = 0
\end{equation}

and that

\begin{equation}
d^2(\mathcal{H} + C_1 + C_2 + ... + C_n)(m) |_{W \times W}
\end{equation}
is a definite (either positive or negative) quadratic form on \( W \times W \), where \( W \) is defined by:

\[
W =\ker dC_1(m) \cap \ker dC_2(m) \cap ... \cap \ker dC_n(m)
\]

Then, \( m \in M \) is stable. If \( W = \{0\} \), \( m \in M \) is also stable.

Proof. See [13]. □

In order to apply the energy–Casimir method, we must declare the mechanical system in question as a Poisson system with a certain Hamiltonian function \( h \). In our case it is given by the formula:

\[
H = \frac{1}{2}(\Pi \cdot \Omega + p \cdot v).
\]

It is also immediate to prove that:

\[
\frac{d\Pi}{dt} = (\Pi + l) \times \nabla_\Pi H + p \times \nabla_p H \quad \frac{dp}{dt} = p \times \nabla_\Pi H
\]

Finally, we provide the following theorem which describes completely Kirchhoff’s equations as a Lie–Poisson system.

**Theorem 2.** \((\mathbb{R}^3 \times \mathbb{R}^3, \{\cdot, \cdot\}, \mathcal{H})\) is a non-canonical Hamiltonian system. The geometrical structure of the Kirchhoff’s equations for a gyrostat in an ideal fluid (when the gyrostatic momentum is constant) is given by the Lie-Poisson bracket, defined in \( \mathbb{R}^3 \times \mathbb{R}^3 \) by the formula \( \{F, G\}(\Pi, p) \) equals to:

\[
-(\Pi + l) \cdot (\nabla_\Pi F \times \nabla_\Pi G) - p \cdot (\nabla_\Pi F \times \nabla_p G + \nabla_p F \times \nabla_\Pi G)
\]

being \( F, G \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \), \( l = (l_1, l_2, l_3) \) the gyrostatic momentum. The Poisson’s tensor corresponding to this bracket is given by the following matrix:

\[
\mathfrak{J}(\Pi, p) = \begin{pmatrix}
0 & -\pi_3 - l_3 & \pi_2 + l_2 & 0 & -p_3 & p_2 \\
\pi_3 + l_3 & 0 & -\pi_1 - l_1 & p_3 & 0 & -p_1 \\
-\pi_2 - l_2 & \pi_1 + l_1 & 0 & -p_2 & p_1 & 0 \\
0 & -p_3 & p_2 & 0 & 0 & 0 \\
p_3 & 0 & -p_1 & 0 & 0 & 0 \\
-p_2 & p_1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

_and the Hamiltonian function is given by (7)._

**Proof.** The proof can be done in a similar way as the results stated in section II (B) of [6], particularly Proposition 6, for a gyrostat with a fixed point as a Langrange–Poisson system obtained via a geometric–mechanics approach. □
3. Determination of equilibrium solutions

We endeavor to determine some equilibrium solutions of the problem for a gyrostat with gyrostatic momentum \( \mathbf{l} = (0, 0, l) \neq 0 \), and the Hamiltonian \( \mathcal{H} \) given by (7).

From the motion equations of our problem, it can be easily deduced that for any equilibrium of the Hamiltonian vector field \( X_{\mathcal{H}} \), we must have \( \Omega = \omega \mathbf{p} \), being \( \Omega = \left( \frac{\pi_1}{I_1}, \frac{\pi_2}{I_2}, \frac{\pi_3}{I_3} \right) \) the angular velocity of the gyrostat and \( \omega \in \mathbb{R} \). Besides, any possible equilibrium solution is subject to the constraints given by the formulas (3).

Using the variational characterization of the equilibria it can be deduced that the two-parametric family

\[
E_{I_3} = (0, 0, 0, 0, 0, p_0)
\]

are equilibrium solutions of our problem.

In the sequel we calculate some equilibrium solutions different from those previously mentioned via the computation of the multipliers at the variational function using a symbolic commercial mathematical package. We have omitted here the long and tedious calculation since they can be reproduced by the reader in a similar way following the procedure previously described. We shall consider two different cases: the symmetric and the triaxial one.

3.1. Symmetric gyrostat. The result in the case of having a symmetric gyrostat is the following. Recall that symmetric means \( I_1 = I_2 \) and \( m_1 = m_2 \).

**Theorem 3.** The only equilibria for Kirchhoff’s equations of a symmetric gyrostat into an ideal fluid, when its gyrostatic momentum is constant and of the form \( \mathbf{l} = (0, 0, l) \), are

\[
E_{I_3} = (0, 0, 0, 0, 0, p_0)
\]

\[
E_\omega = (I_1 \omega u, I_1 \omega v, I_3 \omega w, u, v, w) \ (u \text{ or } v \neq 0).
\]

In the last equilibria the constraints (3) and (12) must be verified.

**Proof.** We assume that \( I_1 = I_2 \), \( m_1 = m_2 \), (the remaining cases are analogous to this); then

\[
E_\omega = (I_1 \omega u, I_1 \omega v, I_3 \omega w, u, v, w)
\]

are equilibria of the problem iff they are critical points of the function

\[
f := \mathcal{H} + \alpha((\pi + 1) \cdot \mathbf{p}) + \beta(\| \mathbf{p} \|^2)
\]

with multipliers \( \alpha, \beta \) to be determined. These equilibria will exist if the first integrals (3) and the following relations are verified:
(12)  \[ \alpha = -\omega, \quad \beta = \frac{\omega^2 I_1 m_1 - 1}{2 m_1}, \quad w = \frac{\omega m_3 m_1 l}{\omega^2 m_1 m_3 (I_1 - I_3) + (m_1 - m_3)} \]

3.2. Triaxial gyrostat. In the most general case where a triaxial gyrostat, i.e. without any symmetry, is considered some equilibrium solutions different from \( E_{I_3} = (0, 0, \pi_0, 0, 0, p_0) \) are obtained as we shall show.

3.2.1. Equilibria in the plane \( I_1 I_3 \).

**Theorem 4.** The following expressions \( E_\omega \) holding the constraints (3) are equilibria for the Kirchhoff’s equations of a triaxial gyrostat into an ideal fluid when its gyrostatic momentum is constant and of the form \( l = (0, 0, l) \) in the plane \( I_1 I_3 \):

(13) \[ E_\omega = (I_1 \omega u, 0, I_3 \omega w, u, 0, w) \quad (u \neq 0 \text{ and } w \neq 0) \]

**Proof.** We assume that an equilibrium solution has the form:

(14) \[ E_\omega = (I_1 \omega u, 0, I_3 \omega w, u, 0, w) \quad (u \neq 0 \text{ and } w \neq 0) \]

By the variational characterization of the equilibria, the equilibrium configurations will be the critical points of the function

\[ f := H + \alpha((\pi + 1) \cdot p) + \beta(||p||^2) \]

with parameters \( \alpha, \beta \) to be determined. Doing the corresponding calculations we have the following relations

\[ \alpha = -\omega, \quad \beta = \frac{\omega^2 I_1 m_1 - 1}{2 m_1}, \quad w = \frac{\omega m_3 m_1 l}{\omega^2 m_1 m_3 (I_1 - I_3) + (m_1 - m_3)} \]

Besides the previous relations, the equilibria \( E_\omega \) have to verify the constraints (3), ending the proof. \( \square \)

**Remark 5.** In a similar way, other families of equilibria in the plane \( I_1 I_2 \) can be obtained.

The equilibria located at the plane \( I_1 I_2 \) will have the form:

(15) \[ E_\omega = (I_1 \omega u, I_2 \omega v, 0, u, v, 0) \quad (u \neq 0 \text{ and } v \neq 0) \]

Doing the corresponding calculations in these equilibria the following conditions are obtained:

\[ \alpha = 0, \quad \beta = -\frac{1}{2 m_2}, \quad \omega = 0, \quad m_2 = m_1 \]

Moreover, the equilibria in a fixed direction will have the form:

(16) \[ E_\omega = (I_1 \omega u, I_2 \omega v, I_3 \omega w, u, v, w) \quad (u \neq 0, v \neq 0 \text{ and } w \neq 0) \]
Repeating the corresponding calculations, these equilibria will exist if the constraints (3) and the following relations are verified:

\[ \beta = \frac{\omega^2 I_2 m_2 - 1}{2m_2} \]

\[ \alpha = -\omega \quad \omega^2 = \frac{m_2 - m_1}{m_1 m_2 (I_1 - I_2)} \quad w = \frac{\omega m_3 m_1 l}{\omega^2 m_2 m_3 (I_2 - I_3) + (m_2 - m_3)} \]

4. Stability of the equilibria \( E_{I_3} \)

In this section we obtain necessary and sufficient conditions of stability of the equilibria \( E_{I_3} \). Remains open to analyze this problem in the case of equilibria of type \( E_\omega \).

4.1. Stability of \( E_{I_3} \) for a triaxial gyrostat.

4.1.1. Necessary conditions of stability. We endeavor to obtain, by using a spectral analysis of the linearized equations, necessary conditions for the stability of these equilibrium solutions. The linearized Lie–Poisson equations at \( E_{I_3} \) can be reduced to the form

\[
\begin{pmatrix}
\dot{\delta\pi_1} \\
\dot{\delta\pi_2} \\
\dot{\delta p_1} \\
\dot{\delta p_2}
\end{pmatrix} =
\begin{pmatrix}
0 & a - g & 0 & b \\
f - a & 0 & c & 0 \\
0 & -d & 0 & a \\
e & 0 & -a & 0
\end{pmatrix}
\begin{pmatrix}
\delta\pi_1 \\
\delta\pi_2 \\
\delta p_1 \\
\delta p_2
\end{pmatrix}
\]

with \( a = \frac{\pi_0}{I_3}, \ b = p_0 \left( \frac{1}{m_3} - \frac{1}{m_2} \right), \ c = p_0 \left( \frac{1}{m_1} - \frac{1}{m_3} \right), \ d = \frac{p_0}{I_2}, \ e = \frac{p_0}{I_1}, \)

\( f = \frac{\pi_0 + l}{I_1}, \ g = \frac{\pi_0 + l}{I_2} \). Its characteristic polynomial is \( \lambda^4 + p\lambda^2 + q \), where \( p \) and \( q \) are given by

\[ p = 2a^2 - ag - fa + fg - eb - dc \]
\[ q = -eca^2 + a^4 + edbc + fdba - ga^3 - dba^2 + fga^2 - fa^3 + ecag, \]

having the following result.

**Theorem 6.** The system is spectrally stable at the equilibrium \( E_{I_3} \) if the following conditions are verified

\[ p \geq 0 \quad q \geq 0 \quad p^2 - 4q \geq 0 \]

4.1.2. Sufficient conditions of stability.

**Theorem 7.** A sufficient condition for equilibrium solutions \( E_{I_3} \) to be (Lyapunov) stable would be to verify the two conditions:

\[ p_0^2 I_2^2 (m_3 - m_2) + \pi_0^2 (I_3 - I_2) m_3 m_2 + \pi_0 I_3 m_3 m_2 > 0 \]

\[ p_0^2 I_3^2 (m_3 - m_1) + \pi_0^2 (I_3 - I_1) m_3 m_1 + \pi_0 I_3 m_3 m_1 > 0 \]
Proof. The energy–Casimir method is applied to obtain sufficient conditions for the stability, so the following function is considered
\[ f := \mathcal{H} + \phi_1((\pi + l) \cdot \mathbf{p}) + \phi_2(\|\mathbf{p}\|^2) \]
with smooth functions \(\phi_1, \phi_2\). From the equation \(d(f)(E_{I_3}) = 0\), we obtain that
\[ \phi_1'((\pi_0 + l)p_0) = -\frac{\pi_0}{I_3p_0} \quad \phi_2'(p_0^2) = \frac{\pi_0^2m_3 + \pi_0m_3l - p_0^2I_3}{2I_3m_3p_0^2} \]
On the other hand, we have to calculate a base of the vector space:
\[ W = \ker d\phi_1((\pi + l) \cdot \mathbf{p})(E_{I_3}) \cap \ker d\phi_2(\|\mathbf{p}\|^2)(E_{I_3}). \]
Doing the appropriate calculations we get \(W = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5 \rangle\), where \(\{\mathbf{e}_i, i = 1 \ldots 6\}\) is the canonical base of \(\mathbb{R}^6\).

The second derivative \(d^2(f)(E_{I_3})|_{W \times W}\) is given by the matrix:
\[
\begin{pmatrix}
\frac{1}{I_1} & \lambda_1 & 0 & 0 \\
\lambda_1 & \frac{1 + 2\lambda_2m_1}{m_1} & 0 & 0 \\
0 & 0 & \frac{1}{I_2} & 0 \\
0 & 0 & 0 & \frac{1 + 2\lambda_2m_2}{m_2}
\end{pmatrix}
\]
where to simplify the notation we will set \(\lambda_1 = \phi_1'((\pi_0 + l)p_0), \lambda_2 = \phi_2'(p_0^2)\).

The quadratic form given by \(d^2(f)(E_{I_3})|_{W \times W}\) is positive definite if and only if
\[ 1 + 2\lambda_2m_1 - \lambda_1^2I_1m_1 > 0 \]
\[ 1 + 2\lambda_2m_2 - \lambda_2^2I_2m_2 > 0 \]
thus the result is proved. \(\square\)

4.2. Stability of \(E_{I_3}\) for a symmetric gyrostat.

4.2.1. Necessary conditions of stability. In a similar way of the results stated in the previous section we deduce the following result:

Theorem 8. A necessary condition for spectral stability (which implies stability in the sense of Lyapunov) of the equilibria \(E_{I_3}\) is that
\[ \frac{(\pi_0 + l)^2}{p_0^2} \geq 4I_1\left(\frac{1}{m_3} - \frac{1}{m_1}\right) \]
4.2.2. Sufficient conditions of stability.

**Theorem 9.** A sufficient condition for equilibrium solutions \( E_{I_3} \) to be (Lyapunov) stable would be to verify that

\[
\frac{(\pi_0 + l)^2}{p_0^2} > 4I_1\left(\frac{1}{m_3} - \frac{1}{m_1}\right)
\]

**Proof.** To apply the energy–Casimir method the following function is considered

\[
f := \mathcal{H} + \phi_1((\pi + l) \cdot p) + \phi_2(\|p\|^2) + \phi_3(\pi_3)
\]

with smooth functions \( \phi_1, \phi_2, \phi_3 \). From the equation \( d(f)(E_{I_3}) = 0 \), introducing \( \lambda_1 = \phi_1'(\pi_0 + l)p_0), \lambda_2 = \phi_2'(p_0^2), \lambda_3 = \phi_3'(\pi_0) \), we obtain:

\[
\phi_1'(\pi_0 + l)p_0 = -\frac{\pi_0 + \lambda_3}{I_3p_0},
\]

and

\[
\phi_2'(p_0^2) = \frac{\pi_0^2m_3 + \lambda_3I_3m_3(\pi_0 + l) + m_3l\pi_0 - p_0^2I_3}{2I_3m_3p_0^2}.
\]

On the other side, we have to calculate a base of the vectorial space:

\[
W = \ker d\phi_1((\pi + l) \cdot p)(E_{I_3}) \cap \ker d\phi_2(\|p\|^2)(E_{I_3}) \cap \ker d\phi_3(\pi_3)(E_{I_3})
\]

and doing the appropriate calculations we have that \( W = \langle \vec{e}_i, i = 1..6 \rangle \) is the canonical base of \( R^6 \).

We obtain the second derivative \( d^2(f)(E_{I_3}) |_{W \times W} \) whose matrix is given by:

\[
d^2(f)(E_{I_3}) |_{W \times W} = \begin{pmatrix}
\frac{1}{I_1} & \lambda_1 & 0 & 0 \\
\lambda_1 & 1 + 2\lambda_2m_1 & 0 & 0 \\
0 & 0 & \frac{1}{I_1} & 0 \\
0 & 0 & 0 & \frac{1 + 2\lambda_3m_1}{m_1}
\end{pmatrix}
\]

Then, the quadratic form given by \( d^2(f)(E_{I_3}) |_{W \times W} \) is positive definite if and only if

\[
1 + 2\lambda_2m_1 - \lambda_3^2I_1m_1 > 0
\]

and substituting the previous values of \( \lambda_1, \lambda_2, \lambda_3 \), we obtain a polynomial of the second order \( p(\lambda_3) \). If its minimum value is positive, then this polynomial is always positive which proves the result. \( \square \)
5. Hamiltonian Hopf Bifurcation When $I_1 = I_2$ and $m_1 = m_2$

In order to study the possible existence of a non-degenerate Hamiltonian Hopf bifurcation in the case of a symmetric gyrostat in an incompressible ideal fluid, we carry out a further reduction, made possible by the axial symmetry of the problem. Since this reduction is singular, see [1] for more details, we will employ invariant theory to describe the reduced space associated with this symmetry.

5.1. Reduction by axial symmetry. Since $L(z) = \pi_3$ if $l = (0,0,l)$ is an integral for the equations of motion, $\pi_3 = \pi_0$ with $\pi_0 \in \mathbb{R}$. Then there exists a circular action $\Phi : S \times M_{k_0, p_0} \to M_{k_0, p_0}$ with

$$M_{k_0, p_0} = \{ (\Pi + l, p) \in \mathbb{R}^6 / (\Pi + l) \cdot p = k_0, \| p \|^2 = p_0^2 \}$$

defined as $\Phi(R_t, z) = (R_t \pi, R_t k)$ with

$$R_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

The invariants associated with this action are given by the following expressions:

$$\rho_1 = p_1^2 + p_2^2, \quad \rho_3 = \pi_1 p_1 + \pi_2 p_2, \quad \rho_5 = p_3$$

$$\rho_2 = \pi_1^2 + \pi_2^2, \quad \rho_4 = p_1 \pi_2 - p_2 \pi_1, \quad \rho_6 = \pi_3.$$  

Consequently, $M_{k_0, p_0}/S$ is diffeomorphic to the semi-algebraic set

$$V_{k_0, p_0}^l = \{ (\rho_2, \rho_4, \rho_5) \in \mathbb{R}^3 \mid R_{k_0, \pi_0}^l(\rho) = 0, \rho_2 \geq 0, |\rho_5| \leq 1 \}$$

with

$$R_{k_0, \pi_0}^l(\rho) = \rho_2^2 + (k_0 - (\pi_0 + l) \rho_5)^2 - (p_0^2 - \rho_5^2) \rho_2.$$  

Observe that

a: $M_{k_0, p_0}/S$ is differentiable for $|k_0| \neq |(\pi_0 + l) p_0|$;

b: $M_{k_0, p_0}/S$ has a singularity for $k_0 = \pm (\pi_0 + l) p_0$ for arbitrary $p_0 \in \mathbb{R}$.

After skipping constant terms the reduced Hamiltonian in $V_{k_0, b}^l$ is given by

$$H = \frac{1}{2} \left( \frac{\rho_2}{I_1} + \frac{(m_1 - m_3) \rho_5^2}{m_1 m_3} \right).$$
5.2. **Sufficient conditions for the existence of a non–degenerate Hopf bifurcation.** In this section we follow the geometric method introduced by Hanssmann and Van der Meer [7] which is based on an analysis of the relative position of the phase space and the energy level sets of the one–degree freedom system after a proper reduction using the symmetry. The second reduction is not regular and leads to a conical singularity. To apply the geometric method certain hypotheses should be checked. 

**H1:** The stratification of the orbit space into the reduced phase spaces must be locally equivalent to the standard case. **H2:** As the bifurcation parameter changes, the energy level set through the vertex of the cone should change too with non–zero speed from passing outside the cone to intersecting the interior of the cone. **H3:** The energy level set must have second order contact at the vertex of the cone at the bifurcation value of the parameter. The first hypothesis ensures that at the critical value of the integral one locally has a cone deforming into a hyperboloid when the value of the integral is varied. This means that locally the invariants form a Lie algebra isomorphic to sl(2; R). The second hypothesis ensures that the unfolding of the linear part is universal and the third one is the non–degeneracy condition.

**Theorem 10.** For a symmetric gyrostat in an incompressible ideal fluid with constant gyrostatic momentum of the form \( l = (0, 0, l) \), a sufficient condition for the existence of a non–degenerate Hamiltonian Hopf bifurcation at the equilibrium \( z_e = (0, 0, \pi_0, 0, 0, p_0) \) is

\[
(\pi_0 + l)^2 = 4 \frac{I_1 (m_1 - m_3)}{m_1 m_3} p_0^2
\]

with \( m_1 \neq m_3 \) and \( k_0 = |(\pi_0 + l)p_0| \).

**Proof.** Consider an equilibrium point of the form \( z_e = (0, 0, \pi_0, 0, 0, p_0) \). Following [7], let us consider the function

\[
f_{\pi_0+l}(p_5) = (\pi_0 + l)^2 \frac{p_0 - p_5}{p_0 + p_5} - \frac{I_1 (m_1 - m_3)}{m_1 m_3} (p_0^2 - p_5^2)
\]

which measures the distance between the energy surface passing through the equilibrium and the reduced phase space in the plane \( \rho_4 = 0 \).

By Theorem 9 we know that the existence of a possible Hamiltonian Hopf bifurcation is linked to a change on the stability of the equilibria. Therefore, we study when

\[
(\pi_0 + l)^2 = 4 \frac{I_1 (m_1 - m_3)}{m_1 m_3} p_0^2
\]

which is the condition that the parameters must hold in order to have a candidate for a non–degenerate Hopf bifurcation to exist.
• Condition H1 is satisfied because, as we showed in Section 5.1, the reduced space for \( k_0 = |(\pi_0 + l)p_0| \) has a conical singularity which implies that the stratification of the orbit space into the reduced one is locally equivalent to the standard case.

• On the other hand, \( f_{\pi_0+l}(p_0) = 0 \), is satisfied since we have

\[
f'_{\pi_0+l}(p_0) = -\frac{(\pi_0 + l)^2}{2p_0} + 2\frac{I_1 (m_1 - m_3)}{m_1 m_3} p_0 = 0.
\]

Thus, for appropriate values of the gyrostatic momentum, condition H2, namely, \( \frac{d}{d(\pi_0 + l)} f'_{\pi_0+l}(\pi_0 + l) \neq 0 \) is satisfied.

• Finally, condition H3, namely, \( f''_{\pi_0+l}(p_0) \neq 0 \), is satisfied provided that \( m_1 \neq m_3 \).

The proof follows from the application of the geometric method stated in [7] and explained in detail in the paragraph preceding this proof. □

The following picture shows the scenario of the generation of the Hamiltonian Hopf bifurcation of the system for some concrete values of the parameters

![Figure 3. Relative positions of the reduces Hamiltonian \( H = h \) (thin–colors) and \( V^l_{k_0,p_0} \) (thick) within \( \{\rho_4 = 0\} \). From left to right the cases \((\pi_0 + l)^2\) less than, equal to and bigger than \( 4\frac{I_1 (m_1 - m_3)}{m_1 m_3} p_0^2 \). The small, the intermediate and the high energy are respectively blue, orange and red.](image)

**6. Conclusions.**

Kirchhoff’s equations for a gyrostat, in an incompressible ideal fluid, have been written as a Lie–Poisson system. The equilibrium solutions have been obtained for a gyrostat with constant gyrostatic momentum, when it adopts the form \( l = (0, 0, l) \), in the triaxial and symmetric case.
Necessary and sufficient conditions for the stability of the equilibria $E_I$ in the aforementioned cases are obtained. Note that these equilibria correspond to uniform translations and stationary rotations around the third axis of inertia of the system. The obtained results generalize other previous ones for rigid bodies (see for instance [8]) and show the importance of the possible choice of the gyrostatic momentum in order to stabilize certain solutions. As open problem remains the study of necessary and sufficient conditions for the stability of the equilibrium solutions $E_\omega$ and obtaining the regions of stability as functions of two or three parameters of the possible non-null components of the gyrostatic momentum. By the application of a recent geometric-mechanic tool introduced by Hanssmann and Van der Meer [7] we are able to give a sufficient condition for the existence of a non-degenerate Hamiltonian Hopf bifurcation at the equilibria of the form $z_e = (0, 0, \pi_0, 0, 0, p_0^0)$ in the case of a symmetric gyrostat in an incompressible ideal fluid with constant gyrostatic momentum of the form $I = (0, 0, l)$. The analysis of the existence of this kind of bifurcation for the equilibria of the form $E_\omega$ is open.

ACKNOWLEDGEMENTS

The authors thank the comments and remarks of the referees which improved the manuscript.

This work was supported partially by MCI (Ministerio de Ciencia e Innovación) and FEDER (Fondo Europeo Desarrollo Regional), grant number MTM2011-22587, Fundación Séneca de la Región de Murcia, grant number 12001/PI/09, and JCCM (Junta de Comunidades de Castilla-La Mancha), grant number PEII09-0220-0222.

REFERENCES


1Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203 Cartagena, Región de Murcia, Spain –Corresponding Author–

2Centro Universitario de la Defensa. Academia General del Aire. Universidad Politécnica de Cartagena. San Javier, Región de Murcia, Spain

E-mail address: juan.garcia@upct.es, juanantonio.vera@upct.es